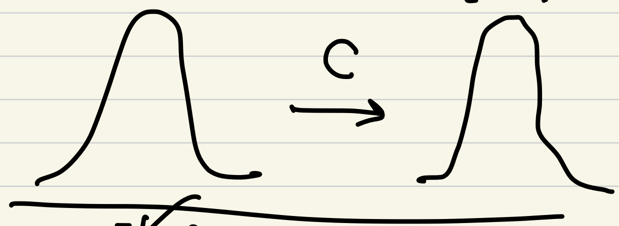
Modified wavenumber analysis

① calculate the modified wavenumber k' for spatial derivative.

② use results from ODE w/ λ replaced with the worst case for k'.

- $\dfrac{\partial u}{\partial t} + c\dfrac{\partial u}{\partial x} = 0$: convection eq.



$$u(x,t) = \psi(t)\, e^{ikx} \rightarrow \frac{d\psi}{dt}\cancel{e^{ikx}} + c\,(ik)\,\psi\, \cancel{e^{ikx}} = 0$$

$$\rightarrow \boxed{\frac{d\psi}{dt} = -ikc\,\psi}$$

SD (CD2) : $\dfrac{\partial u_j}{\partial t} + c\,\dfrac{u_{j+1} - u_{j-1}}{2\Delta x} = 0$

Assume $u_j = \psi(x)\, e^{ikx_j}$

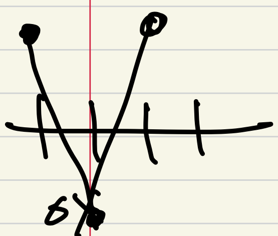
$$\rightarrow \frac{d\psi}{dt} e^{ikx_j} + c\,\frac{\psi e^{ikx_{j+1}} - \psi e^{ikx_{j-1}}}{2\Delta x} = 0$$

$$\rightarrow \boxed{\frac{d\psi}{dt} = -i\,\frac{\sin k\Delta x}{\Delta x}\,c\,\psi}$$

purely imaginary

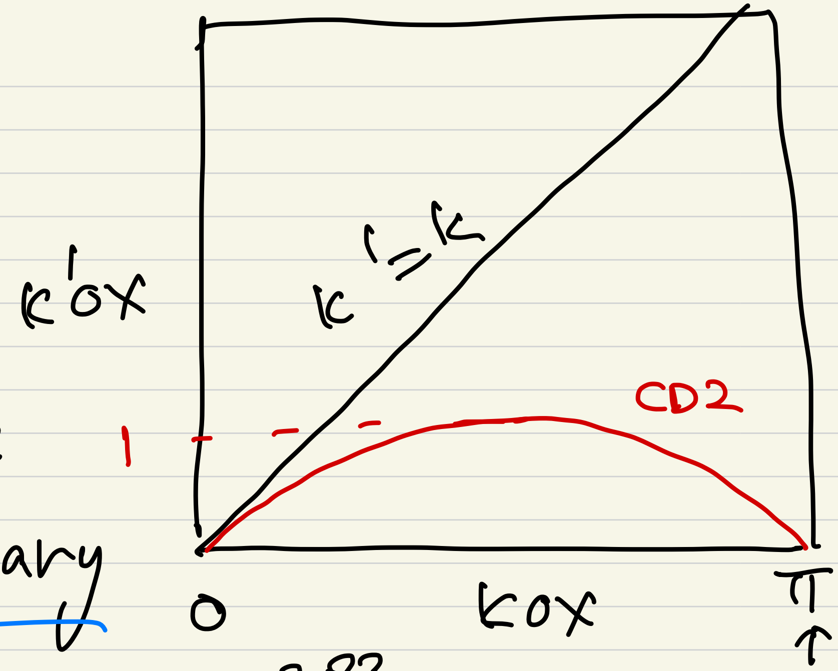
k' : modified wavenumber

$k'\Delta x = \sin k\Delta x$ ✓



$K(2\Delta x) = 2\pi$

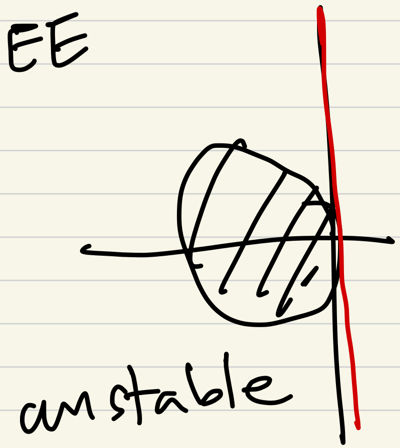
$K\Delta x = \pi$

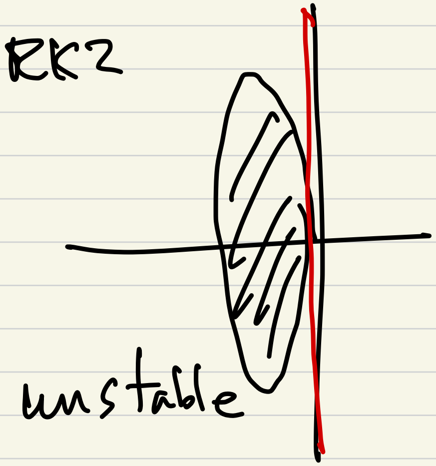


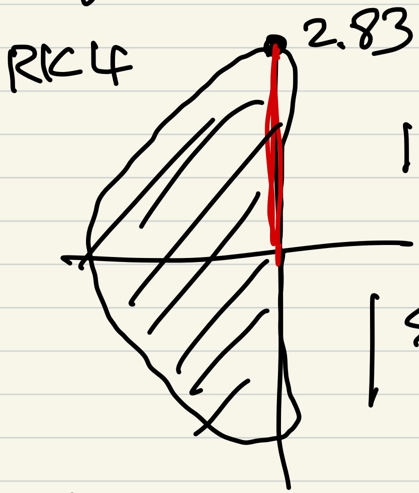
$\rightarrow \dfrac{d\psi}{dt} = \omega\psi$

$\omega = -i\dfrac{\sin k\Delta x}{\Delta x}c$

purely imaginary







$|\lambda_I \Delta t| \le 2.83$

$\downarrow$

$\left|\dfrac{\sin k\Delta x}{\Delta x} c\Delta t\right| \le 2.83$

worst case : $|\sin k\Delta x| = 1$

$\rightarrow \dfrac{c\Delta t}{\Delta x} \le \dfrac{2.83}{|\sin k\Delta x|}$

$(c>0)$

$\rightarrow \boxed{\dfrac{c\Delta t}{\Delta x} \le 2.83}$ RK4

$\Delta t \le \dfrac{2.83\,\Delta x}{c}$

$\Delta t \sim \Delta x$

$\Delta x \to \Delta x/2$, $\Delta t \to \Delta t/2$) CPU 4 times leapfrog method

$|\omega \Delta t| \le 1$

$$\Delta t \le \frac{1}{|\omega|} = \frac{\Delta x}{c\,|\sin k\Delta x|}$$

worst case: $|\sin k\Delta x| = 1$ → $\Delta t \le \frac{\Delta x}{c}$

$$\boxed{\frac{c\Delta t}{\Delta x} \le 1}$$ leapfrog.

much better than diff. eq.

$\frac{c\Delta t}{\Delta x}$: non-dimensional variable

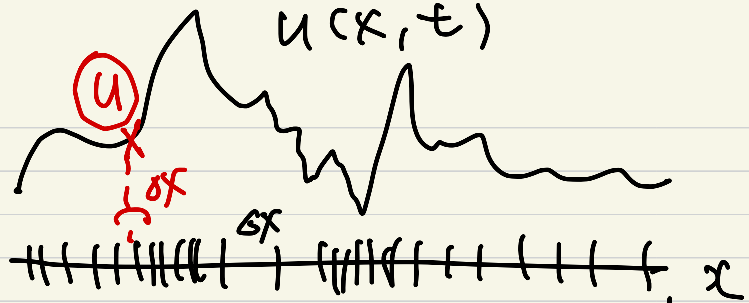
CFL (Courant, Friedrich & Lewy) number

$$\left(\frac{\partial u}{\partial t} + c\frac{\partial y}{\partial x} = 0 \right)$$

velocity

RK4 : CFL ≤ 2.83

Leapfrog : CFL ≤ 1



non-uniform mesh

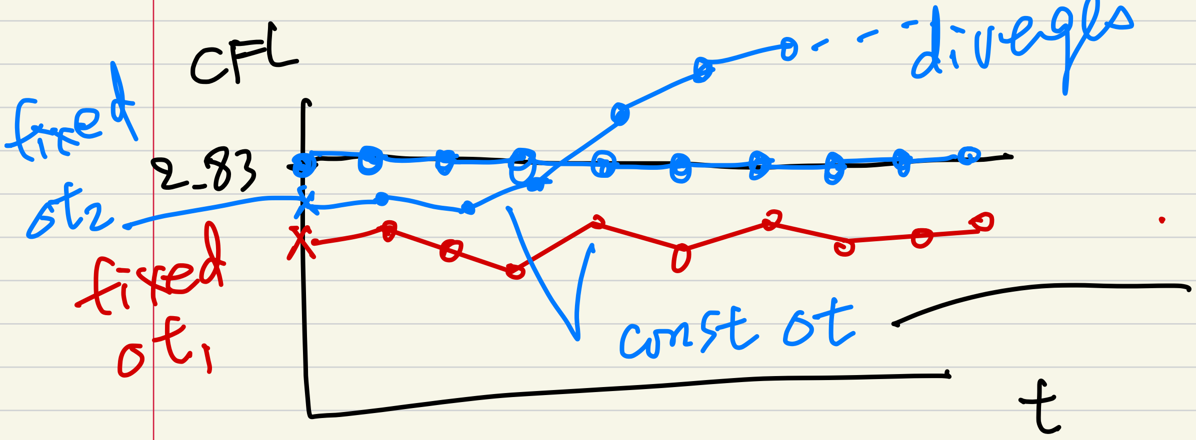
$$CFL = \left|\frac{u\Delta t}{\Delta x}\right| = \left|\frac{u(x,t)\,\Delta t}{\Delta x(x)}\right| \le 2.83$$

for RK4

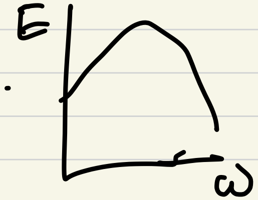
$$\rightarrow \underbrace{\Delta t}_{(t)} \le \frac{2.83\,\Delta x(x)}{|u(x,t)|}$$

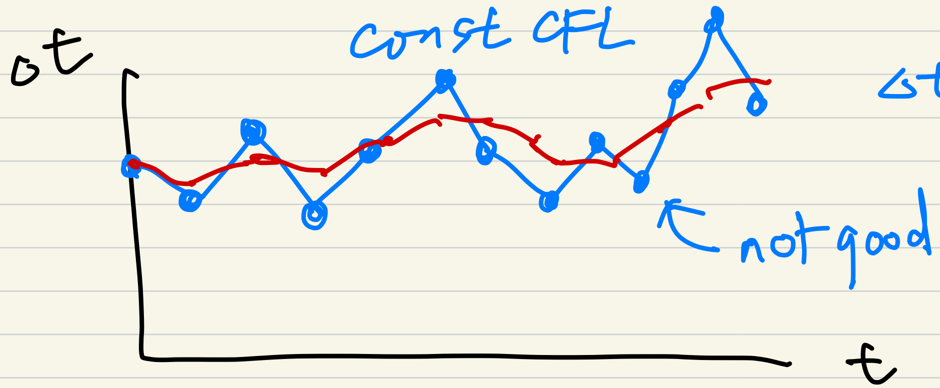
worst case : $\Delta x/|u|$ minimum

$$\Delta t_{max}(t) = 2.83 \left.\frac{\Delta x}{|u|}\right|_{min}$$



is required for FFT.





$$\Delta t = \alpha\,\Delta t_{new} + (1-\alpha)\,\Delta t_{old} \quad (0 \le \alpha \le 1)$$

5.4 Implicit time advancement

$$\frac{\partial \phi}{\partial t} = \alpha \frac{\partial^2 \phi}{\partial x^2} \xrightarrow[\text{CD2}]{\text{EE}} \Delta t \le \frac{\Delta x^2}{2\alpha}$$ too restrictive

$\downarrow$

implicit method

- Crank-Nicolson method (CN) — trapezoidal method

very popular

$$\left(y' = \lambda y \qquad \frac{y^{n+1} - y^n}{\Delta t} = \frac{1}{2}\lambda (y^{n+1} + y^n) \right.$$

C-N : $$\frac{\phi^{n+1} - \phi^n}{\Delta t} = \frac{\alpha}{2}\left[\frac{\partial^2 \phi^{n+1}}{\partial x^2} + \frac{\partial^2 \phi^n}{\partial x^2} \right) + O(\Delta t^2)$$

CD2 : $$\frac{\phi_j^{n+1} - \phi_j^n}{\Delta t} = \frac{\alpha}{2}\left[\frac{\phi_{j+1}^{n+1} - 2\phi_j^{n+1} + \phi_{j-1}^{n+1}}{\Delta x^2} + \frac{\phi_{j+1}^n - 2\phi_j^n + \phi_{j-1}^n}{\Delta x^2} \right] + O(\Delta t^2) + O(\Delta x^2)$$

$$(\beta \equiv \alpha \Delta t / \Delta x^2)$$

$$\rightarrow -\beta \underline{\phi_{j+1}^{n+1}} + (1+2\beta) \underline{\phi_j^{n+1}} - \beta \underline{\phi_{j-1}^{n+1}} = \beta \phi_{j+1}^n + (1-2\beta)\phi_j^n + \beta \phi_{j-1}^n$$

$j = 1, 2, \cdots, N-1$

tri-diagonal system of eqs. ☺

Solve this sys. of eqs. to get ϕ_j^{n+1} w/ $O(N)$ operations.

$$\left(y' = \lambda y \xrightarrow{TR} y_n = \sigma^n y_0 , \quad \sigma = \frac{1 + \lambda \Delta t/2}{1 - \lambda \Delta t/2} \right)$$

$$\frac{\partial \phi}{\partial t} = \alpha \frac{\partial^2 \phi}{\partial x^2} \xrightarrow{CD2} \frac{d\psi}{dt} = -\alpha k'^2 \psi \qquad k'^2 = \frac{2(1-\cos k\Delta x)}{\Delta x^2}$$

Then, $\sigma = \dfrac{1 - \alpha \frac{\Delta t}{\Delta x^2}(1-\cos k\Delta x)}{1 + \alpha \frac{\Delta t}{\Delta x^2}(1-\cos k\Delta x)} \Rightarrow |\sigma| \leq 1$ unconditionally stable

$\Delta x \rightarrow \frac{\Delta x}{2}$, $\Delta t \Rightarrow \Delta t$ CPU time twice

For large Δt, $\sigma \rightarrow -1$. $\phi^n = \sigma^n \phi^0 = (-1)^n \phi^0$

unphysical but never diverges.

dangerous

① reduce Δt

② apply different method like IE.

5.5 Accuracy via modified equation

Since the numerical sol. is an approx. of the exact sol., it does not satisfy the continuous PDE at hand, but satisfies a modified PDE.

Let $\tilde{\phi}$ be the exact sol. and ϕ be the numerical sol. obtained from EE and CD2.

$$\frac{\partial \tilde{\phi}}{\partial t} = \alpha \frac{\partial^2 \tilde{\phi}}{\partial x^2}$$

$$\frac{\phi_j^{n+1} - \phi_j^n}{\Delta t} = \alpha \frac{\phi_{j+1}^n - 2\phi_j^n + \phi_{j-1}^n}{\Delta x^2}$$

$$L(\phi_j^n) \equiv \frac{\phi_j^{n+1} - \phi_j^n}{\Delta t} - \alpha \frac{\phi_{j+1}^n - 2\phi_j^n + \phi_{j-1}^n}{\Delta x^2} = 0$$

Taylor series expansion

$$\phi_j^{n+1} = \phi_j^n + \Delta t \frac{\partial \phi_j^n}{\partial t} + \frac{1}{2}\Delta t^2 \frac{\partial^2 \phi_j^n}{\partial t^2} + \cdots \qquad = \frac{\partial \phi_j^n}{\partial t} + \frac{1}{2}\Delta t \frac{\partial^2 \phi_j^n}{\partial t^2} + \cdots$$

Similarly, $\dfrac{\phi_{j+1}^n - 2\phi_j^n + \phi_{j-1}^n}{\Delta x^2} = \left.\dfrac{\partial^2 \phi^n}{\partial x^2}\right|_j + \dfrac{\Delta x^2}{12}\left.\dfrac{\partial^4 \phi^n}{\partial x^4}\right|_j + \cdots$

$$\Rightarrow L(\phi_j^n) = \frac{\partial \phi_j^n}{\partial t} + \frac{1}{2}\Delta t \frac{\partial^2 \phi_j^n}{\partial t^2} - \alpha \left.\frac{\partial^2 \phi^n}{\partial x^2}\right|_j - \alpha \frac{\Delta x^2}{12}\left.\frac{\partial^4 \phi^n}{\partial x^4}\right|_j + \cdots$$

remove n and j.

$$L(\phi) = \frac{\partial \phi}{\partial t} + \frac{1}{2}\Delta t \frac{\partial^2 \phi}{\partial t^2} - \alpha \frac{\partial^2 \phi}{\partial x^2} - \alpha \frac{\Delta x^2}{12}\frac{\partial^4 \phi}{\partial x^4} + \cdots = 0$$

Thus, the numerical sol. actually satifies the following modified PDE.

$$\boxed{\frac{\partial \phi}{\partial t} - \alpha \frac{\partial^2 \phi}{\partial x^2} = \alpha \frac{\Delta x^2}{12}\frac{\partial^4 \phi}{\partial x^4} - \frac{1}{2}\Delta t \frac{\partial^2 \phi}{\partial t^2} + \cdots}$$

EE

+CD2

As Δt & $\Delta x \to 0$, $\dfrac{\partial \tilde{\phi}}{\partial t} - \alpha \dfrac{\partial^2 \tilde{\phi}}{\partial x^2} = 0$

$\alpha^2 \dfrac{\partial^4 \tilde{\phi}}{\partial x^4}$

$$\text{error } \varepsilon = L(\tilde{\phi}) = -\alpha \frac{\Delta x^2}{12}\frac{\partial^4 \tilde{\phi}}{\partial x^4} + \frac{1}{2}\Delta t \frac{\partial^2 \tilde{\phi}}{\partial t^2} + \cdots$$

//

$$\frac{\partial}{\partial t}\left(\frac{\partial \tilde{\phi}}{\partial t}\right) = \frac{\partial}{\partial t}\left(\alpha \frac{\partial^2 \tilde{\phi}}{\partial x^2}\right) = \frac{\partial^2}{\partial x^2}\left(\alpha \frac{\partial \tilde{\phi}}{\partial t}\right)$$

$$\rightarrow \quad \varepsilon = \underbrace{\left(-\alpha \frac{\Delta x^2}{12} + \alpha^2 \frac{\Delta t}{2} \right)}_{=0 \text{ by choosing } \Delta t = \Delta x^2 / 6\alpha} \frac{\partial^4 \phi}{\partial x^4} + \cdots$$

Then, we can increase accuracy.

Stability limit for EE+CD2 : $\Delta t \le \dfrac{\Delta x^2}{2\alpha}$

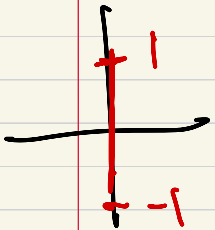
but $\Delta t = \Delta x^2 / 6\alpha$ $\rightarrow$ too restrictive (3 times!)

- Dufort-Frankel method. : an inconsistent numerical method.

• Dufort-Frankel method : an inconsistent numerical method

$$\frac{\partial\phi}{\partial t} = \alpha\frac{\partial^2\phi}{\partial x^2}$$

Leapfrog method + CD2



$$\frac{\phi_j^{n+1}-\phi_j^{n-1}}{2\Delta t} = \alpha\,\frac{\phi_{j+1}^{n} - 2\phi_j^{n} + \phi_{j-1}^{n}}{\Delta x^2} + O(\Delta t^2) + O(\Delta x^2)$$

→ is unconditionally unstable for real & negative λ.

$$\phi_j^n = \frac{1}{2}\left(\phi_j^{n+1}+\phi_j^{n-1}\right) + O(\Delta t^2)$$

$$\rightarrow \phi_j^{n+1} - \phi_j^{n-1} = \underbrace{\frac{2\alpha\Delta t}{\Delta x^2}}_{\beta}\left(\phi_{j+1}^{n} - \phi_j^{n+1} - \phi_j^{n-1} + \phi_{j-1}^{n}\right)$$

$$\rightarrow \boxed{(1+2\beta)\,\phi_j^{n+1} = (1-2\beta)\,\phi_j^{n-1} + 2\beta\phi_{j+1}^{n} + 2\beta\,\phi_{j-1}^{n}}$$ Dufort-Frankel method.

stability analysis ($\phi_j^n = \sigma^n e^{ikx_j}$) → unconditionally stable.

no matrix inversion is required.

2nd-order order accurate

⇒ too good to be true!

What is the modified PDE for Dufort-Frankel method?

$$\phi_j^{n+1} = \phi_j^n + \Delta t \frac{\partial \phi}{\partial t}\Big|_j^n + \cdots$$

$$\phi_{j+1}^n = \phi_j^n + \Delta x \frac{\partial \phi}{\partial x}\Big|_j^n + \cdots$$

$$\Rightarrow \frac{\partial \phi}{\partial t} - \alpha \frac{\partial^2 \phi}{\partial x^2} = -\frac{\Delta t^2}{6}\frac{\partial^3 \phi}{\partial t^3} + \alpha \frac{\Delta x^2}{12}\frac{\partial^4 \phi}{\partial x^4} - \alpha \boxed{\frac{\Delta t^2}{\Delta x^2}} \frac{\partial^2 \phi}{\partial t^2} - \alpha \boxed{\frac{\Delta t^4}{120 \Delta x^2}} \frac{\partial^4 \phi}{\partial x^4} + \cdots$$

For a given Δt, the error actually increases when we refine Δx.

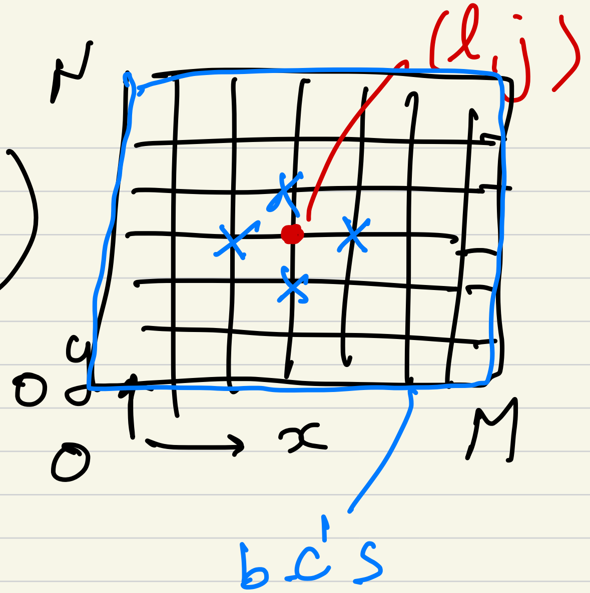
Thus, one cannot increase the accuracy of numerical sol. by arbitrarily letting $\Delta x \to 0$ and $\Delta t \to 0$.

The third term on RHS approaches zero only if $\Delta t \to 0$ faster than $\Delta x \to 0$.

This is an example of <u>inconsistent numerical method</u>.

5.7 Higher dimensions

2D diffusion eq. $\frac{\partial \phi}{\partial t} = \alpha \left(\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} \right)$



CD2: $\frac{\partial \phi_{l,j}}{\partial t} = \alpha \left(\frac{\phi_{l+1,j} - 2\phi_{l,j} + \phi_{l-1,j}}{\Delta x^2} + \frac{\phi_{l,j+1} - 2\phi_{l,j} + \phi_{l,j-1}}{\Delta y^2} \right)$

EE: $\frac{\phi_{l,j}^{n+1} - \phi_{l,j}^{n}}{\Delta t} = \alpha \left(\frac{\phi_{l+1,j}^{n} - 2\phi_{l,j}^{n} + \phi_{l-1,j}^{n}}{\Delta x^2} + \frac{\phi_{l,j+1}^{n} - 2\phi_{l,j}^{n} + \phi_{l,j-1}^{n}}{\Delta y^2} \right)$

$l = 1, 2, \cdots, M-1 \; ; \; j = 1, 2, \cdots, N-1$

Start from initial condition $\phi_{l,j}^{0}$

and then march in time using b.c's.

Δt ?

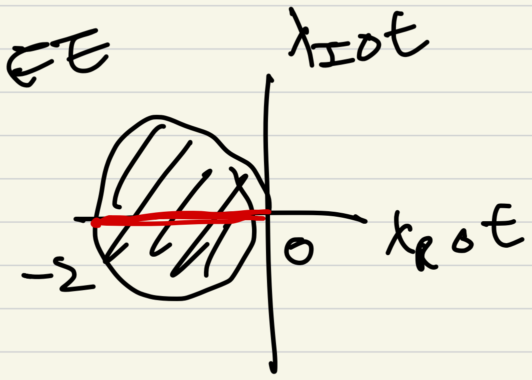
Stability CD2: modified wavenumber k_1' & k_2'
(x) (y)

$$\phi(x,y,t) = \psi(t)\, e^{ik_1 x_l}\, e^{ik_2 y_j}$$

$$\rightarrow \frac{d\psi}{dt} = \boxed{\alpha\left(-k_1'^2 - k_2'^2\right)}\,\psi \quad \text{where} \quad k_1'^2 = \frac{2(1-\cos k_1 \Delta x)}{\Delta x^2}, \quad k_2'^2 = \frac{2(1-\cos k_2 \Delta y)}{\Delta y^2}$$

$\|$

λ : real & negative



$|\lambda_R \Delta t| \le 2$ for stability

$$\Delta t \le \frac{2}{|\lambda|} = \frac{2}{\alpha\left[\frac{2(1-\cos k_1\Delta x)}{\Delta x^2} + \frac{2(1-\cos k_2 \Delta y)}{\Delta y^2}\right]}$$

worst case : $\cos k_1 \Delta x = \cos k_2 \Delta y = -1$

$$\Rightarrow \boxed{\Delta t \le \frac{1}{2\alpha\left(\frac{1}{\Delta x^2} + \frac{1}{\Delta y^2}\right)}}$$

EE + CD2 for 2D diff. eq.

if $\Delta x = \Delta y$, $\quad \Delta t \le \frac{\Delta x^2}{4\alpha}$ (2D) $\quad \Delta t \le \frac{\Delta x^2}{2\alpha}$ (1D)

$\Delta t \le \frac{\Delta x^2}{6\alpha}$ (3D) $\Rightarrow$ too restrictive

$\Rightarrow$ use implicit methods!

5.8 Implicit methods in high dimensions

$$\frac{\partial \phi}{\partial t} = \alpha \left(\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} \right)$$

Crank-Nicolson method (CN)

$$\frac{\phi^{n+1} - \phi^{n}}{\Delta t} = \frac{\alpha}{2} \left(\frac{\partial^2 \phi^{n+1}}{\partial x^2} + \frac{\partial^2 \phi^{n}}{\partial x^2} + \frac{\partial^2 \phi^{n+1}}{\partial y^2} + \frac{\partial^2 \phi^{n}}{\partial y^2} \right)$$

$\Delta x = \Delta y = h$

CD2 :
$$\phi^{n+1}_{l,j} - \phi^{n}_{l,j} = \underbrace{\frac{\alpha \Delta t}{2h^2}}_{\beta} \left(\phi^{n+1}_{l+1,j} - 2\phi^{n+1}_{l,j} + \phi^{n+1}_{l-1,j} \right)$$
$$+ \; '' \; \left(\phi^{n}_{l+1,j} - 2\phi^{n}_{l,j} + \phi^{n}_{l-1,j} \right)$$
$$+ \; '' \; \left(\phi^{n+1}_{l,j+1} - 2\phi^{n+1}_{l,j} + \phi^{n+1}_{l,j-1} \right)$$
$$+ \; '' \; \left(\phi^{n}_{l,j+1} - 2\phi^{n}_{l,j} + \phi^{n}_{l,j-1} \right)$$

$$\Rightarrow \boxed{-\beta \phi^{n+1}_{l+1,j} + (1+4\beta)\phi^{n+1}_{l,j} - \beta \phi^{n+1}_{l-1,j} - \beta \phi^{n+1}_{l,j+1} - \beta \phi^{n+1}_{l,j-1} = F^{n}_{l,j}}$$

CN + CD2 $\quad l = 1, 2, \cdots, M-1$; $j = 1, 2, \cdots, N-1$

$O(\Delta t^2)$ $\quad O(\Delta x^2)$

$\Rightarrow$ Sys. of eqs

$$\underbrace{\begin{bmatrix} B & C & & & O \\ A & B & C & & \\ & A & B & C & \\ & O & & \ddots & \ddots \\ & & & A & B \end{bmatrix}}_{(M-1)(N-1) \times (M-1)(N-1)} \begin{bmatrix} \phi_{11} \\ \phi_{21} \\ \vdots \\ \phi_{lj} \\ \vdots \\ \phi_{M-1N-1} \end{bmatrix} = \begin{bmatrix} F_{11} \\ F_{21} \\ \vdots \\ F_{lj} \\ \vdots \\ F_{M-1N-1} \end{bmatrix}$$

Block-tridiagonal matrix.

$M = N = 100$: # of elts in the matrix $= 10^8$

$\rightarrow$ too difficult to solve

direct inversion requires $O(M^3N^3)$ operations.

too expensive!

$\rightarrow$ may have to introduce an iterative method.

$\rightarrow$ but actually NOT! $\rightarrow$ we use ADI method!

5.9 Alternating directional implicit (ADI) method and approximate factorization

$$\frac{\partial \phi}{\partial t} = \alpha\left(\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2}\right)$$

$$\frac{\phi_{l+1,j}^{n+1} - 2\phi_{l,j}^{n+1} + \phi_{l-1,j}^{n+1}}{\Delta x^2}$$

CN + CD2 : $\dfrac{\phi^{n+1} - \phi^{n}}{\Delta t} = \dfrac{\alpha}{2}\left(A_x \phi^{n+1} + A_x \phi^{n} + A_y \phi^{n+1} + A_y \phi^{n}\right)$

vector

$$+ O(\Delta t^2) + O(\Delta x^2) + O(\Delta y^2)$$

A_x, A_y : difference operators having 2nd-order accuracy representing derivatives in x and y directions.

$$\rightarrow \left[I - \frac{\alpha \Delta t}{2} A_x - \frac{\alpha \Delta t}{2} A_y\right] \phi^{n+1} = \left[I + \frac{\alpha \Delta t}{2} A_x + \frac{\alpha \Delta t}{2} A_y\right] \phi^{n} + \Delta t\left(O(\Delta t^2) + O(\Delta x^2) + O(\Delta y^2)\right)$$

$$\left(I - \frac{\alpha \Delta t}{2} A_x\right)\left(I - \frac{\alpha \Delta t}{2} A_y\right) - \frac{\alpha^2 \Delta t^2}{4} A_x A_y \qquad \left(I + \frac{\alpha \Delta t}{2} A_x\right)\left(I + \frac{\alpha \Delta t}{2} A_y\right) - \frac{\alpha^2 \Delta t^2}{4} A_x A_y$$

$$\to \boxed{\left(I-\frac{\alpha\Delta t}{2}A_x\right)\left(I-\frac{\alpha\Delta t}{2}A_y\right)}\phi^{n+1}=\boxed{\left(I+\frac{\alpha\Delta t}{2}A_x\right)\left(I+\frac{\alpha\Delta t}{2}A_y\right)}\phi^{n}$$

$$+\boxed{\frac{\alpha^2\Delta t^2}{4}A_xA_y\left(\phi^{n+1}-\phi^{n}\right)}+\Delta t\left(O(\Delta t^2)+O(\Delta x^2)+O(\Delta y^2)\right)$$

$$= \Delta t\frac{\partial\phi}{\partial t}+\cdots$$

Δt^3 $\qquad$ $O(\Delta t^3)$

$\therefore$ neglect $\square$ term w/o losing accuracy

$\Rightarrow$ approximate factorization (AF)

$$\Rightarrow \boxed{\left(I-\frac{\alpha\Delta t}{2}A_x\right)\underbrace{\left(I-\frac{\alpha\Delta t}{2}A_y\right)\phi^{n+1}}_{Z}=\underbrace{\left(I+\frac{\alpha\Delta t}{2}A_x\right)\left(I+\frac{\alpha\Delta t}{2}A_y\right)\phi^{n}}_{F}}$$

CN + CD2 + AF

$$\left(I-\frac{\alpha\Delta t}{2}A_x\right)Z=F$$

$$\to Z_{l,j}-\frac{\alpha\Delta t}{2}\,\frac{Z_{l+1,j}-2Z_{l,j}+Z_{l-1,j}}{\Delta x^2}=F_{l,j}\qquad \text{(*)}$$

$l=1,2,\cdots,M-1$

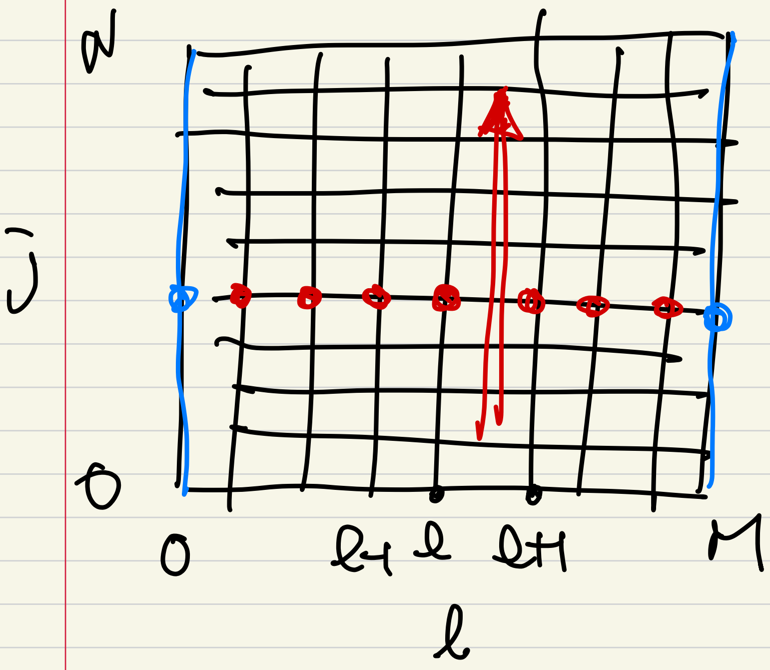
$j=1,2,\cdots,N-1$

tri-diagonal matrix for l

For each j, solve a tri-diagonal matrix for $z_{l,j}$.

$\underbrace{\text{For each } j}_{N}\,,\ \underbrace{\text{solve a tri-diagonal matrix}}_{O(M)}$

$\boxed{O(MN) \text{ operations}}$



b.c.'s → for z!

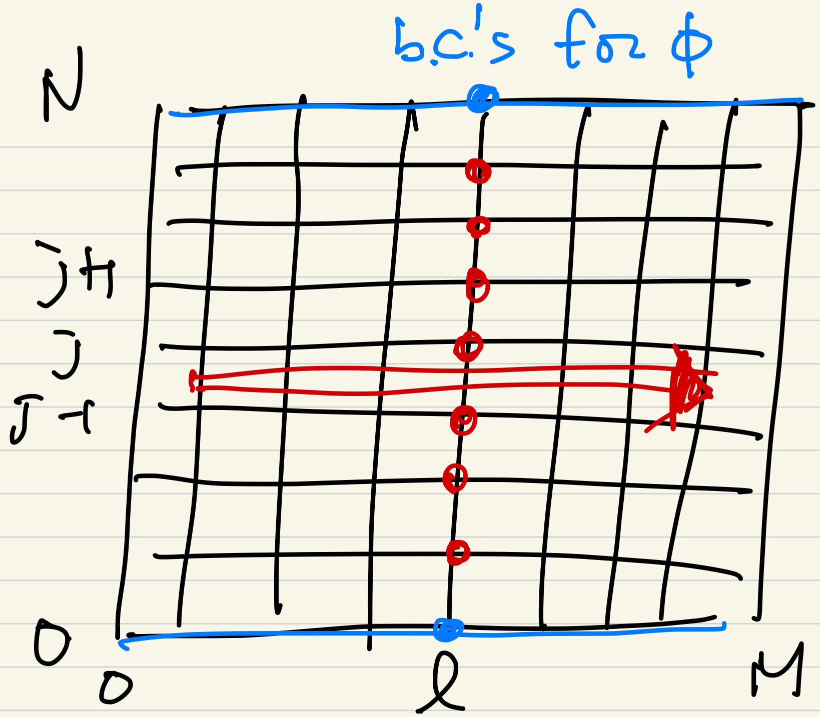
Having solved for $z_{l,j}$, $\left(I - \frac{\alpha\Delta t}{2}A_y\right)\phi^{n+1} = z$.

$$\rightarrow \phi^{n+1}_{l,j} - \frac{\alpha\Delta t}{2}\,\frac{\phi^{n+1}_{l,j+1} - 2\phi^{n+1}_{l,j} + \phi^{n+1}_{l,j-1}}{\Delta y^2} = z_{l,j} \quad \text{—} (*)$$

for each l, solve a tri-diagonal matrix for $\phi^{n+1}_{l,j}$.

$\underbrace{\text{for each } l}_{M}\,,\ \underbrace{\text{solve a tri-diagonal matrix}}_{O(N)}$

$\boxed{O(MN) \text{ operations}}$



$\therefore$ total $O(2MN)$ operations are required.

Alternating directional implicit (ADI) method!

$\circledast$ eq. requires b.c.'s for $z_{0,j}$ & $z_{M,j}$ for $j = 1, 2, \cdots, N-1$.

$\circledast\circledast$ eq. ⓐ $l = 0$: $z_{0,j} = \phi_{0,j}^{n+1} - \dfrac{\alpha \Delta t}{2} \dfrac{\phi_{0,j+1}^{n+1} - 2\phi_{0,j}^{n+1} + \phi_{0,j-1}^{n+1}}{\Delta y^2}$

ⓑ $l = M$: $z_{M,j} = \text{- - - - - -}$

obtain from b.c.'s for ϕ

$\Rightarrow$ ADI method w/ approximate factorization

no iteration required $\Rightarrow$ Great!

implicit method