Laplace Transform (1)

– If $f(t)$ is defined for all t ≥ 0, its Laplace transform is

$$F(s) = \mathcal{L}(f) = \int_0^{\infty} e^{-st} f(s)\,dt$$

– Integral transform with kernel

$$F(s) = \int_0^{\infty} k(s,t)\, f(s)\,dt \qquad k(s,t) = e^{-st}$$

– Inverse transform

$$f(t) = \mathcal{L}^{-1}(F)$$

– Example: $f(t) = 1$

$$\mathcal{L}(f) = \mathcal{L}(1) = \int_0^{\infty} e^{-st}\,dt = \left| -\frac{1}{s} e^{-st} \right|_0^{\infty} = \frac{1}{s} \quad (s > 0)$$

Laplace Transform (2)

– Example: $f(t) = e^{at}$

$$\mathcal{L}\left(e^{at}\right) = \int_0^\infty e^{-st} e^{at} dt = \left|\frac{1}{a-s} e^{-(s-a)t}\right|_0^\infty = \frac{1}{s-a} \quad \left(s-a>0\right)$$

– Linearity of Laplace transform

$$\mathcal{L}\left\{af\left(t\right)+bg\left(t\right)\right\} = a\mathcal{L}\left\{f\left(t\right)\right\} + b\mathcal{L}\left\{g\left(t\right)\right\}$$

– Application: hyperbolic function

$$\mathcal{L}\left(\cosh at\right) = \frac{1}{2}\left\{\mathcal{L}\left(e^{at}\right)+\mathcal{L}\left(e^{-at}\right)\right\} = \frac{1}{2}\left(\frac{1}{s-a}+\frac{1}{s+a}\right) = \frac{s}{s^2-a^2},$$

$$\mathcal{L}\left(\sinh at\right) = \frac{1}{2}\left\{\mathcal{L}\left(e^{at}\right)-\mathcal{L}\left(e^{-at}\right)\right\} = \frac{1}{2}\left(\frac{1}{s-a}-\frac{1}{s+a}\right) = \frac{a}{s^2-a^2}$$

Laplace Transform (3)

– Application: trigonometric function

$$L_c = \int_0^\infty e^{-st}\cos\omega t dt = \left|\frac{e^{-st}}{-s}\cos\omega t\right|_0^\infty - \frac{\omega}{s}\int_0^\infty e^{-st}\sin\omega t dt = \frac{1}{s} - \frac{\omega}{s}L_s,$$

$$L_s = \int_0^\infty e^{-st}\sin\omega t dt = \left|\frac{e^{-st}}{-s}\sin\omega t\right|_0^\infty + \frac{\omega}{s}\int_0^\infty e^{-st}\cos\omega t dt = \frac{\omega}{s}L_c$$

– By substitution

$$L_c = \frac{1}{s} - \frac{\omega}{s}\left(\frac{\omega}{s}L_c\right), L_c = \frac{s}{s^2+\omega^2},$$

$$L_s = \frac{\omega}{s}\left(\frac{1}{s} - \frac{\omega}{s}L_c\right), L_s = \frac{\omega}{s^2+\omega^2}$$

Laplace Transform (4)

– Or by complex method

$$\mathcal{L}\left(e^{i\omega t}\right)=\frac{1}{s-i\omega}=\frac{s+i\omega}{(s-i\omega)(s+i\omega)}=\frac{s+i\omega}{s^2+\omega^2}=\frac{s}{s^2+\omega^2}+i\frac{\omega}{s^2+\omega^2}$$

$$\mathcal{L}\left(e^{i\omega t}\right)=\mathcal{L}\left(\cos\omega t+i\sin\omega t\right)=\mathcal{L}\left(\cos\omega t\right)+i\mathcal{L}\left(\sin\omega t\right)$$

– Basic functions and their transforms

$f(t)$	$L(f)$	$f(t)$	$L(f)$
1	$1/s$	$t^a, (a:\text{positive})$	$\Gamma(a+1)/s^{a+1}$
t	$1/s^2$	e^{at}	$1/(s-a)$
t^2	$2!/s^3$	$\cos\omega t$	$s/(s^2+\omega^2)$
$t^n, (n=0,1,\cdots)$	$n!/s^{n+1}$	$\sin\omega t$	$\omega/(s^2+\omega^2)$

Laplace Transform (5)

– Basic functions and their transforms (cont'd)

$f(t)$	$L(f)$	$f(t)$	$L(f)$
$\cosh at$	$\dfrac{s}{s^2-a^2}$	$e^{at}\cos\omega t$	$\dfrac{s-a}{(s-a)^2+\omega^2}$
$\sinh at$	$\dfrac{a}{s^2-a^2}$	$e^{at}\sin\omega t$	$\dfrac{\omega}{(s-a)^2+\omega^2}$

– Proofs

$$\mathrm{L}\left(t^{n+1}\right)=\int_0^\infty e^{-st}t^{n+1}dt=\left|-\frac{1}{s}e^{-st}t^{n+1}\right|_0^\infty+\frac{n+1}{s}\int_0^\infty e^{-st}t^{n}dt$$

$$=\frac{n+1}{s}\mathrm{L}\left(t^{n}\right)=\frac{n+1}{s}\frac{n!}{s^{n+1}}=\frac{(n+1)!}{s^{n+2}}$$

s-Shifting (1)

– Proofs --- by setting $st = x$

$$\mathcal{L}\left(t^a\right) = \int_0^\infty e^{-st} t^a dt = \int_0^\infty e^{-x}\left(\frac{x}{s}\right)^a \frac{dx}{s} = \frac{1}{s^{a+1}}\int_0^\infty e^{-x} x^n dx = \frac{1}{s^{a+1}}\Gamma\left(a+1\right)$$

– s-Shifting --- If the transform of $f(t)$ exists for some s greater than some k, then the transform of $e^{at}f(t)$ exists for s-a>k.

$$\mathcal{L}\left\{e^{at} f\left(t\right)\right\} = F\left(s-a\right)$$

$$e^{at} f\left(t\right) = \mathcal{L}^{-1}\left\{F\left(s-a\right)\right\}$$

– Proof

$$F\left(s-a\right) = \int_0^\infty e^{-(s-a)t} f\left(t\right) dt = \int_0^\infty e^{-st}\left[e^{at} f\left(t\right)\right] dt = \mathcal{L}\left\{e^{at} f\left(t\right)\right\}$$

Existence/Uniqueness of Laplace Transform

- Existence --- If $f(t)$ is defined and piecewise continuous on the semi-axis $t \geq 0$, and satisfies the relation below for all $t \geq 0$ and some constants M and k, then its Laplace transform exists for all $s > k$.

$$\left| f(t) \right| \leq M e^{kt}$$

- Proof

$$\left| \mathcal{L}(f) \right| = \left| \int_0^\infty e^{-st} f(t)\, dt \right| \leq \int_0^\infty \left| f(t) \right| e^{-st} dt \leq \int_0^\infty M e^{kt} e^{-st} dt = \frac{M}{s-k}$$

- Uniqueness --- If two continuous functions have the same transform, then they are completely identical.

Transform of Derivatives (1)

$$\mathcal{L}(f') = s\mathcal{L}(f) - f(0)$$
$$\mathcal{L}(f'') = s^2\mathcal{L}(f) - sf(0) - f'(0)$$

– Proof

$$\mathcal{L}(f') = \int_0^\infty e^{-st} f'(t)\,dt = \left[e^{-st} f(t)\right]\Big|_0^\infty + s\int_0^\infty e^{-st} f(t)\,dt$$

$$\mathcal{L}(f'') = s\mathcal{L}(f') - f'(0) = s\left[s\mathcal{L}(f) - f(0)\right] - f'(0)$$

$$= s^2\mathcal{L}(f) - sf(0) - f'(0)$$

– For any order of derivative

$$\mathcal{L}\left(f^{(n)}\right) = s^n\mathcal{L}(f) - s^{n-1}f(0) - s^{n-2}f'(0) - \cdots - f^{(n-1)}(0)$$

Transform of Derivatives (2)

– Example 1

$$f(t) = t\sin\omega t,\ f(0) = 0,$$

$$f'(t) = \sin\omega t + \omega t\cos\omega t,\ f'(0) = 0,$$

$$f''(t) = 2\omega\cos\omega t - \omega^2 t\sin\omega t$$

$$\mathrm{L}(f'') = 2\omega\frac{s}{s^2+\omega^2} - \omega\mathrm{L}(f) = s^2\mathrm{L}(f)$$

$$\mathrm{L}(f) = \mathrm{L}(f) = \frac{2\omega s}{\left(s^2+\omega^2\right)^2}$$

– Example 2

$$f(t) = \cos\omega t,\ f(0) = 1,\ f'(0) = 0,\ f''(t) = -\omega^2\cos\omega t$$

$$\mathrm{L}(f'') = s^2\mathrm{L}(f) - s = -\omega^2\mathrm{L}(f),\ \mathrm{L}(f) = \frac{s}{s^2+\omega^2}$$

Solution of DE by Laplace Transform (1)

– Initial Value Problem

$$y'' + ay' + by = r(t),\ y(0) = K_0,\ y'(0) = K_1$$

– Step 1: Apply Laplace transform (Subsidiary eqn.)

$$\left[s^2Y - sy(0) - y'(0)\right] + a\left[sY - y(0)\right] + bY = R(s)$$

$$\left(s^2Y + as + b\right)Y = (s+a)\,y(0) + y'(0) + R(s)$$

– Step 2: Solve algebraically for *Y* (Transfer function)

$$Q(s) = \frac{1}{s^2 + as + b}$$

Solution

$$Y(s) = \left[(s+a)\,y(0) + y'(0)\right]Q(s) + R(s)Q(s)$$

Solution of DE by Laplace Transform (2)

– If $y(0)=0, y'(0)=0$

$$Q(s)=\frac{Y(s)}{R(s)}=\frac{\mathcal{L}(\text{output})}{\mathcal{L}(\text{input})}$$

Transfer Function

– Step 3: Reduce *Y* to a sum of terms (partial fractions) and apply inverse transform

– Example: IVP

$$y''-y=t, y(0)=1, y'(0)=1$$

– Step 1:

$$s^2Y-sy(0)-y'(0)-Y=1/s^2,$$

$$(s^2-1)Y=s+1+1/s^2$$

Solution of DE by Laplace Transform (3)

– Step 2:

$$Q(s)=\frac{1}{s^2-1}$$

$$Y=(s+1)Q+\frac{1}{s^2}Q=\frac{s+1}{s^2-1}+\frac{1}{s^2(s^2-1)}=\frac{1}{s-1}+\left(\frac{1}{s^2-1}-\frac{1}{s^2}\right)$$

– Step 3:

$$y(t)=\mathcal{L}^{-1}(Y)=\mathcal{L}^{-1}\left\{\frac{1}{s-1}\right\}+\mathcal{L}^{-1}\left\{\frac{1}{s^2-1}\right\}-\mathcal{L}^{-1}\left\{\frac{1}{s^2}\right\}$$

$$=e^t+\sinh t-t$$

– Advantages of the Laplace transform method for solving DE

1) No need to determine a general sol. for a homogeneous eqn.

2) No need to determine an arbitrary constants in a general sol.

Transform of Integrals (1)

$$\mathcal{L}\left\{\int_0^t f(\tau)\,d\tau\right\} = \frac{1}{s}F(s) \qquad (s>0, s>k)$$

$$\int_0^t f(\tau)\,d\tau = \mathcal{L}^{-1}\left\{\frac{1}{s}F(s)\right\}$$

– Proof

$$g(t) = \int_0^t f(\tau)\,d\tau$$

$$|g(t)| \le \int_0^t |f(\tau)|\,d\tau \le M\int_0^t e^{k\tau}\,d\tau = \frac{M}{k}\left(e^{kt}-1\right) \le \frac{M}{k}e^{kt} \qquad (k>0)$$

$$\mathcal{L}\{f(t)\} = \mathcal{L}\{g'(t)\} = s\mathcal{L}\{g(t)\} - g(0) \qquad (s>k)$$

Transform of Integrals (2)

– Example 1

$$\mathcal{L}(f) = \frac{1}{s(s^2+\omega^2)},\ f(t) = ?$$

$$\mathcal{L}^{-1}\left(\frac{1}{s^2+\omega^2}\right) = \frac{1}{\omega}\sin\omega t$$

$$\mathcal{L}^{-1}\left\{\frac{1}{s}\left(\frac{1}{s^2+\omega^2}\right)\right\} = \frac{1}{\omega}\int_0^t \sin\omega\tau\, d\tau = \frac{1}{\omega^2}(1-\cos\omega t)$$

– Example 2: Shifted data problem

$$y'' + y = 2t,\ y\left(\frac{1}{4}\pi\right) = \frac{1}{2}\pi,\ y'\left(\frac{1}{4}\pi\right) = 2-\sqrt{2}$$

Transform of Integrals (3)

- Set $t_0 = 1/4\pi, t = \tilde{t} + 1/4\pi$

$$\tilde{y}'' + \tilde{y} = 2\left(\tilde{t} + \frac{1}{4}\pi\right), \tilde{y}(0) = \frac{1}{2}\pi, \tilde{y}'(0) = 2 - \sqrt{2}$$

- Step 1:

$$s^2\tilde{Y} - s\tilde{y}(0) - \tilde{y}'(0) + \tilde{Y} = \frac{2}{s^2} + \frac{\pi/2}{s},$$

- Step 2:

$$\tilde{Y} = \frac{2}{\left(s^2+1\right)s^2} + \frac{\pi/2}{s\left(s^2+1\right)} + \tilde{y}(0)\frac{s}{s^2+1} + \tilde{y}'(0)\frac{1}{s^2+1}$$

$$\tilde{y}(t) = 2\left(\tilde{t} - \sin\tilde{t}\right) + \frac{1}{2}\pi\left(1 - \cos\tilde{t}\right) + \frac{1}{2}\pi\cos\tilde{t} + \left(2 - \sqrt{2}\right)\sin\tilde{t}$$

Unit Step Function (1)

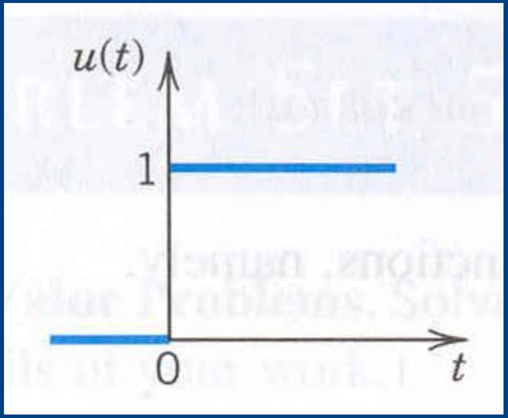
– Set back to $\tilde{t} = t - 1/4\,\pi$

$$y(t) = 2t - \sin t + \cos t$$

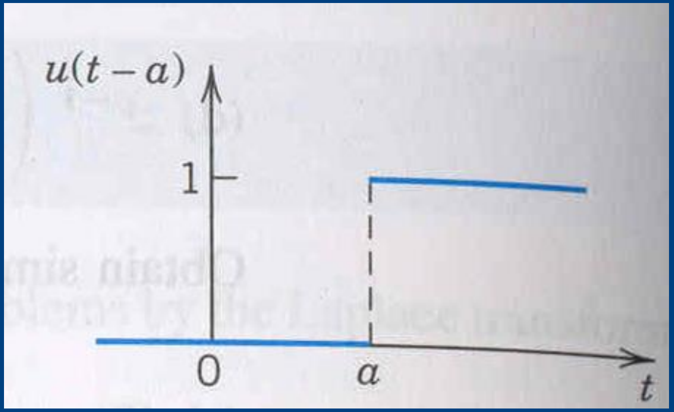
– Unit step function (Heaviside function)

$$u(t-a) = \begin{cases} 0 & \text{if} \quad t < a \\ 1 & \text{if} \quad t > a \end{cases} \qquad (a \geq 0)$$

– $u(t)$

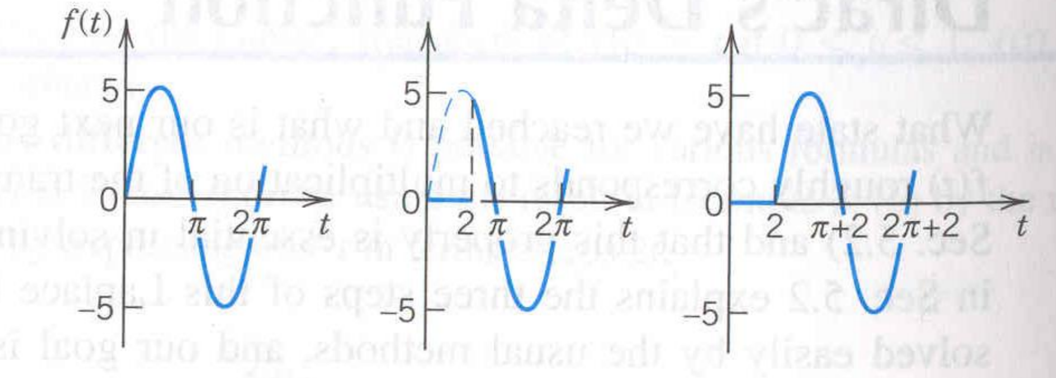


$u(t\text{-}a)$

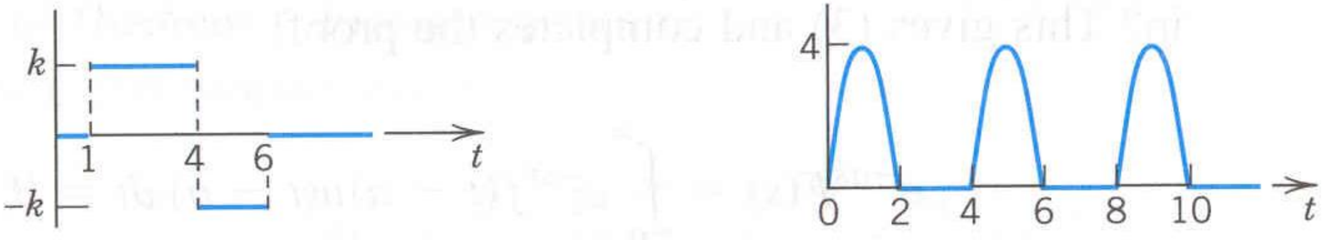


Unit Step Function (2)

– Usages



(a) $f(t) = 5 \sin t$ (b) $f(t)u(t-2)$ (c) $f(t-2)u(t-2)$



(A) $k[u(t-1) - 2u(t-4) + u(t-6)]$ (B) $4 \sin (\frac{1}{2}\pi t)[u(t) - u(t-2) + u(t-4) - + \cdots]$

Time Shifting Theorem (1)

– "Shifted function"

$$\tilde{f}(t) = f(t-a)u(t-a) = \begin{cases} 0 & \text{if} \quad t < a \\ f(t-a) & \text{if} \quad t > a \end{cases}$$

$$\mathcal{L}\{f(t-a)u(t-a)\} = e^{-as}F(s)$$

$$f(t-a)u(t-a) = \mathcal{L}^{-1}\{e^{-as}F(s)\}$$

– Proof

$$e^{-as}F(s) = e^{-as}\int_0^\infty e^{-s\tau} f(\tau)\,d\tau = \int_0^\infty e^{-s(\tau+a)} f(\tau)\,d\tau$$

– Set $\tau + a = t, \tau = t - a, d\tau = dt$

$$e^{-as}F(s) = e^{-as}\int_a^\infty e^{-st} f(t-a)\,dt$$

Time Shifting Theorem (2)

– RHS is equivalent to

$$e^{-as}F(s) = e^{-as}\int_0^\infty e^{-st} f(t-a)u(t-a)\,dt = \int_0^\infty e^{-st}\tilde{f}(t)\,dt$$

Application of Shifting Theorems (1)

– Example 1: Find the transform.

$$f(t)=\begin{cases} 2 & \text{if} \quad 0<t<1 \\ t^2/2 & \text{if} \quad 1<t<\pi/2 \\ \cos t & \text{if} \quad t>\pi/2 \end{cases}$$

– Step 1:

$$f(t)=2\{1-u(t-1)\}+\frac{1}{2}t^2\left\{u(t-1)-u\left(t-\frac{1}{2}\pi\right)\right\}$$
$$+(\cos t)u\left(t-\frac{1}{2}\pi\right)$$

– Step 2: Arrange each term in order for t-shifting theorem

$$\mathcal{L}\left\{\frac{1}{2}t^2u(t-1)\right\}=\mathcal{L}\left\{\left(\frac{1}{2}(t-1)^2+(t-1)-\frac{1}{2}\right)u(t-1)\right\}$$

Application of Shifting Theorems (2)

$$=\left(\frac{1}{s^3}+\frac{1}{s^2}+\frac{1}{2s}\right)e^{-s}$$

$$\mathcal{L}\left\{\frac{1}{2}t^2u\left(t-\frac{1}{2}\pi\right)\right\}=\mathcal{L}\left\{\left(\frac{1}{2}\left(t-\frac{1}{2}\pi\right)^2+\frac{\pi}{2}\left(t-\frac{1}{2}\pi\right)+\frac{\pi^2}{8}\right)u\left(t-\frac{1}{2}\pi\right)\right\}$$

$$=\left(\frac{1}{s^3}+\frac{\pi}{2s^2}+\frac{\pi^2}{8s}\right)e^{-\pi s/2}$$

$$\mathcal{L}\left\{(\cos t)u\left(t-\frac{1}{2}\pi\right)\right\}=\mathcal{L}\left\{-\left(\sin\left(t-\frac{1}{2}\pi\right)\right)u\left(t-\frac{1}{2}\pi\right)\right\}$$

$$=-\frac{1}{s^2+1}e^{-\pi s/2}$$

Application of Shifting Theorems (3)

– Add together

$$\mathcal{L}(f) = \frac{2}{s} - \frac{2}{s}e^{-s} + \left(\frac{1}{s^3} + \frac{1}{s^2} + \frac{1}{2s}\right)e^{-s} - \left(\frac{1}{s^3} + \frac{\pi}{2s^2} + \frac{\pi^2}{8s}\right)e^{-\pi s/2}$$

$$- \frac{1}{s^2+1}e^{-\pi s/2}$$

– Or other convenient form of t-shifting theorem

$$\boxed{\mathcal{L}\{f(t)u(t-a)\} = e^{-as}\mathcal{L}\{f(t+a)\}}$$

– Proof

$$f(t+a) = g(t),\ f(t) = g(t-a)$$

Application of Shifting Theorems (4)

– Then

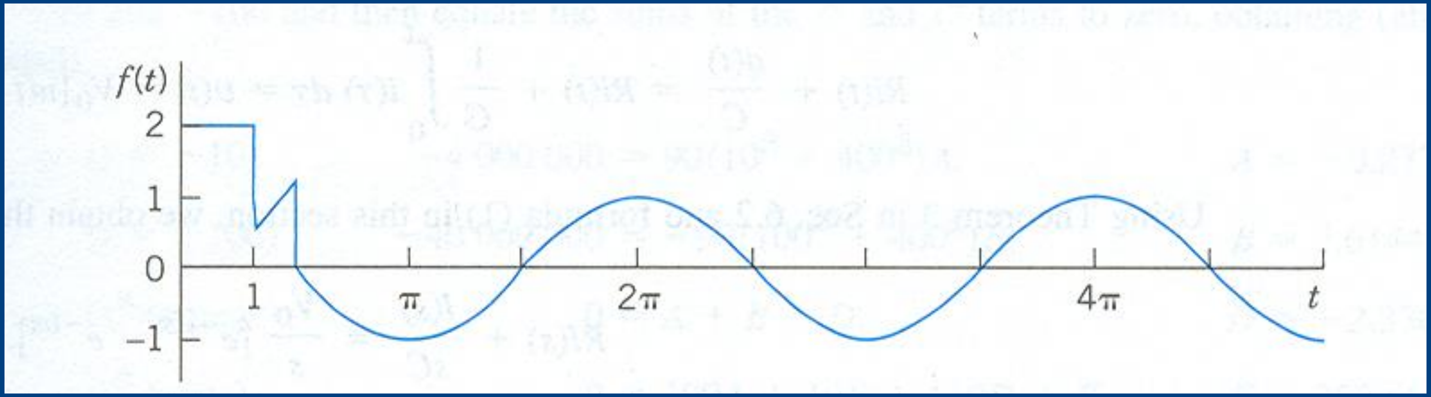
$$\mathcal{L}\left\{\frac{1}{2}t^2u(t-1)\right\}=e^{-s}\mathcal{L}\left\{\frac{1}{2}(t+1)^2\right\}=e^{-s}\mathcal{L}\left\{\frac{1}{2}t^2+t+\frac{1}{2}\right\}$$

$$=\left(\frac{1}{s^3}+\frac{1}{s^2}+\frac{1}{2s}\right)e^{-s}$$

$$\mathcal{L}\left\{(\cos t)u\left(t-\frac{1}{2}\pi\right)\right\}=e^{-\pi s/2}\mathcal{L}\left\{\cos\left(t+\frac{1}{2}\pi\right)\right\}=e^{-\pi s/2}\mathcal{L}\{-\sin t\}$$

$$=-e^{-\pi s/2}\frac{1}{s^2+1}$$

Application of Shifting Theorems (5)



Application of Shifting Theorems (6)

– Example 2: Find the inverse transform.

$$F(s) = \frac{e^{-s}}{s^2+\pi^2} + \frac{e^{-2s}}{s^2+\pi^2} + \frac{e^{-3s}}{(s+2)^2}$$

– Without the exponential functions in the numerator

$$\frac{1}{\pi}\sin \pi t, \frac{1}{\pi}\sin \pi t, te^{-2t}$$

– By the t-shifting theorem

$$f(t) = \frac{1}{\pi}\sin\{\pi(t-1)\}u(t-1) + \frac{1}{\pi}\sin\{\pi(t-2)\}u(t-2) +$$

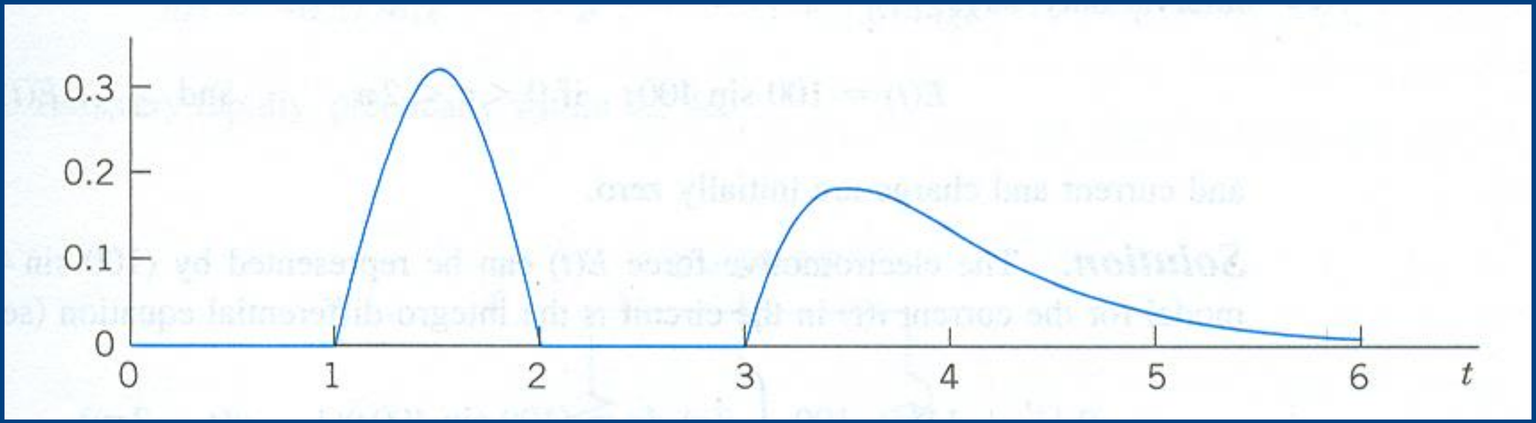
$$(t-3)e^{-2(t-3)}u(t-3)$$

– The second and third terms cancel each other when $t>2$

Application of Shifting Theorems (7)

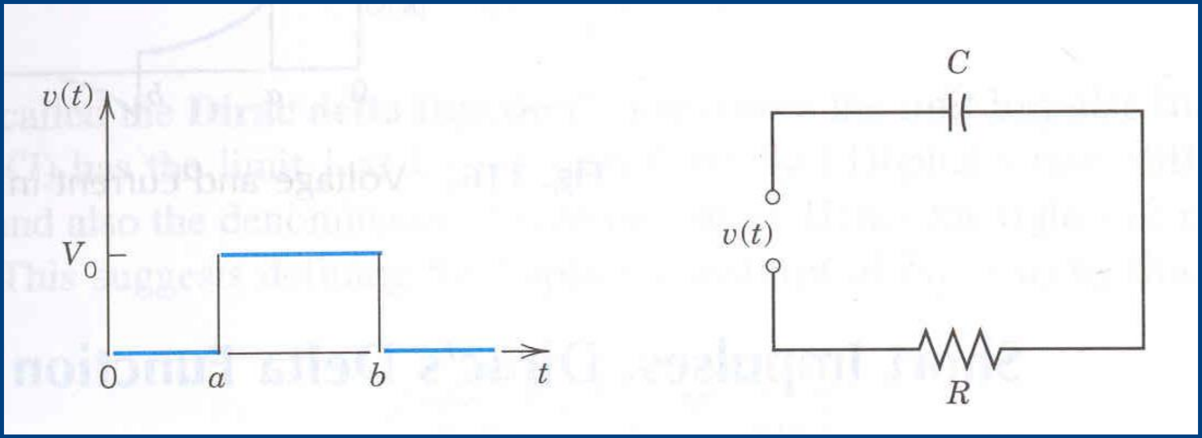
– Finally

$$f(t)=\begin{cases}0 & \text{if } \quad 0<t<1\\ -(\sin \pi t)/\pi & \text{if } \quad 1<t<2\\ 0 & \text{if } \quad 2<t<3\\ (t-3)e^{-2(t-3)} & \text{if } \quad t>3\end{cases}$$



Application of Shifting Theorems (8)

– Example 3: Response of RC-circuit to a single rectangular wave.



– Input: $V_0\left[u(t-a)-u(t-b)\right]$

– Governing DE

$$Ri(t)+\frac{q(t)}{C}=Ri(t)+\frac{1}{C}\int_0^t i(\tau)\,d\tau=v(t)=V_0\left[u(t-a)-u(t-b)\right]$$

Application of Shifting Theorems (9)

– Subsidiary eqn.

$$RI(s)+\frac{I(s)}{sC}=\frac{V_0}{s}\left[e^{-as}-e^{-bs}\right]$$

– Solve algebraically for *I*(*s*)

$$I(s)=F(s)\left(e^{-as}-e^{-bs}\right), F(s)=\frac{V_0/R}{s+1/(RC)}, \mathcal{L}^{-1}(F)=\frac{V_0}{R}e^{-t/(RC)}$$

– By the t-shifting theorem

$$i(t)=\mathcal{L}^{-1}(I)=\mathcal{L}^{-1}\left\{e^{-as}F(s)-e^{-bs}F(s)\right\}$$

$$=\frac{V_0}{R}\left[e^{-(t-a)/(RC)}u(t-a)-e^{-(t-b)/(RC)}u(t-b)\right]$$

Application of Shifting Theorems (10)

– Finally

$$i(t) = \begin{cases} K_1 e^{-t/(RC)} & \text{if} \quad a < t < b \\ (K_1 - K_2) e^{-t/(RC)} & \text{if} \quad t > b \end{cases}$$

$$K_1 = V_0 e^{a/(RC)} / R, K_2 = V_0 e^{b/(RC)} / R$$