③ delta form $(\delta\phi \equiv \phi^{n+1} - \phi^{n})$

$$\frac{\partial \phi}{\partial t} = \alpha\left(\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2}\right)$$

CN+CD2 : $\dfrac{\phi^{n+1} - \phi^{n}}{\Delta t} = \dfrac{\alpha}{2}\left(A_x \phi^{n+1} + A_x \phi^{n} + A_y \phi^{n+1} + A_y \phi^{n}\right) + O(\Delta t^2) + O(\Delta x^2) + O(\Delta y^2)$

$(\delta\phi = \phi^{n+1} - \phi^{n})$

$\rightarrow \dfrac{\delta\phi}{\Delta t} = \dfrac{\alpha}{2}\left(A_x(\delta\phi + \phi^{n}) + A_x \phi^{n} + A_y(\delta\phi + \phi^{n}) + A_y \phi^{n}\right)$

$\rightarrow \underbrace{\left(I - \dfrac{\alpha \Delta t}{2} A_x - \dfrac{\alpha \Delta t}{2} A_y\right)\delta\phi}_{\parallel} = \alpha \Delta t \left(A_x \phi^{n} + A_y \phi^{n}\right) + \Delta t\left[O(\Delta t^2) + O(\Delta x^2) + O(\Delta y^2)\right]$

$\left(I - \dfrac{\alpha \Delta t}{2} A_x\right)\left(I - \dfrac{\alpha \Delta t}{2} A_y\right)\delta\phi - \underbrace{\dfrac{1}{4}\alpha^2 \Delta t^2 A_x A_y \delta\phi}_{\text{neglect w/o losing accuracy}}$

$\delta\phi = \phi^{n+1} - \phi^{n} = \Delta t \dfrac{\partial \phi^{n}}{\partial t} + \cdots$

$O(\Delta t^3)$

$$\rightarrow \boxed{\left(I-\frac{\alpha\Delta t}{2}A_x\right)\left(I-\frac{\alpha\Delta t}{2}A_y\right)\delta\phi = \alpha\Delta t\left(A_x\phi^n + A_y\phi^n\right)}$$

ADI in δ-form

Let $\left(I-\frac{\alpha\Delta t}{2}A_y\right)\delta\phi = z$.

Then $\left(I-\frac{\alpha\Delta t}{2}A_x\right)z = \alpha\Delta t\left(A_x\phi^n + A_y\phi^n\right)$

$$\rightarrow z_{l,j} - \frac{\alpha\Delta t}{2}\,\frac{z_{l+1,j} - 2z_{l,j} + z_{l-1,j}}{\Delta x^2} = RHS_{l,j} \qquad \begin{aligned} l &= 1,2,\cdots,M-1 \\ j &= 1,2,\cdots,N-1 \end{aligned}$$

for each j, solve tri-diagonal matrix to get $z_{l,j}$.

(we need $z_{0,j}$ & $z_{M,j}$ as boundary conditions)

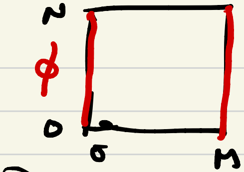
Then, $\left(I-\frac{\alpha\Delta t}{2}A_y\right)\delta\phi = z$

$$\rightarrow \delta\phi_{l,j} - \frac{\alpha\Delta t}{2}\,\frac{\delta\phi_{l,j+1} - 2\delta\phi_{l,j} + \delta\phi_{l,j-1}}{\Delta y^2} = z_{l,j}$$

for each l, solve tri-diagonal matrix to get $\delta\phi_{l,j}$.

b.c.'s : $\left.\begin{aligned} \delta\phi_{l,0} &= \phi^{n+1}_{l,0} - \phi^{n}_{l,0} \\ \delta\phi_{l,N} &= \phi^{n+1}_{l,N} - \phi^{n}_{l,N} \end{aligned}\right]$ known.

Once $\delta\phi$ is obtained, $\phi^{n+1} = \delta\phi + \phi^{n}$.



How about $z_{0,j}$ & $z_{M,j}$?

@ $l=0$: $z_{0,j} = \delta\phi_{0,j} - \frac{\alpha\Delta t}{2}\,\frac{\delta\phi_{0,j+1} - 2\delta\phi_{0,j} + \delta\phi_{0,j-1}}{\Delta y^2}$

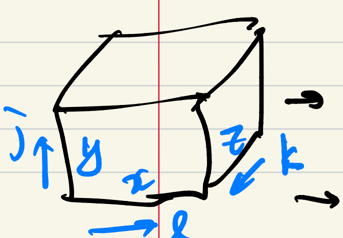
@ $l=M$: $z_{M,j} = \delta\phi_{M,j} - \frac{\alpha\Delta t}{2}\,\frac{\delta\phi_{M,j+1} - 2\delta\phi_{M,j} + \delta\phi_{M,j-1}}{\Delta y^2}$

If b.c.'s do not change in time, $\left.\begin{array}{l}\delta\phi_{0,j} = 0\\ \delta\phi_{M,j} = 0\end{array}\right] \rightarrow \begin{array}{l} z_{0,j} = 0\\ z_{M,j} = 0\end{array}$

- How about 3D ?

$$\frac{\partial\phi}{\partial t} = \alpha\left(\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} + \frac{\partial^2\phi}{\partial z^2}\right) \qquad \left.\begin{array}{l}\text{EE}\\ \text{CD2}\end{array}\right] \rightarrow \Delta t \propto \Delta x^2 \ \times$$

CN+CD2 : $\underbrace{\left(I - \beta\Delta t A_x - \beta\Delta t A_y - \beta\Delta t A_z\right)}\underset{\delta\phi}{\phi^{n+1}} = \underline{RHS}$



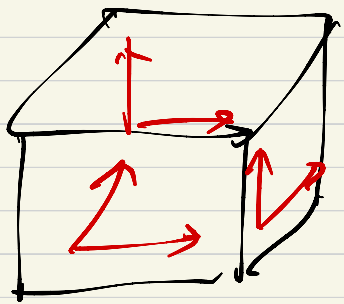
$\rightarrow (I - \beta\Delta t A_x)\underbrace{(I - \beta\Delta t A_y)(I - \beta\Delta t A_z)\underset{\delta\phi}{\phi^{n+1}}}_{= z} = \underset{RHS}{RHS'}$

$\rightarrow (I - \beta\Delta t A_x)\, z = RHS'$: tri-diagonal matrix in x for each j and k

$$\underbrace{(I-\beta_0\Delta t A_y)(I-\beta_0\Delta t A_z)\phi^{n+1}}_{=Y} = Z$$

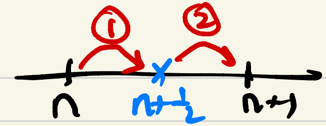
$\rightarrow (I-\beta_0\Delta t A_y)Y = Z$: tri-diagonal matrix in y for each l and k.

$\rightarrow (I-\beta_0\Delta t A_z)\phi^{n+1} = Y$: " " in z for each l and j



ADI method

⊙ Peaceman & Rachford (1955)



① $$\frac{\phi^{n+\frac{1}{2}}-\phi^{n}}{\Delta t/2} = \alpha\left(\frac{\partial^2\phi^{n+\frac{1}{2}}}{\partial x^2}+\frac{\partial^2\phi^{n}}{\partial y^2}\right)$$: implicit only in x direction

② $$\frac{\phi^{n+1}-\phi^{n+\frac{1}{2}}}{\Delta t/2} = \alpha\left(\frac{\partial^2\phi^{n+\frac{1}{2}}}{\partial x^2}+\frac{\partial^2\phi^{n+1}}{\partial y^2}\right)$$: " " " y "

→ Alternating directional implicit method.

CD2: ① $(I-\alpha\frac{\Delta t}{2}A_x)\phi^{n+\frac{1}{2}} = (I+\frac{\alpha\Delta t}{2}A_y)\phi^{n}$: tri-diagonal matrix in x for each j

② $(I-\alpha\frac{\Delta t}{2}A_y)\phi^{n+1} = (I+\frac{\alpha\Delta t}{2}A_x)\phi^{n+\frac{1}{2}}$: " in y for each l

① → $\phi^{n+\frac{1}{2}} = (I-\frac{\alpha\Delta t}{2}A_x)^{-1}(I+\frac{\alpha\Delta t}{2}A_y)\phi^{n}$

→ ② : $(I-\alpha\frac{\Delta t}{2}A_y)\phi^{n+1} = (I+\frac{\alpha\Delta t}{2}A_x)(I-\frac{\alpha\Delta t}{2}A_x)^{-1}(I+\frac{\alpha\Delta t}{2}A_y)\phi^{n}$

→ $$\boxed{\begin{aligned}&(I-\alpha\tfrac{\Delta t}{2}A_x)(I-\alpha\tfrac{\Delta t}{2}A_y)\phi^{n}\\ &= (I+\tfrac{\alpha\Delta t}{2}A_x)(I+\tfrac{\alpha\Delta t}{2}A_y)\phi^{n}\end{aligned}} = \underbrace{\cancel{(I-\alpha\tfrac{\Delta t}{2}A_x)}(I+\tfrac{\alpha\Delta t}{2}A_x)}_{\text{commute}}\cancel{(I-\alpha\tfrac{\Delta t}{2}A_x)^{-1}}(I+\tfrac{\alpha\Delta t}{2}A_y)\phi^{n}$$

$\therefore$ same as before from approx. factorization!

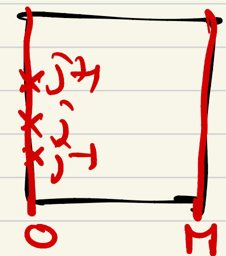
$\rightarrow$ second-order accurate in time.

b.c.'s for $\phi^{n+\frac{1}{2}}$

$$① \quad (I - \tfrac{\Delta t}{2} A_x)\phi_B^{n+\frac{1}{2}} = (I + \tfrac{\Delta t}{2} A_y)\phi_B^{n}$$

$$② \quad (I - \tfrac{\Delta t}{2} A_y)\phi_B^{n+1} = (I + \tfrac{\Delta t}{2} A_x)\phi_B^{n+\frac{1}{2}}$$

$$2\phi_B^{n+\frac{1}{2}} = (I + \tfrac{\Delta t}{2} \underbrace{A_y)\phi_B^{n}}_{\frac{\phi^n_{B,j+1} - 2\phi^n_{B,j} + \phi^n_{B,j-1}}{\Delta y^2}} + (I - \tfrac{\Delta t}{2} A_y)\phi_B^{n+1} : \text{known}$$



5.9.3 Mixed time advancement

$$\boxed{\frac{\partial \phi}{\partial t} = -c\frac{\partial \phi}{\partial x} + \alpha \frac{\partial^2 \phi}{\partial x^2}}$$ convection-diffusion eq.

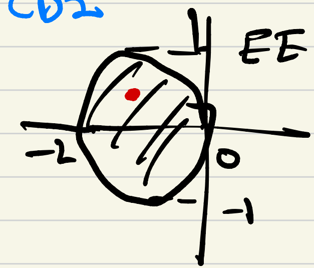
$\phi = \psi e^{ikx_j}$

$$\frac{\partial \phi}{\partial t} = -c\frac{\partial \phi}{\partial x} \rightarrow \frac{d\psi}{dt} = -ick'\psi$$: purely imaginary

$$\frac{\partial \phi}{\partial t} = \alpha\frac{\partial^2 \phi}{\partial x^2} \rightarrow \frac{d\psi}{dt} = -\alpha k'^2\psi$$: real & negative

CD2

$$\frac{d\psi}{dt} = \underbrace{\left(-\alpha k_D'^2 - ick_c'\right)}_{\text{complex number}}\psi$$



- EE on conv. term + on diff. term

$$\frac{\phi^{n+1}-\phi^n}{\Delta t} = -c\underbrace{\frac{\partial \phi^n}{\partial x}}_{\Delta t \propto \Delta x} + \alpha \underbrace{\frac{\partial^2 \phi^n}{\partial x^2}}_{\Delta t \propto \Delta x^2} + O(\Delta t)$$

$\Rightarrow \Delta t \propto \Delta x^2$ not good!

- EE on conv. term + CN on diff. term

$$\frac{\phi^{n+1}-\phi^n}{\Delta t} = \underbrace{-c\frac{\partial \phi^n}{\partial x}}_{\Delta t \,\propto\, \Delta x} + \underbrace{\frac{1}{2}\alpha\left(\frac{\partial^2 \phi^{n+1}}{\partial x^2} + \frac{\partial^2 \phi^n}{\partial x^2}\right)}_{\text{CD2} \rightarrow \text{TDMA}} + O(\Delta t)$$

$\Delta t \propto \Delta x$

not bad!

- AB2 on conv. term + CN on diff. term

$$\frac{\phi^{n+1}-\phi^n}{\Delta t} = -\frac{c}{3}\left(3\frac{\partial \phi^n}{\partial x} - \frac{\partial \phi^{n-1}}{\partial x}\right) + \frac{\alpha}{2}\left(\frac{\partial^2 \phi^{n+1}}{\partial x^2} + \frac{\partial^2 \phi^n}{\partial x^2}\right) + O(\Delta t^2)$$

CD2 → TDMA

stability analysis : conditionally stable → $\boxed{\frac{\Delta t\, c}{\Delta x} \le 1}$

store ϕ^n & ϕ^{n-1}, not self-starting, Δt fixed

- CN on all terms

Kim, Moin & Moser (1987) JFM

$$\frac{\phi^{n+1}-\phi^n}{\Delta t} = -\frac{c}{2}\left(\frac{\partial \phi^{n+1}}{\partial x} + \frac{\partial \phi^n}{\partial x}\right) + \frac{\alpha}{2}\left(\frac{\partial^2 \phi^{n+1}}{\partial x^2} + \frac{\partial^2 \phi^n}{\partial x^2}\right) + O(\Delta t^2)$$

CD2 → TDMA

absolutely stable

How about $c = \phi$?

$$\frac{\partial \phi}{\partial t} = -\phi \frac{\partial \phi}{\partial x} + \alpha \frac{\partial^2 \phi}{\partial x^2}$$: nonlinear conv.-diff eq.

- CN on all terms

$$\frac{\phi^{n+1} - \phi^n}{\Delta t} = -\frac{1}{2}\left(\underline{\phi^{n+1} \frac{\partial \phi^{n+1}}{\partial x}} + \phi^n \frac{\partial \phi^n}{\partial x}\right) + \frac{\alpha}{2}\left(\frac{\partial^2 \phi^{n+1}}{\partial x^2} + \frac{\partial^2 \phi^n}{\partial x^2}\right) + O(\Delta t^2)$$

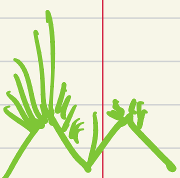
fully implicit method → absolutely stable

CD2

↳ nonlinear algebraic eq.

↳ i) linearize this eq.

ii) solve this eq. with iteration



- AB2 on conv. term + CN on diff. term

$$\frac{\phi^{n+1} - \phi^n}{\Delta t} = \underbrace{-\frac{1}{2}\left(3\phi^n \frac{\partial \phi^n}{\partial x} - \phi^{n-1} \frac{\partial \phi^{n-1}}{\partial x}\right)}_{\text{explicit}} + \underbrace{\frac{\alpha}{2}\left(\frac{\partial^2 \phi^{n+1}}{\partial x^2} + \frac{\partial^2 \phi^n}{\partial x^2}\right)}_{\text{CD2} \rightarrow \text{TDMA}} + O(\Delta t^2)$$

($\Delta t \propto \Delta x$ ($\frac{\Delta t \phi}{\Delta x} \lesssim 1$) explicit — CD2 → TDMA implicit

no iteration

semi-implicit method

not self starting, fixed Δt, store ϕ^n & ϕ^{n-1}

- AB2 on conv. & diff terms

$$\frac{\phi^{n+1}-\phi^{n}}{\Delta t} = \underbrace{-\frac{1}{2}\left(3\phi^{n}\frac{\partial\phi^{n}}{\partial x} - \phi^{n-1}\frac{\partial\phi^{n-1}}{\partial x}\right)}_{\Delta t \,\propto\, \Delta x} + \underbrace{\frac{\alpha}{2}\left(3\frac{\partial^2\phi^{n}}{\partial x^2} - \frac{\partial^2\phi^{n-1}}{\partial x^2}\right)}_{\Delta t \,\propto\, \Delta x^2 \;\; X} + O(\Delta t^2)$$

fully explicit method

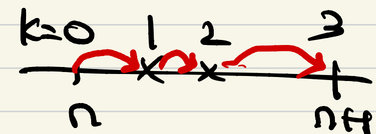
- RK3 on conv. term + CN on diff. term

$$\frac{\phi^{k}-\phi^{k-1}}{\Delta t} = \underbrace{\gamma_k \phi^{k-1}\frac{\partial\phi^{k-1}}{\partial x} + \rho_k \phi^{k-2}\frac{\partial\phi^{k-2}}{\partial x}}_{O(\Delta t^3)} + \underbrace{\alpha_k \frac{\partial^2\phi^{k}}{\partial x^2} + \alpha_k \frac{\partial^2\phi^{k-1}}{\partial x^2}}_{O(\Delta t^2)} + O(\Delta t^2)$$

$K=1,2,3$

$$\phi^0 = \phi^n, \quad \phi^3 = \phi^{n+1}$$

$$\left(\begin{array}{lll} \alpha_1 = \frac{4}{15}, & \gamma_1 = \frac{8}{15}, & \boxed{\rho_1 = 0} \\ \alpha_2 = \frac{1}{15}, & \gamma_2 = \frac{5}{12}, & \rho_2 = -\frac{17}{60} \\ \alpha_3 = \frac{1}{6}, & \gamma_2 = \frac{3}{4}, & \rho_3 = -\frac{5}{12} \end{array}\right)$$



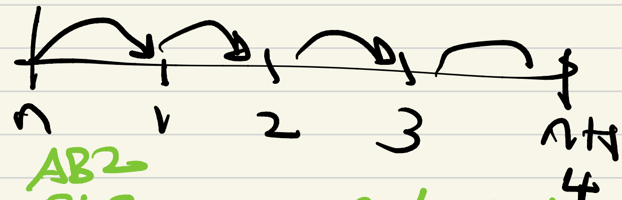
self starting

$$\frac{\Delta t\,\phi}{\Delta x} \le \sqrt{3} \quad (\Delta t \propto \Delta x)$$

semi-implicit method
change Δt during computation
store ϕ^{k-1} & ϕ^{k-2}

(No class next week.
Supplementary lectures will be uploaded later.)

- RK4 on conv. term & CN on diff. term



too much

AB2
RK3
CN

CN w/ ADI

* Navier-Stokes eqs.

$$\begin{cases} \rho \dfrac{\partial u_i}{\partial t} + \boxed{\rho u_j \dfrac{\partial u_i}{\partial x_j}} = -\dfrac{\partial p}{\partial x_i} + \dfrac{\partial}{\partial x_j}\left(\mu \dfrac{\partial u_i}{\partial x_j}\right) \\ \dfrac{\partial u_i}{\partial x_i} = 0 \end{cases}$$

$i = 1, 2, 3$
$j = 1, 2, 3$

$i = 1$: u_1 u_1, u_2, u_3

nonlinear

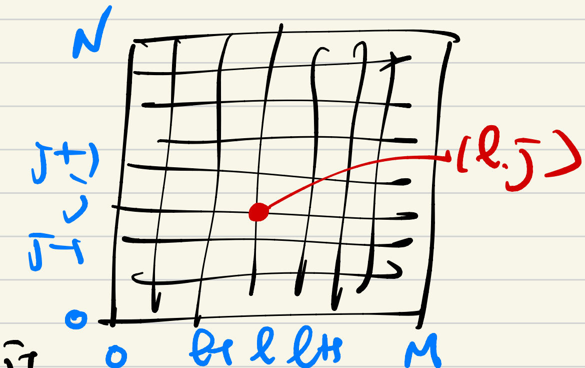
coupled

2nd-order PDE
3-dimensional

4 eqs. for u_i & p

5.10 Elliptic PDEs

$$\begin{cases} \nabla^2\phi = 0 & \text{Laplace eq.} \\ \nabla^2\phi = f & \text{Poisson eq.} \\ \nabla^2\phi + k^2\phi = f & \text{Helmholtz eq.} \end{cases}$$



- $\nabla^2\phi = f$

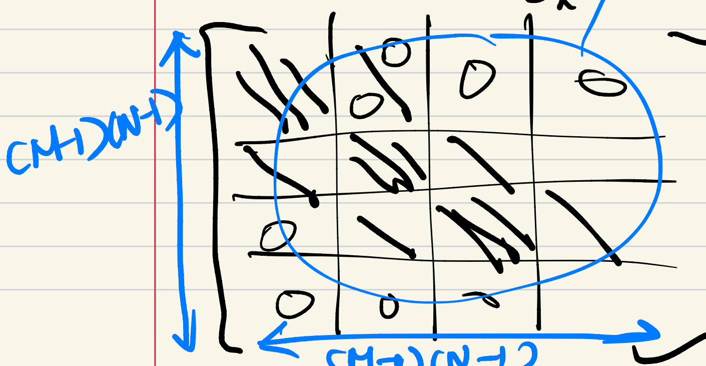
in 2D, $\dfrac{\partial^2\phi}{\partial x^2} + \dfrac{\partial^2\phi}{\partial y^2} = f$

$(M-1)(N-1) \times (M-1)(N-1)$

CD2: $\dfrac{\phi_{l+1,j} - 2\phi_{l,j} + \phi_{l-1,j}}{\Delta x^2} + \dfrac{\phi_{l,j+1} - 2\phi_{l,j} + \phi_{l,j-1}}{\Delta y^2} = f_{l,j}$

$l = 1, 2, \cdots, M-1$

$j = 1, 2, \cdots, N-1$



$$\begin{bmatrix} \phi_{1,1} \\ \phi_{2,1} \\ \phi_{3,1} \\ \vdots \\ \phi_{l,j} \\ \vdots \\ \phi_{M-1,N-1} \end{bmatrix} = \begin{bmatrix} \vdots \end{bmatrix}$$

Block-tridiagonal matrix

difficult to solve directly

↓

introduce iterative methods

$$\frac{\phi^{k}_{\ell+1,j} - 2\phi^{k+1}_{\ell,j} + \phi^{k}_{\ell-1,j}}{\Delta x^2} + \frac{\phi^{k}_{\ell,j+1} - 2\phi^{k+1}_{\ell,j} + \phi^{k}_{\ell,j-1}}{\Delta y^2} = f_{\ell,j}$$

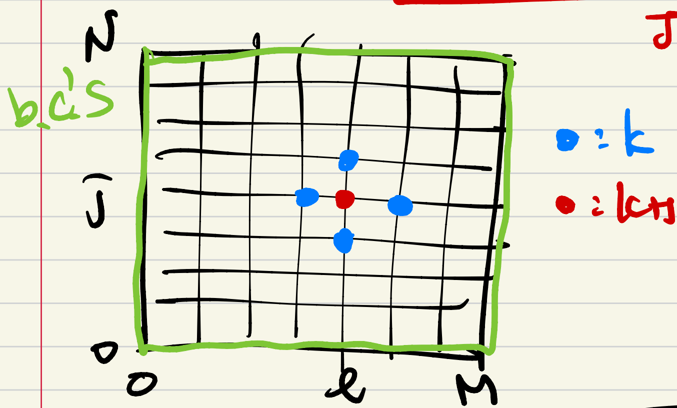
$(\Delta x = \Delta y) \rightarrow$ $$\boxed{\phi^{k+1}_{\ell,j} = \frac{1}{4}\left(\phi^{k}_{\ell+1,j} + \phi^{k}_{\ell-1,j} + \phi^{k}_{\ell,j+1} + \phi^{k}_{\ell,j-1}\right) - \frac{\Delta x^2}{4} f_{\ell,j}}$$

k : iteration index

Jacobi iteration

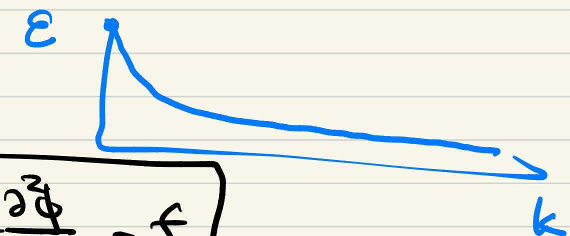
$\ell = 1, 2, \cdots, M-1$

$j = 1, 2, \cdots, N-1$



↓

slow convergence



$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = f \longrightarrow \boxed{\frac{\partial \phi}{\partial t} = \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} - f}$$

ellip. eq. para. eq.

As $t \to \infty$, steady sol. $\to$ sol. of Poisson eq.

$\left(\frac{\partial \phi}{\partial t} = 0\right)$

We integrate para. eq. until reaching steady state ($\frac{\partial \phi}{\partial t}=0$).

EE (Δx = Δy) CD2:

$$\frac{\phi^{n+1}_{l,j}-\phi^{n}_{l,j}}{\Delta t} = \frac{1}{\Delta x^2}\left(\phi^n_{l+1,j} - 2\phi^n_{l,j} + \phi^n_{l-1,j} + \phi^n_{l,j+1} - 2\phi^n_{l,j} + \phi^n_{l,j-1}\right) - f^n_{l,j}$$

stability limit: $\Delta t \le \frac{\Delta x^2}{4}$

$\Delta t_{max} = \Delta x^2/4$

not popular

$$\rightarrow \phi^{n+1}_{l,j} = \frac{1}{4}\left(\phi^n_{l+1,j} + \phi^n_{l-1,j} + \phi^n_{l,j+1} + \phi^n_{l,j-1}\right) - \frac{\Delta x^2}{4} f_{l,j}$$

same as Jacobi iteration → very slow convergence

CN + CD2: no limit in Δt.

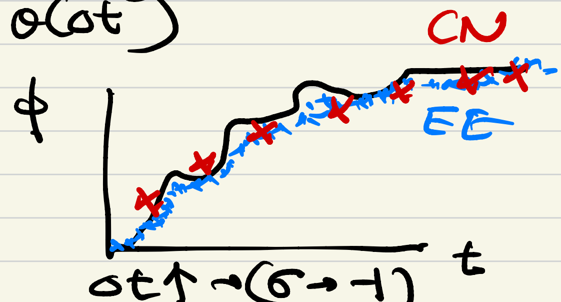
$$\frac{\phi^{n+1}-\phi^{n}}{\Delta t} = \frac{1}{2}\nabla^2\left(\phi^{n+1}+\phi^{n}\right) - f + O(\Delta t^2)$$

ADI

IE + CD2: no limit in Δt

$$\frac{\phi^{n+1}-\phi^{n}}{\Delta t} = \nabla^2\phi^{n+1} - f + O(\Delta t)$$

this is better.



5.10.1 Iterative solution methods

$Ax = b \rightarrow x = A^{-1}b$ direct sol. $\leftarrow O(M^3N^3)$ operations expensive

$$A = A_1 - A_2$$

$$A_1 x = A_2 x + b \quad \text{— } (*)$$

$$A_1 x^{k+1} = A_2 x^k + b \quad \text{— } (**)$$

k : iteration index

$$\rightarrow \boxed{x^{k+1} = A_1^{-1} A_2 x^k + A_1^{-1} b}$$

A_1 should be easy to invert.

Error : $\varepsilon^k = x - x^k$

$(*) - (**)$: $A_1(x - x^{k+1}) = A_2(x - x^k)$

$$\rightarrow A_1 \varepsilon^{k+1} = A_2 \varepsilon^k \rightarrow \varepsilon^{k+1} = A_1^{-1} A_2 \varepsilon^k$$

$$\rightarrow \varepsilon^k = (A_1^{-1} A_2)^k \varepsilon^0$$

For convergence, $\varepsilon^k \rightarrow 0$

$\| A_1^{-1} A_2 \| < 1$: eigenvalues of $A_1^{-1} A_2$ should have modulus less than 1.

$|\lambda_{max}| < 1$

5.10.2 Point Jacobi iteration

$\nabla^2\phi = f \xrightarrow{CD2} A\phi = b$

$$A = \begin{bmatrix} \begin{matrix}-4 & 1 & 0\\ 1 & -4 & 1\\ & \ddots & \ddots\end{matrix} & \begin{matrix}1&0&0\\0&1&0\\0&0&1\ddots\end{matrix} & 0 \\ \begin{matrix}1 & & \\ & 1 & \\ & & 1\ddots\end{matrix} & \begin{matrix}-4&1&\\1&-4&1\\&\ddots&\ddots\end{matrix} & \begin{matrix}1&&\\&1&\\&&1\ddots\end{matrix} \\ & & \end{bmatrix}$$

$A = A_1 - A_2 \Rightarrow A_1\phi^{k+1} = A_2\phi^k + b$

A_1 : diagonal matrix D

↑ simplest choice for A_1

$$A_1 = D = \begin{bmatrix} -4 & & 0 \\ & -4 & \\ 0 & & \ddots -4 \end{bmatrix} \rightarrow A_1^{-1} = D^{-1} = \begin{bmatrix} -\frac{1}{4} & & 0 \\ & -\frac{1}{4} & \\ 0 & & \ddots -\frac{1}{4} \end{bmatrix}$$

$$A_2 = \begin{bmatrix} \begin{matrix}0 & -1 & \\ -1 & 0 & -1\\ & & \end{matrix} & \begin{matrix}-1 & & \\ & -1 & \\ & & -1\end{matrix} & 0 \\ \begin{matrix}-1 & & \\ & -1 & \\ & & -1\end{matrix} & \begin{matrix}0 & -1 & \\ -1 & 0 & -1\\ & -1 & 0 & -1 \\ & & \ddots\end{matrix} & \\ & & \end{bmatrix}$$

$A_1^{-1}A_2 = D^{-1}A_2 = -\frac{1}{4}A_2$

point Jacobi method

$$\rightarrow \phi^{k+1} = -\frac{1}{4}A_2\phi^k - \frac{1}{4}b \Rightarrow \boxed{\phi_{l,j}^{k+1} = \frac{1}{4}\left[\phi_{l+1,j}^k + \phi_{l-1,j}^k + \phi_{l,j+1}^k + \phi_{l,j-1}^k\right] - \frac{1}{4}b_{l,j}}$$

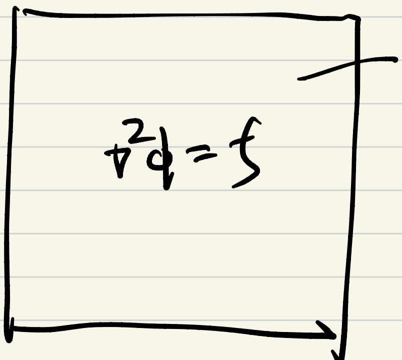
λ for $D^{-1}A_2$: $\overset{''\lambda_j}{\lambda_{l,j}} = \frac{1}{2}\left[\cos\frac{l\pi}{M} + \cos\frac{j\pi}{N}\right]$ $\quad l = 1, 2, \cdots, M-1$, $\quad j = 1, 2, \cdots, N-1$

$$\lambda_{max} = \frac{1}{2}\left[\cos\frac{\pi}{M} + \cos\frac{\pi}{N}\right] = 1 - \frac{1}{4}\left(\frac{\pi^2}{M^2} + \frac{\pi^2}{N^2}\right) + \cdots$$

$\left(\cos x = 1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 + \cdots\right)$ $\qquad \left(\varepsilon^k \doteq (\lambda_{max})^k \varepsilon^0\right)$

For large M and N, max eigenvalues is close to 1.

$\varepsilon^k \sim \lambda_{max}^k \varepsilon^0 \rightarrow$ slow convergence.



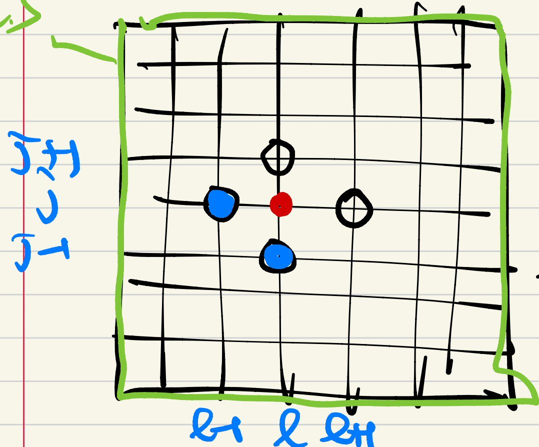
$M = 20$, $N = 20$

$M = 1000$, $N = 1000$ $\rightarrow$ look for 'marginal' solution.

5.10.3 Gauss-Seidel method (GS)

CD2: $\phi_{l,j}^{k+1} = \frac{1}{4}\left(\phi_{l+1,j}^{k} + \phi_{l-1,j}^{k+1} + \phi_{l,j+1}^{k} + \phi_{l,j-1}^{k+1}\right) - \frac{1}{4} b_{l,j}$

6 c.'s



$$A = A_1 - A_2 = (D-L) - U$$

$$(D-L)\phi^{k+1} = U\phi^{k} + b$$

$$\rightarrow \phi^{k+1} = (D-L)^{-1} U \phi^{k} + (D-L)^{-1} b$$

Eigenvalues of $A_1^{-1}A_2$:

$$\lambda_{l,j} = \frac{1}{4}\left(\cos\frac{l\pi}{M} + \cos\frac{j\pi}{N}\right)^2 \qquad l=1,2,\cdots,M-1 \quad j=1,2,\cdots,N-1$$

$$\lambda_{GS} = \lambda_J^2$$

$$\varepsilon_{GS}^{k} = \lambda_{GS}^{k}\varepsilon^{0} = (\lambda_J^2)^k \varepsilon^0 = \lambda_J^{2k}\varepsilon^0 = \varepsilon_J^{2k}$$

$\therefore$ GS is twice faster than Jacobi.