Frobenius Method (1)

– Let *b*(*x*) and *c*(*x*) be any analytic functions at *x* = 0. Then, ODE

$$y'' + \frac{b(x)}{x} y' + \frac{c(x)}{x^2} y = 0$$

has at least one sol. represented in the form

$$y(x) = x^r \sum_{m=0}^{\infty} a_m x^m = x^r \left(a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots \right) \qquad (a_0 \neq 0)$$

where *r* may be any (real or complex) number.

– The ODE has a second sol. (which is linearly independent) similar to above series form with a different *r* and different coeff. or may contain a logarithm term.

Frobenius Method (2)

– Application example 1 of Frobenius method: Bessel's eqn.

$$y'' + \frac{1}{x} y' + \left(\frac{x^2 - \nu^2}{x^2} \right) y = 0 \qquad b(x) = 1, c(x) = x^2 - \nu^2$$

– Application example 2: hyergeometric DE

– Regular point --- a point x_0 at which $p(x)$ and $q(x)$ are analytic. Then, power series method can be applied.

$$y'' + p(x) y' + q(x) y = 0$$

Or, a point x_0 at which $\tilde{h}, \tilde{p}, \tilde{q}$ are analytic, and $\tilde{h}(x_0) \neq 0$

$$\tilde{h}(x) y'' + \tilde{p}(x) y' + \tilde{q}(x) y = 0$$

– Singular point --- not regular

Indicial Equation (1)

– Multiplying x^2 on the given DE

$$x^2 y'' + xb(x)y' + c(x)y = 0$$

– Expand $b(x)$ and $c(x)$ in power series

$$b(x) = b_0 + b_1 x + b_2 x^2 + \cdots, c(x) = c_0 + c_1 x + c_2 x^2 + \cdots$$

– Term by term differentiation of the assumed sol.

$$y'(x) = \sum_{m=0}^{\infty} (m+r) a_m x^{m+r-1} = x^{r-1}\left[r a_0 + (r+1) a_1 x + \cdots \right]$$

$$y''(x) = \sum_{m=0}^{\infty} (m+r)(m+r-1) a_m x^{m+r-2}$$

$$= x^{r-2}\left[r(r-1) a_0 + (r+1) r a_1 x + \cdots \right]$$

Indicial Equation (2)

– Substituting into the given DE

$$x^r\left[r\left(r-1\right)a_0+\cdots\right]+\left(b_0+b_1x+\cdots\right)x^r\left(ra_0+\cdots\right)$$
$$+\left(c_0+c_1x+\cdots\right)x^r\left(a_0+a_1x+\cdots\right)=0$$

– Sum of the coeff. corresponding to x^r

$$r\left(r-1\right)+b_0r+c_0=0$$ ➡ *Indicial equation of ODE*

– Two roots of the indicial eqn.: r_1 and r_2

Indicial Equation (3)

– Case I: Distinct roots not differing by an integer

$$y_1(x) = x^{r_1}\left(a_0 + a_1 x + a_2 x^2 + \cdots\right)$$

$$y_2(x) = x^{r_2}\left(A_0 + A_1 x + A_2 x^2 + \cdots\right)$$

– Case II: Double root $r_1 = r_2 = r$

$$y_1(x) = x^{r}\left(a_0 + a_1 x + a_2 x^2 + \cdots\right)$$

$$y_2(x) = y_1(x)\ln x + x^{r}\left(A_1 x + A_2 x^2 + \cdots\right)$$

– Case III: Roots differing by an integer

$$y_1(x) = x^{r_1}\left(a_0 + a_1 x + a_2 x^2 + \cdots\right)$$

$$y_2(x) = k y_1(x)\ln x + x^{r_2}\left(A_0 + A_1 x + A_2 x^2 + \cdots\right)$$

Example of Case III (1)

– DE

$$\left(x^2-x\right)y''-xy'+y=0$$

– Applying Frobenius method

$$\left(x^2-x\right)\sum_{m=0}^{\infty}(m+r)(m+r-1)a_m x^{m+r-2}-x\sum_{m=0}^{\infty}(m+r)a_m x^{m+r-1}+$$

$$\sum_{m=0}^{\infty}a_m x^{m+r}=0$$

– Collect all the terms with power x^{m+r} and simplify

$$\sum_{m=0}^{\infty}(m+r-1)^2 a_m x^{m+r}-\sum_{m=0}^{\infty}(m+r)(m+r-1)a_m x^{m+r-1}=0$$

Example of Case III (2)

– Set $m = s$ in the first series, and $m = s + 1$ in the second

$$\sum_{s=0}^{\infty}(s+r-1)^2 a_s x^{s+r} - \sum_{s=-1}^{\infty}(s+r+1)(s+r)a_{s+1}x^{s+r} = 0$$

– Lowest power is x^{r-1}, and the resulting indicial eqn.

$$r(r-1) = 0$$

– $r_1 = 1, r_2 = 0$, differ by an integer, Case III

– First sol. y_1 corresponding to $r_1 = 1$

$$\sum_{s=0}^{\infty}\left[s^2 a_s - (s+2)(s+1)a_{s+1}\right]x^{s+1} = 0$$

– Recurrence formula

$$a_{s+1} = \frac{s^2}{(s+2)(s+1)}a_s$$

Example of Case III (3)

- $a_1 = 0, a_2 = 0, \cdots$, Taking $a_0 = 1$

$$y_1 = x^{r_1} a_0 = x$$

- Second sol. y_2 --- can be obtained by the reduction of order

$$y_2 = y_1 u = xu, y_2' = xu' + u, y_2'' = xu'' + 2u'$$

- Substituting

$$\left(x^2 - x\right)\left(xu'' + 2u'\right) - x\left(xu' + u\right) + xu = 0$$

- Dividing by x and simplifying

$$\left(x^2 - x\right)u'' + \left(x - 2\right)u' = 0$$

Example of Case III (4)

– Partial fractions and integrating

$$\frac{u''}{u'} = -\frac{x-2}{x^2-x} = -\frac{2}{x} + \frac{1}{x-1}, \qquad \ln u' = \ln\left|\frac{x-1}{x^2}\right|$$

– Taking exponents and integrating

$$u' = \frac{x-1}{x^2} = \frac{1}{x} - \frac{1}{x^2}, u = \ln x + \frac{1}{x},$$

$$y_2 = xu = x\ln x + 1$$

– y_1, y_2: linearly independent

Bessel's Eqn. and Functions (1)

– Bessel's DE

$$x^2 y'' + xy' + \left(x^2 - \nu^2\right) y = 0$$

– Electrical field, vibrations, heat conduction with cylindrical symmetry
v: real and nonnegative

– Can apply Frobenius method

$$y(x) = x^r \sum_{m=0}^{\infty} a_m x^{m+r} \qquad \left(a_0 \neq 0\right)$$

– Substituting

$$\sum_{m=0}^{\infty} (m+r)(m+r-1) a_m x^{m+r} + \sum_{m=0}^{\infty} (m+r) a_m x^{m+r}$$

$$+\sum_{m=0}^{\infty} a_m x^{m+r+2} - \nu^2 \sum_{m=0}^{\infty} a_m x^{m+r} = 0$$

Bessel's Eqn. and Functions (2)

– Collecting the coeff. of x^{s+r}

$$r(r-1)a_0 + ra_0 - \nu^2 a_0 = 0 \qquad (s=0)$$

$$(r+1)ra_1 + (r+1)a_1 - \nu^2 a_1 = 0 \qquad (s=1)$$

$$(s+r)(s+r-1)a_s + (s+r)a_s + a_{s-2} - \nu^2 a_s = 0 \quad (s=2,3,\ldots)$$

– Indicial eqn.

$$\boxed{(r+\nu)(r-\nu)=0}$$

– Two roots: $r_1 = \nu(\geq 0),\, r_2 = -\nu$

– Coeff. recurrence for the first root $r = r_1 = \nu$

$$a_1 = 0$$

$$(s+r+\nu)(s+r-\nu)a_s + a_{s-2} = 0$$

Bessel's Eqn. and Functions (3)

$$(s+2\nu)sa_s + a_{s-2} = 0$$

– Resultantly, $a_3 = 0, a_5 = 0, \cdots$

– We only have the coeff. of even numbers *s* = 2*m*

$$(2m+2\nu)2ma_{2m} + a_{2m-2} = 0$$

$$a_{2m} = -\frac{1}{2^2 m(\nu+m)} a_{2m-2}, \qquad m = 1, 2, \cdots$$

– Even numbered coeff.

$$a_2 = -\frac{a_0}{2^2(\nu+1)}, a_4 = \frac{a_0}{2^4 2!(\nu+1)(\nu+2)}$$

$$a_{2m} = \frac{(-1)^m a_0}{2^{2m} m!(\nu+1)(\nu+2)\cdots(\nu+m)}, \qquad m = 1, 2, \cdots$$

Bessel Function of the 1st Kind $J_n(x)$ (1)

– Integer *v* now denoted by *n*

$$a_{2m} = \frac{(-1)^m a_0}{2^{2m} m!(n+1)(n+2)\cdots(n+m)}, \qquad m = 1, 2, \cdots$$

– Arbitrary a_0 determined as follows:

$$a_0 = \frac{1}{2^n n!}, \qquad a_{2m} = \frac{(-1)^m}{2^{2m+n} m!(n+m)!} \qquad m = 1, 2, \cdots$$

– Bessel function of the 1st kind of order *n* --- converges for all *x*

$$J_n(x) = x^n \sum_{m=0}^{\infty} \frac{(-1)^m}{2^{2m+n} m!} \frac{x^{2m}}{(n+m)!}$$

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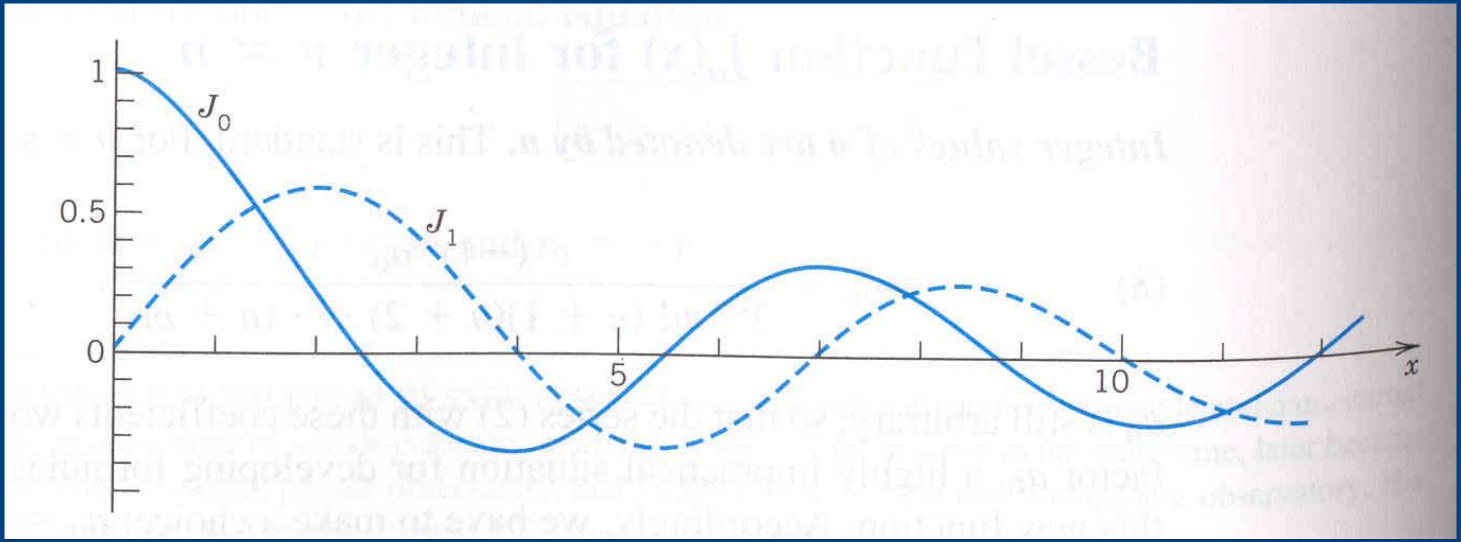
Bessel Function of the 1st Kind $J_n(x)$ (2)

– Bessel function of order 0

$$J_0(x)=\sum_{m=0}^{\infty}\frac{(-1)^m x^{2m}}{2^{2m}(m!)^2}=1-\frac{x^2}{2^2(1!)^2}+\frac{x^4}{2^4(2!)^2}-\frac{x^6}{2^6(3!)^2}+-\cdots$$



Bessel Function of the 1st Kind $J_v(x)$ (1)

– Extend from integer $v = n$ to any $v \geq 0$

– Gamma function $\Gamma(v)$

$$\Gamma(\nu) = \int_0^\infty e^{-t} t^{\nu-1} dt$$

– Integrating by parts

$$\Gamma(\nu+1) = \int_0^\infty e^{-t} t^{\nu} dt = \left| e^{-t} t^{\nu} \right|_0^\infty + \nu \int_0^\infty e^{-t} t^{\nu-1} dt$$

$$\Gamma(\nu+1) = \nu\Gamma(\nu)$$

$$\Gamma(1) = \int_0^\infty e^{-t} dt = \left| -e^{-t} \right|_0^\infty = 0 - (-1) = 1$$

$$\Gamma(2) = \Gamma(1) = 1!, \Gamma(3) = 2\Gamma(2) = 2!, \cdots$$

$$\Gamma(n+1) = n! \qquad (n = 0, 1, \cdots)$$

Bessel Function of the 1st Kind $J_v(x)$ (2)

– Now for any *v*

$$a_0 = \frac{1}{2^v v!} = \frac{1}{2^v \Gamma(v+1)}$$

– The even numbered coeff.

$$a_{2m} = \frac{(-1)^m}{2^{2m+v} m! \Gamma(v+m+1)}$$

– Bessel function of the 1st kind of order *v* --- converges for all *x*

$$J_v(x) = x^v \sum_{m=0}^{\infty} \frac{(-1)^m}{2^{2m+v} m!} \frac{x^{2m}}{\Gamma(v+m+1)}$$

General sol. Of Bessel's DE

– Replacing v by - v

$$J_{-\nu}(x) = x^{-\nu} \sum_{m=0}^{\infty} \frac{(-1)^m}{2^{2m-\nu} m!} \frac{x^{2m}}{\Gamma(m-\nu+1)}$$

– General sol. of Bessel's DE for non-integer v

$$y(x) = c_1 J_\nu(x) + c_2 J_{-\nu}(x)$$

– If v is an integer, above form is not a general sol. because they are linearly dependent.

$$J_{-n}(x) = (-1)^n J_n(x) \qquad (n = 1, 2, \cdots)$$

$$J_{-n}(x) = \sum_{m=n}^{\infty} \frac{(-1)^m}{2^{2m-n} m!} \frac{x^{2m-n}}{(m-n)!} = \sum_{s=0}^{\infty} \frac{(-1)^{n+s}}{2^{2s+n}} \frac{x^{2s-n}}{(n+s)! s!}$$

Properties of Bessel function

– Derivatives

$$\left[x^{\nu}J_{\nu}(x)\right]' = x^{\nu}J_{\nu-1}(x)$$

$$\left[x^{-\nu}J_{\nu}(x)\right]' = -x^{-\nu}J_{\nu+1}(x)$$

– Recurrence relation

$$J_{\nu-1}(x) + J_{\nu+1}(x) = \frac{2\nu}{x}J_{\nu}(x)$$

$$J_{\nu-1}(x) - J_{\nu+1}(x) = 2J_{\nu}'(x)$$

– For half-integer order *v*

$$J_{1/2}(x) = \sqrt{\frac{2}{\pi x}}\sin x,\ J_{-1/2}(x) = \sqrt{\frac{2}{\pi x}}\cos x$$

Bessel Function of the 2nd Kind $Y_0(x)$ (1)

- When v is an integer, need a second linearly independent sol.
  --- Bessel function of the 2nd kind $Y_n(x)$
- $n = 0$: Bessel function of the 2nd kind $Y_0(x)$

$$xy'' + y' + xy = 0$$

- Indicial eqn. has double root, $r = 0$, Case II
- Second desired sol. of the form

$$y_2(x) = J_0(x)\ln x + \sum_{m=1}^{\infty} A_m x^m$$

$$y_2' = J_0' \ln x + \frac{J_0}{x} + \sum_{m=1}^{\infty} m A_m x^{m-1}$$

$$y_2'' = J_0'' \ln x + \frac{2J_0'}{x} - \frac{J_0}{x^2} + \sum_{m=1}^{\infty} m(m-1) A_m x^{m-2}$$

Bessel Function of the 2nd Kind $Y_0(x)$ (2)

– Substituting into the DE and resulting eqn.

$$2J_0' + \sum_{m=1}^{\infty} m(m-1)A_m x^{m-1} + \sum_{m=1}^{\infty} mA_m x^{m-1} + \sum_{m=1}^{\infty} A_m x^m = 0$$

– Expression for $J'_0(x)$

$$J_0'(x) = \sum_{m=1}^{\infty} \frac{(-1)^m 2mx^{2m-1}}{2^{2m}(m!)^2} = \sum_{m=1}^{\infty} \frac{(-1)^m x^{2m-1}}{2^{2m-1}m!(m-1)!}$$

– Resulting eqn.

$$\sum_{m=1}^{\infty} \frac{(-1)^m x^{2m-1}}{2^{2m-2}m!(m-1)!} + \sum_{m=1}^{\infty} m^2 A_m x^{m-1} + \sum_{m=1}^{\infty} A_m x^{m+1} = 0$$

– Collecting the power of x^0, x^{2s} $\quad A_1 = 0, A_3 = 0, A_5 = 0, \cdots$

Bessel Function of the 2nd Kind $Y_0(x)$ (3)

– Even numbered coeff.

$$A_{2m} = \frac{(-1)^{m-1}}{2^{2m}(m!)^2}\left(1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{m}\right), \qquad m=1,2,\cdots$$

$$h_1 = 1, h_m = 1+\frac{1}{2}+\cdots+\frac{1}{m} \qquad m=2,3,\cdots$$

– Expression for the second linearly independent sol. $y_2(x)$

$$y_2(x) = J_0(x)\ln x + \sum_{m=1}^{\infty}\frac{(-1)^{m-1}h_m}{2^{2m}(m!)^2}x^{2m}$$

– Another basis if y_2 replaced by an independent particular sol. With

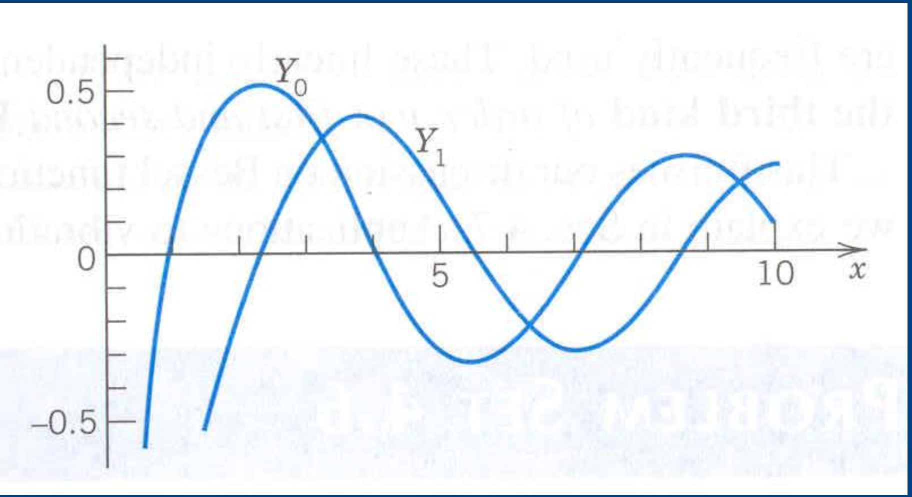
$$a(y_2 + bJ_0), a = \frac{2}{\pi}, b = \gamma - \ln 2$$

(γ: Euler const.)

Bessel Function of the 2nd Kind $Y_0(x)$ (4)

– Bessel function of the 2nd kind of order 0 --- Neumann's function of order 0

$$Y_0(x) = \frac{2}{\pi}\left[J_0(x)\left(\ln\frac{x}{2} + \gamma \right) + \sum_{m=1}^{\infty} \frac{(-1)^{m-1} h_m}{2^{2m}(m!)^2} x^{2m} \right]$$



Bessel Function of the 2nd Kind $Y_n(x)$ (1)

– When $v = n = 1,2,\ldots$, y_2 can be obtained as in Case III
– Standard second sol. for all v

$$Y_\nu(x) = \frac{1}{\sin \pi x}\left[J_\nu(x)\cos \nu\pi - J_{-\nu}(x)\right]$$

$$Y_n(x) = \lim_{\nu \to n} Y_\nu(x)$$

--- Bessel function of the 2nd kind of order v, Neumann's function of order v

– $n = 1,2,\ldots$: Bessel function of the 2nd kind of order n $Y_n(x)$

$$Y_n(x) = \frac{2}{\pi} J_n(x)\left(\ln\frac{x}{2} + \gamma\right) + \frac{x^n}{\pi}\sum_{m=0}^{\infty}\frac{(-1)^{m-1}\left(h_m + h_{m-n}\right)}{2^{2m+n} m!(m+n)!} x^{2m}$$

Bessel Function of the 2nd Kind $Y_n(x)$ (2)

$$-\frac{x^{-n}}{\pi}\sum_{m=0}^{n-1}\frac{(n-m-1)!}{2^{2m+n}m!}x^{2m}$$

– Furthermore

$$Y_{-n}(x)=(-1)^n Y_n(x)$$

– General sol. of Bessel's DE

$$y(x)=c_1 J_\nu(x)+c_2 Y_\nu(x)$$

– Bessel function of the 3rd kind of order *v*, 1st and 2nd Henkel function of order *v*

$$H_\nu^{(1)}(x)=J_\nu(x)+iY_\nu(x)$$

$$H_\nu^{(2)}(x)=J_\nu(x)-iY_\nu(x)$$