Simulation of rigid origami

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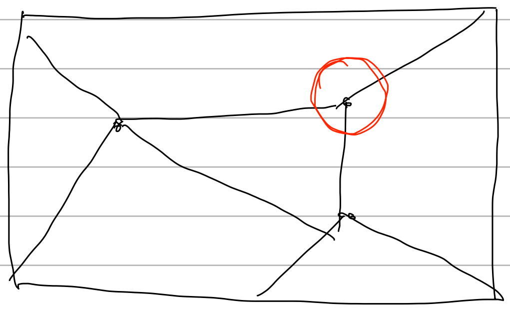
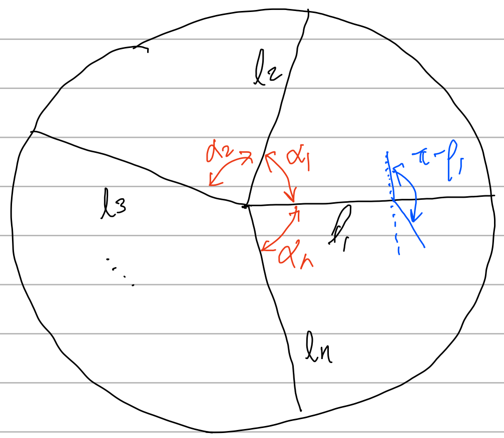
: use crease angles of all folds as variables to represent configuration of an origami model





n edges : $l_1, l_2, \cdots, l_n$

n fold angles : $\rho_1, \cdots, \rho_n$

Recall $\displaystyle\prod_{k=1}^{n_j} R_1(\theta_{jk})\, R_3(\alpha_{jk}) = I_3, \quad j=1, \cdots, N_I$

$$R_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & c\phi & -s\phi \\ 0 & s\phi & c\phi \end{bmatrix} \qquad R_3 = \begin{bmatrix} c\alpha & -s\alpha & 0 \\ s\alpha & c\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

3×3 matrix constraint function F

$$F(\rho_1, \rho_2, \cdots, \rho_n) = X_1 X_2 \cdots X_{n-1} X_n = I$$

X_n : rotation by fold lines

$$X_1 = C_1 B_{12}$$
$$X_2 = C_2 B_{23}$$
$$\vdots$$
$$X_n = C_n B_{n1}$$

$$\chi_j = \begin{bmatrix} \cos\rho_j & 0 & \sin\rho_j \\ 0 & 1 & 0 \\ -\sin\rho_j & 0 & \cos\rho_j \end{bmatrix} \begin{bmatrix} \cos\alpha_j & -\sin\alpha_j & 0 \\ \sin\alpha_j & \cos\alpha_j & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

If we take a derivative of F with respect to ρ_j

$$\frac{dF}{dt} = \frac{\partial F}{\partial \rho_1}\cdot\dot{\rho}_1 + \frac{\partial F}{\partial \rho_2}\cdot\dot{\rho}_2 + \cdots + \frac{\partial F}{\partial \rho_j}\cdot\dot{\rho}_j + \cdots + \frac{\partial F}{\partial \rho_n}\cdot\dot{\rho}_n = 0_3$$

We obtain 9 equations

$$\begin{bmatrix} \frac{\partial F}{\partial \rho_1}(1,1) & \cdots & \frac{\partial F}{\partial \rho_n}(1,1) \\ \frac{\partial F}{\partial \rho_1}(1,2) & \cdots & \frac{\partial F}{\partial \rho_n}(1,2) \\ \\ \frac{\partial F}{\partial \rho_1}(3,3) & \cdots & \frac{\partial F}{\partial \rho_n}(3,3) \end{bmatrix} \begin{bmatrix} \dot{\rho}_1 \\ \dot{\rho}_2 \\ \vdots \\ \dot{\rho}_n \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

9 rows $\rightarrow$ 9 equations
But only 3 equations are independent

When $F = I$

$$\frac{\partial F}{\partial \rho_j} + \left(\frac{\partial F}{\partial \rho_j}\right)^T = \frac{\partial F}{\partial \rho_j} F^T + F\left(\frac{\partial F}{\partial \rho_j}\right)^T = \frac{\partial}{\partial \rho_j}(FF^T) = 0$$

Thus we can express

$$\frac{\partial F}{\partial \rho_j} = \begin{bmatrix} 0 & -a_j & c_j \\ a_j & 0 & -b_j \\ -c_j & b_j & 0 \end{bmatrix}$$

We finally get $3\times n$ matrix instead of $9\times n$ matrix

$$\begin{bmatrix} a_1 & a_2 & \cdots & a_n \\ b_1 & b_2 & \cdots & b_n \\ c_1 & c_2 & \cdots & c_n \end{bmatrix} \begin{bmatrix} \dot{\rho}_1 \\ \dot{\rho}_2 \\ \vdots \\ \dot{\rho}_n \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}$$

Rewrite the matrix using global edge number.
Assume edge i is connected to vertex k, but not edge j

Constraint by vertex k : C_k

$$[C_k]\begin{bmatrix}\dot\rho_1\\ \dot\rho_2\\ \vdots\\ \dot\rho_N\end{bmatrix} = \begin{array}{c} \begin{array}{ccccc} & i\text{th} & & j\text{th} & \end{array} \\ \begin{bmatrix} \cdots & a_{ki} & \cdots & 0 & \cdots \\ \cdots & b_{ki} & & 0 & \\ \cdots & c_{ki} & & 0 & \end{bmatrix} \end{array} \begin{bmatrix}\dot\rho_1\\ \dot\rho_2\\ \vdots\\ \dot\rho_N\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}$$

N : total crease number

$$C\dot\rho = \begin{bmatrix}[C_1]\\ [C_2]\\ \vdots\\ [C_M]\end{bmatrix}\begin{bmatrix}\dot\rho_1\\ \dot\rho_2\\ \vdots\\ \dot\rho_N\end{bmatrix} = \begin{bmatrix}0\\0\\ \vdots\\ 0\end{bmatrix}$$

M : total vertex number

If and only if the rank of matrix C is less than N
the linear equation has nontrivial solution.

Moore-Penrose inverse matrix A^+
: pseudo inverse matrix for $m\times n$ matrix
(always exist and unique)

SVD (singular value decomposition)

$$A = U\Sigma V^T$$

U : $m\times m$ orthogonal matrix $(U^{-1} = U^T)$
V : $n\times n$ 〃 $(V^{-1} = V^T)$

$$\Sigma = \begin{bmatrix}\Sigma_r & O_{r\times(n-r)}\\ O_{(m-r)\times r} & O_{(m-r)\times(n-r)}\end{bmatrix}$$

$$\Sigma_r = \begin{bmatrix} \sigma_1 & & 0 \\ & \ddots & \\ 0 & & \sigma_r \end{bmatrix}$$

r : rank of A

$$\underline{A^+ = V \Sigma^+ U^T}$$

$$\Sigma^+ = \begin{bmatrix} \Sigma_r^{-1} & 0_{r \times (m-r)} \\ 0_{(n-r) \times r} & 0_{(n-r) \times (m-r)} \end{bmatrix}$$

If $m > n$ and column vectors are linearly independent

$$\underline{A^+} = \underline{(A^T A)^{-1} A^T}$$: left inverse

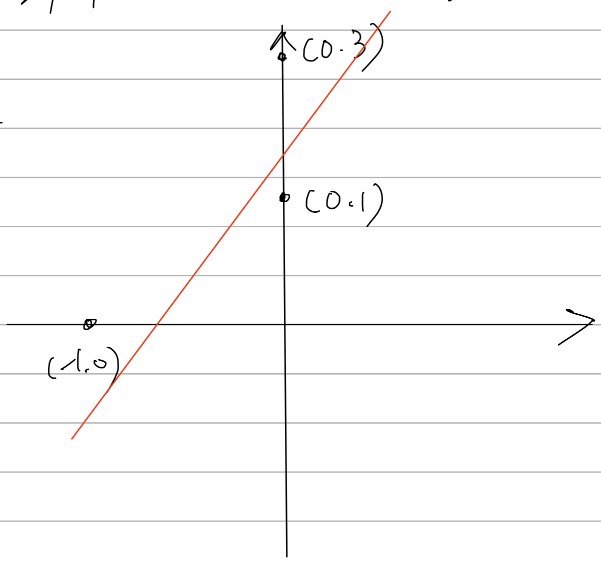
$$A^+ A = (A^T A)^{-1} A^T A = I$$

If $n > m$ and row vectors are linearly independent

$$A^+ = A^T (A A^T)^{-1}$$: right inverse

$$A A^+ = A A^T (A A^T)^{-1} = I$$

Example.



model

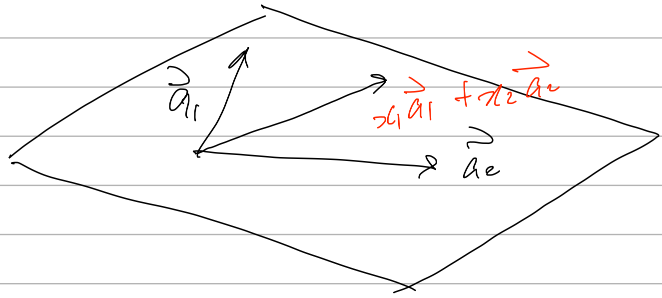
$f(x) = mx + b$

$$f(-1) = -m + b = 0$$
$$f(0) = 0 + b = 1$$
$$f(0) = 0 + b = 3$$

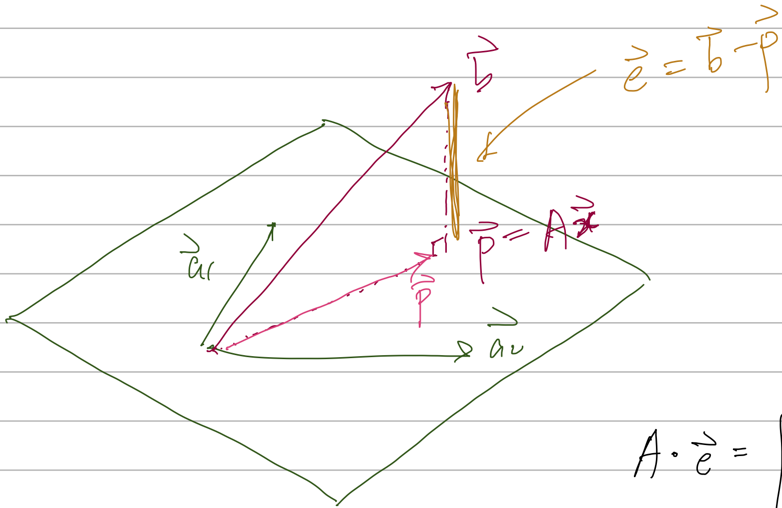
$$\begin{bmatrix} -1 & 1 \\ 0 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} m \\ b \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 3 \end{bmatrix}$$

$$A\vec{x} = \vec{b}$$

$$\begin{bmatrix} | & | \\ \vec{a}_1 & \vec{a}_2 \\ | & | \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = x_1 \cdot \begin{bmatrix} | \\ \vec{a}_1 \\ | \end{bmatrix} + x_2 \begin{bmatrix} | \\ \vec{a}_2 \\ | \end{bmatrix} = \begin{bmatrix} | \\ \vec{b} \\ | \end{bmatrix}$$



Solution exists if $\vec{b} = \text{span}(\vec{a}_1, \vec{a}_2)$



$$A \cdot \vec{e} = \begin{bmatrix} | & | \\ \vec{a}_1 & \vec{a}_2 \\ | & | \end{bmatrix} \cdot \vec{e} = \begin{bmatrix} 0 & 0 \end{bmatrix}$$

$$A^T e = A^T(\vec{b} - A\vec{x}) = 0$$

$$A^T\vec{b} - A^T A\vec{x} = 0$$

$$A^T A\vec{x} = A^T\vec{b}$$

$$\vec{x} = (A^T A)^{-1} A^T \vec{b} = A^{+}\vec{b}$$

Project unconstrained angles movement to constrained space by using orthogonal projection matrix

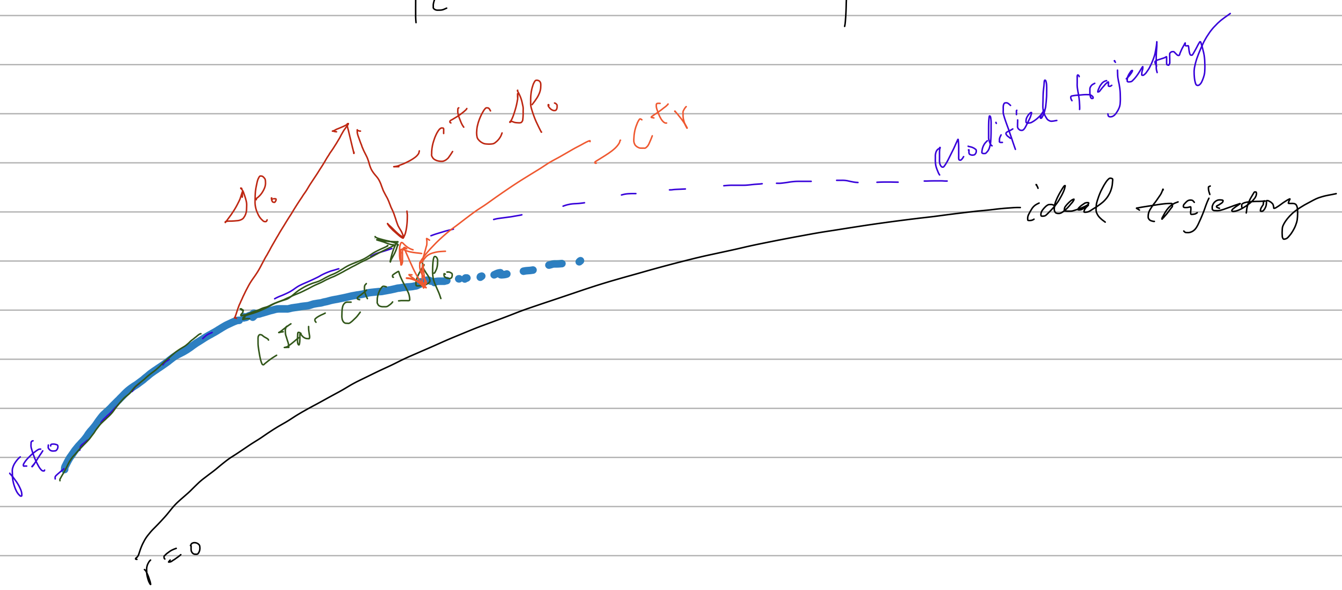
$$\dot{\rho} = \underbrace{[I_N - C^+ C]}_{\text{orthogonal projection matrix}} \dot{\rho}_0$$

$\dot{\rho}_0$: unconstrained value of the angle velocities calculated by mountain-valley values

$$\Delta\rho = \dot{\rho}(t)\,\Delta t$$

$$= [I_N - C^+ C]\,\Delta\rho_0$$

$$\Delta\rho_0 \rightarrow \frac{\partial E}{\partial \rho_i} \rightarrow [C] \rightarrow \Delta\rho$$



Euler integration results in accumulation of numerical errors

$$\underline{r = -C\dot{\rho}}$$

Modified solution

$$\Delta\rho = -C^+ r + [I_N - C^+ C]\,\Delta\rho_0$$

$C^+ = C^T(CC^T)^+$ if C is full rank and $3M < N$

$$\Delta p = -C^T(CC^T)^+ r + [I_N - C^T(CC^T)^+ C]\Delta p_0$$

$$= \Delta p_0 - C^T\{(CC^T) \backslash (Cr + C\Delta p_0)\}$$