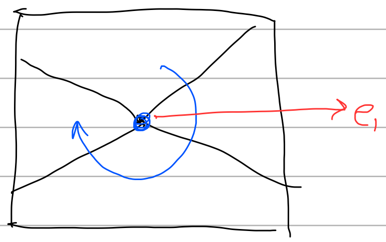
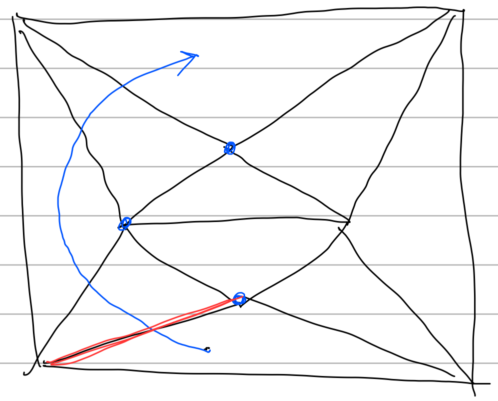
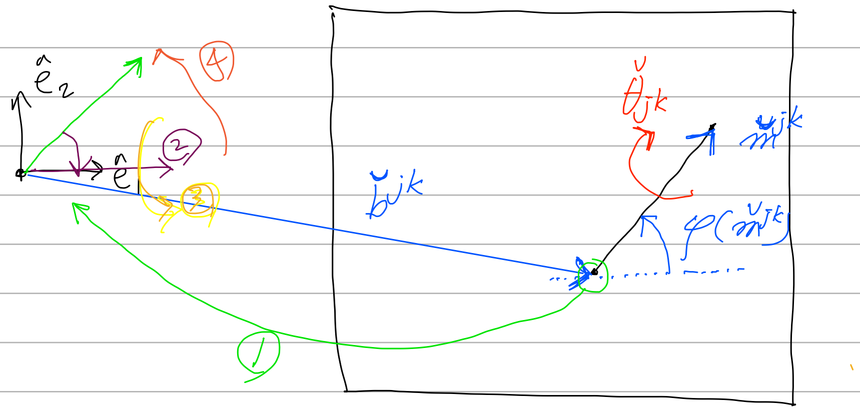
Folding map formation







1. The axis of rotation is translated such that it crosses the origin (translation by $-\breve{b}^{jk}$)

2. The axis of rotation is then aligned with $\hat{e}_1$ (via a rotation of $-\varphi(\breve{m}^{jk})$ about $\hat{e}_3$)

3. $\breve{\theta}_{jk}$ about $\hat{e}_1$ is applied.

4. The axis of rotation is aligned back to its initial orientation. ($\varphi(\breve{m}^{jk})$ about $\hat{e}_3$)

5. The axis of rotation is finally translated to its initial position. (translated by $\breve{b}^{jk}$)

Translation and rotation

$T(b) \in \mathbb{R}^{4\times4}$

$$T(b) = \left[\begin{array}{ccc|c} 1 & 0 & 0 & b_1 \\ 0 & 1 & 0 & b_2 \\ 0 & 0 & 1 & b_3 \\ \hline 0 & 0 & 0 & 1 \end{array}\right] = \begin{bmatrix} I_3 & b \\ 0_3^T & 1 \end{bmatrix}$$

$$T(b) \begin{bmatrix} y \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & b_1 \\ 0 & 1 & 0 & b_2 \\ 0 & 0 & 1 & b_3 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} y_1 + b_1 \\ y_2 + b_2 \\ y_3 + b_3 \\ 1 \end{bmatrix} = \begin{bmatrix} y + b \\ 1 \end{bmatrix}$$

$Q_1(\phi) \in \mathbb{R}^{4\times 4}$

$$Q_1(\phi) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\phi & -s\phi & 0 \\ 0 & s\phi & c\phi & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} R_1(\phi) & 0_3 \\ 0_3^T & 1 \end{bmatrix}$$

$$Q_1(\phi) \begin{bmatrix} y \\ 1 \end{bmatrix} = \begin{bmatrix} R_1(\phi)\, y \\ 1 \end{bmatrix}$$

$$Q_3(\phi) = \begin{bmatrix} c\phi & -s\phi & 0 & 0 \\ s\phi & c\phi & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} R_3(\phi) & 0_3 \\ 0_3^T & 1 \end{bmatrix}$$

Folding transformation matrix

$$\check{H}_{jk} = T(\check{b}^{jk})\, Q_3(\varphi(\check{m}^{jk}))\, Q_1(\check{\theta}_{jk})\, Q_3^{-1}(\varphi(\check{m}^{jk})) \cdot T^{-1}(\check{b}_{jk})$$

translation by $-\check{b}^{jk}$

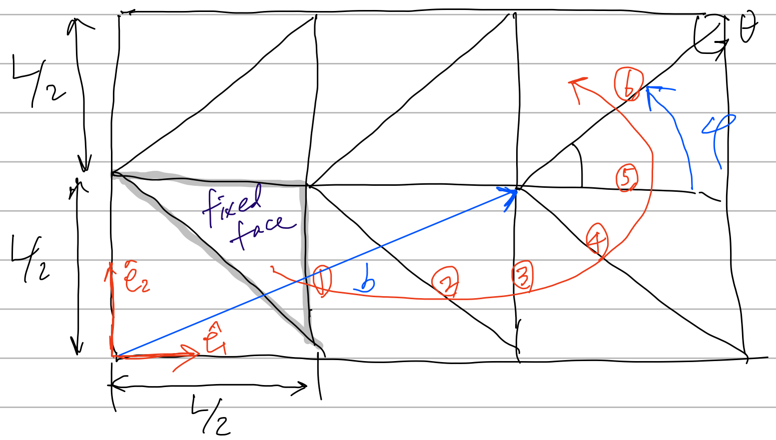
→ rotation by $-\varphi(\breve{m}^{jk})$ about $\hat{e}_3$

→ rotation by $\breve{\theta}_{jk}$ about $\hat{e}_1$

→ rotation by $\varphi(\breve{m}^{jk})$

→ translation by $\breve{b}^{jk}$

Example. Determine the folding transformation matrix associated with the six fold crossed by the path $\breve{\gamma}^j(\eta)$ shown below.



$\varphi = \frac{\pi}{4}$

$b = [L \;\; \frac{L}{2} \;\; 0]^T$

$$T(b) = \begin{bmatrix} 1 & 0 & 0 & L \\ 0 & 1 & 0 & L/2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

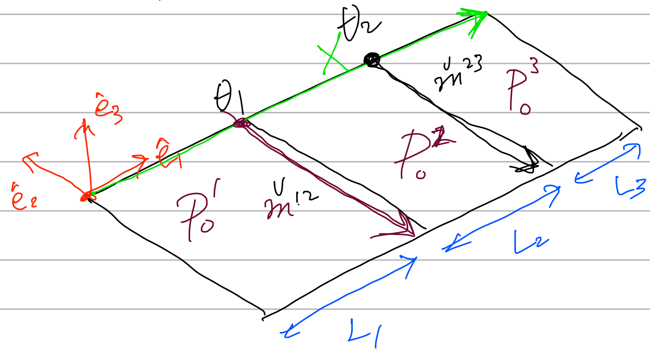
$$Q_1(\theta) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & c\theta & -s\theta & 0 \\ 0 & s\theta & c\theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad Q_3(\varphi) = \begin{bmatrix} \sqrt{2}/2 & -\sqrt{2}/2 & 0 & 0 \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

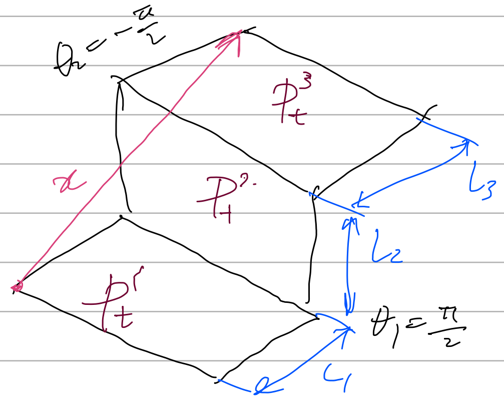
$$\breve{H}_6 = T(b)\, Q_3(\varphi)\, Q_1(\theta)\, Q_3^T(\varphi) \cdot T^{-1}(b)$$

$$
= \begin{bmatrix} \frac{1}{2}+\frac{1}{2}c\theta & \frac{1}{2}-\frac{1}{2}c\theta & \frac{\sqrt{2}}{2}s\theta & -\frac{L}{4}(c\theta-1) \\ \frac{1}{2}-\frac{1}{2}c\theta & \frac{1}{2}+\frac{1}{2}c\theta & -\frac{\sqrt{2}}{2}s\theta & \frac{L}{4}(c\theta-1) \\ -\frac{\sqrt{2}}{2}s\theta & \frac{\sqrt{2}}{2}s\theta & c\theta & \frac{\sqrt{2}}{4}L\,s\theta \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$

$$
\begin{bmatrix} x \\ 1 \end{bmatrix} = \left(\prod_{k=1}^{6} \breve{H}_k \right) \begin{bmatrix} X \\ 1 \end{bmatrix}
$$

Example





$$
X = \begin{bmatrix} L_1+L_2+L_3 \\ 0 \\ 0 \end{bmatrix}
$$

1. Folding along $\breve{m}^{23}$

$$
\breve{H}_{23} = T((L_1+L_2, 0, 0)^T)\, Q_3(\tfrac{3}{2}\pi)\, Q_1(\theta_2)\, Q_3^{-1}(\tfrac{3}{2}\pi) \cdot T^{-1}((L_1+L_2, 0, 0)^T)
$$

$$
= \begin{bmatrix} c\theta_2 & 0 & -s\theta_2 & (L_1+L_2)(1-c\theta_2) \\ 0 & 1 & 0 & 0 \\ s\theta_2 & 0 & c\theta_2 & -(L_1+L_2)s\theta_2 \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$

2. Folding along $\breve{m}^{12}$

$$
{}^{U}H_{12} = \begin{bmatrix} c\theta_1 & 0 & -s\theta_1 & L_1(1-c\theta_1) \\ 0 & 1 & 0 & 0 \\ s\theta_1 & 0 & c\theta_1 & -L_1 s\theta_1 \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$

$$
\begin{bmatrix} x \\ \\ 1 \end{bmatrix} = {}^{U}H_{12}\big|_{\theta_1=\frac{\pi}{2}} \cdot {}^{U}H_{23}\big|_{\theta_2=-\frac{\pi}{2}} \cdot \begin{bmatrix} L_1+L_2+L_3 \\ 0 \\ 0 \\ 1 \end{bmatrix}
$$

$$
= \begin{bmatrix} 0 & 0 & -1 & L_1 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & -L_1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 & L_1+L_2 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & L_1+L_2 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} L_1+L_2+L_3 \\ 0 \\ 0 \\ 1 \end{bmatrix}
$$

$$
= \begin{bmatrix} L_1+L_3 \\ 0 \\ L_2 \\ 1 \end{bmatrix}
$$