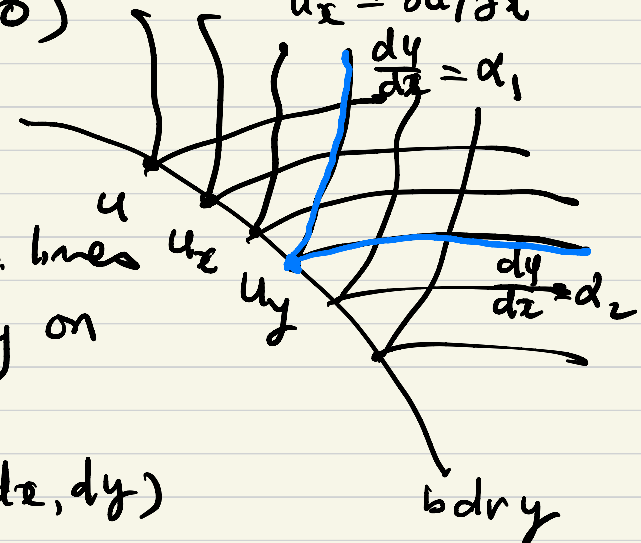
⓪ Hyperbolic PDE

$$a\frac{\partial^2 u}{\partial x^2} + 2b\frac{\partial^2 u}{\partial x \partial y} + c\frac{\partial^2 u}{\partial y^2} = F \qquad a, b, c = f(x, y, u, u_x, u_y)$$

$$(b^2 - ac > 0) \qquad u_x = \partial u / \partial x$$

Characteristic lines

$$\frac{dy}{dx} = \frac{b \pm \sqrt{b^2 - ac}}{a} = \alpha_1, \alpha_2$$



We need to know locations of char. lines (x, y) and variations of u_x and u_y on char. lines.

Along any diff'l line element (dx, dy)

$$d\left(\frac{\partial u}{\partial x}\right) = du_x = \frac{\partial^2 u}{\partial x^2}dx + \frac{\partial^2 u}{\partial x \partial y}dy \quad \text{— ①}$$

$$d\left(\frac{\partial u}{\partial y}\right) = du_y = \frac{\partial^2 u}{\partial x \partial y}dx + \frac{\partial^2 u}{\partial y^2}dy \quad \text{— ②}$$

$$\text{GE}: \quad a u_{xx} + 2b u_{xy} + c u_{yy} = F$$

$$① : \quad dx\, u_{xx} + dy\, u_{xy} = du_x$$

$$② : \quad dx\, u_{xy} + dy\, u_{yy} = du_y$$

$$\begin{vmatrix} a & 2b & c \\ dx & dy & 0 \\ 0 & dx & dy \end{vmatrix} = 0 \rightarrow$$

eq. for char. lines

$$\frac{dy}{dx} = \frac{b \pm \sqrt{b^2 - ac}}{a} = \alpha_1, \alpha_2$$

For the existence of u_{xx}, u_{xy} & u_{yy},
$$\begin{vmatrix} a & F & c \\ dx & du_x & 0 \\ 0 & du_y & dy \end{vmatrix} = 0$$

$$\rightarrow dy\,(a\,du_x - F\,dx) - du_y\,(-c\,dx) = 0$$

$$\rightarrow \frac{dy}{dx} a\, du_x + c\, du_y - F\, dy = 0$$

char. lines $dy/dx = \alpha_1, \alpha_2$ — i) & ii)

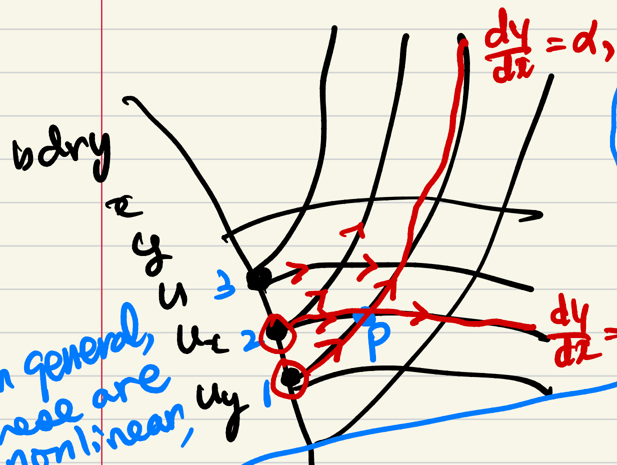
$$\alpha_1 a\, du_x + c\, du_y - F\, dy = 0 \quad \text{on } dy/dx = \alpha_1 \quad \text{— iii)}$$

$$\alpha_2 a\, du_x + c\, du_y - F\, dy = 0 \quad \text{on } dy/dx = \alpha_2 \quad \text{— iv)}$$

from chain rule, $du = u_x\,dx + u_y\,dy \cdot \sim v)$

$\Rightarrow$ 5 eqs for 5 unknowns $(\underline{x, y}, u, u_x, u_y)$ $\rightarrow$ char. lines

PDE $\rightarrow$ ODEs : method of characteristics (MOC)



In general, these are nonlinear,

$\downarrow$

iterative method

- $\frac{dy}{dx} = \alpha_1 : \frac{y(P) - y(1)}{x(P) - x(1)} = \frac{1}{2}[\alpha_1(P) + \alpha_1(1)]$

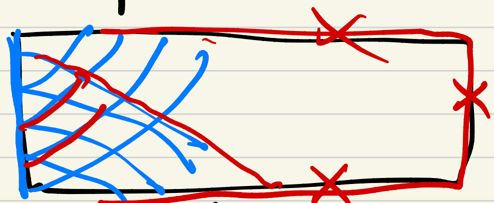
- $\frac{dy}{dx} = \alpha_2 : \frac{y(P) - y(2)}{x(P) - x(2)} = \frac{1}{2}[\alpha_2(P) + \alpha_2(2)]$

- $u(P) - u(1) = \frac{1}{2}[u_x(P) + u_x(1)][x(P) - x(1)] + \frac{1}{2}[u_y(P) + u_y(1)][y(P) - y(1)]$

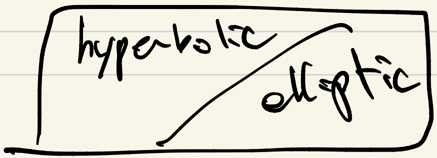
- $\frac{1}{2}[\alpha_1(P)a(P) + \alpha_1(1)a(1)][u_x(P) - u_x(1)] + \frac{1}{2}[c(P) + c(1)][u_y(P) - u_y(1)] - \frac{1}{2}[F(P) + F(1)][y(P) - y(1)] = 0$

- $\frac{1}{2}[\alpha_2(P)a(P) + \alpha_2(2)a(2)][u_x(P) - u_x(2)] + \frac{1}{2}[c(P) + c(2)][u_y(P) - u_y(2)] - \frac{1}{2}[F(P) + F(2)][y(P) - y(2)] = 0$

* Advantages of MOC
  - important properties of exact sol. are preserved in numerical sol.
  - method is easily adapted to the computation of problems that contain discontinuities.
  - ability to compute the sol. over a long span of indep. variable.



* Disadvantages of MOC
  - difficulties of keeping track of the locations of the char. lines and the values of variables in 3D.
  - difficulties in handling mixed-type PDE : e.g. hyperbolic in one area and elliptic in other area.



④ Explicit methods for hyperbolic eqs.

convection eq. $\frac{\partial \phi}{\partial t} + c\frac{\partial \phi}{\partial x} = 0$ $\qquad x - ct = \text{const}$

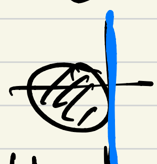
wave eq. $\frac{\partial^2 \phi}{\partial t^2} = c^2 \frac{\partial^2 \phi}{\partial x^2}$

$$\rightarrow \left(\frac{\partial}{\partial t} - c\frac{\partial}{\partial x}\right)\underbrace{\left(\frac{\partial}{\partial t} + c\frac{\partial}{\partial x}\right)\phi}_{u} = 0$$

$x - ct = \text{const}$ $\quad \left(\frac{dx}{dt} = \pm c\right)$
$x + ct = \text{const}$

$$\begin{cases} \frac{\partial u}{\partial t} - c\frac{\partial u}{\partial x} = 0 \\ \frac{\partial \phi}{\partial t} + c\frac{\partial \phi}{\partial x} = u \end{cases} \quad \text{or} \quad \begin{cases} \frac{\partial \phi}{\partial t} = c\frac{\partial u}{\partial x} \\ \frac{\partial u}{\partial t} = c\frac{\partial \phi}{\partial x} \end{cases}$$

For $\frac{\partial \phi}{\partial t} + c\frac{\partial \phi}{\partial x} = 0$ $\quad (c > 0)$

EE + CD2 :  unstable

leapfrog : stable (CFL<1), no amp. error, spurious root

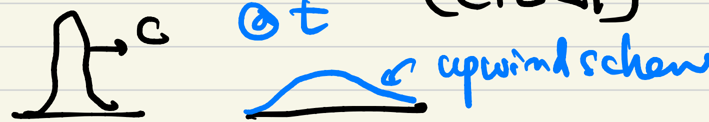
EE + upwind scheme : $\dfrac{\phi_j^{n+1}-\phi_j^{n}}{\Delta t} + c\,\dfrac{\phi_j^{n}-\phi_{j-1}^{n}}{\Delta x} = 0$

↳ complex eigenvalue

Von Neumann stability analysis $\rightarrow \dfrac{c\Delta t}{\Delta x} < 1$ for stability

(CFL < 1)

too dissipative



$c<0$, $\dfrac{\partial \phi}{\partial x} = \dfrac{\phi_{j+1}-\phi_j}{\Delta x}$ (downwind)

For wave eq., two char. lines

$x - ct = \text{const}$ $\xrightarrow{c}$

$x + ct = \text{const}$ $\xleftarrow{}$ c

$\Rightarrow$ no upwind scheme!

CD2 + EE : unstable

+ leapfrog : stable for CFL < 1, spurious root

+ RK4 : conditionally stable.

* Lax-Wendroff scheme

$$\frac{\partial \phi}{\partial t} + c\frac{\partial \phi}{\partial x} = 0$$

$$\phi(x, t+\Delta t) = \phi(x,t) + \Delta t \underbrace{\frac{\partial \phi}{\partial t}(x,t)}_{= -c\frac{\partial \phi}{\partial x}} + \frac{1}{2}\Delta t^2 \underbrace{\frac{\partial^2 \phi}{\partial t^2}(x,t)}_{= \frac{\partial}{\partial t}\left(\frac{\partial \phi}{\partial t}\right) = \frac{\partial}{\partial t}\left(-c\frac{\partial \phi}{\partial x}\right) = -c\frac{\partial}{\partial x}\frac{\partial \phi}{\partial t} = c^2\frac{\partial^2 \phi}{\partial x^2}} + \frac{1}{6}\Delta t^3 \frac{\partial^3 \phi}{\partial t^3}(x,t) + \cdots$$

$\parallel$

CD2 :

$$\boxed{\phi_j^{n+1} = \phi_j^n - c\Delta t \frac{\phi_{j+1}^n - \phi_{j-1}^n}{2\Delta x} + \frac{1}{2}c^2\Delta t^2 \frac{\phi_{j+1}^n - 2\phi_j^n + \phi_{j-1}^n}{\Delta x^2}} \underbrace{+ \alpha \Delta t^2 \frac{\partial^3 \phi}{\partial t^3} + \cdots}_{\text{dispersive error}} + O(\Delta x^2)$$

Lax-Wendroff scheme

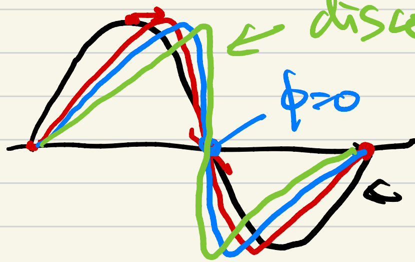
explicit
stable for CFL < 1
2nd-order accurate in space & time

disadvantage : c is not constant, f(t), nonlinear, $c(\phi)$, higher dimension → difficult.

This method can be applied to

$$\frac{\partial^2\phi}{\partial t^2} = c^2 \frac{\partial^2\phi}{\partial x^2} \longrightarrow \begin{cases} \dfrac{\partial\phi}{\partial t} = c\,\dfrac{\partial u}{\partial x} \\ \dfrac{\partial u}{\partial t} = c\,\dfrac{\partial \phi}{\partial x} \end{cases}.$$

- $\dfrac{\partial\phi}{\partial t} + \phi\dfrac{d\phi}{dx} = 0$ nonlinear eq.

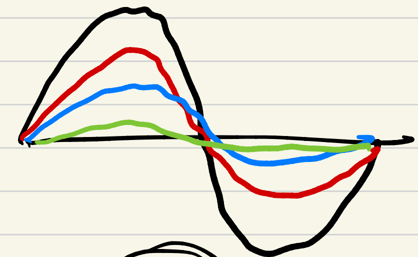


$\Uparrow$

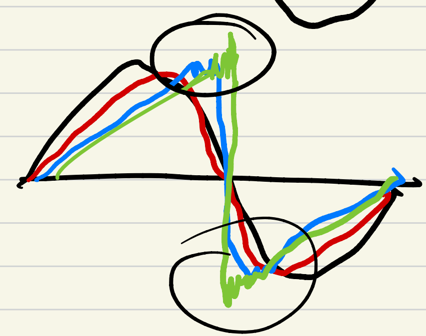
TVD
ENO

schemes !

upwind



CD2



⊙ Implicit method for hyperbolic eqs.

$$\frac{\partial \phi}{\partial t} + c\frac{\partial \phi}{\partial x} = 0$$

CD2 + CN : $\dfrac{\phi_j^{n+1} - \phi_j^n}{\Delta t} + \dfrac{1}{2}c\left[\dfrac{\phi_{j+1}^{n+1} - \phi_{j-1}^{n+1}}{2\Delta x} + \dfrac{\phi_{j+1}^{n} - \phi_{j-1}^{n}}{2\Delta x}\right] = 0$

(no amp. error (no dissipation)
dispersive error
stable

$$\frac{\partial^2 \phi}{\partial t^2} = c^2\frac{\partial^2 \phi}{\partial x^2}$$

① splitting into two eqs. ϕ & u
and applying CN to two eqs.

② applying CN directly to wave eq.

$$\frac{\phi_j^{n+1} - 2\phi_j^{n} + \phi_j^{n-1}}{\Delta t^2} = \frac{1}{2}c^2\left[\frac{\phi_{j+1}^{n+1} - 2\phi_j^{n+1} + \phi_{j-1}^{n+1}}{\Delta x^2} + \frac{\phi_{j+1}^{n} - 2\phi_j^{n} + \phi_{j-1}^{n}}{\Delta x^2}\right]$$

(stable but spurious root
a little more accurate than the method applied to two splitted eqs.

- higher dimension (2D, 3D) → ADI
- nonlinear eq. $\frac{\partial \phi}{\partial t} + \phi \frac{\partial \phi}{\partial x} = 0$ → iterative method for implicit scheme.

* MacCormack scheme – very popular

downwind

$$\frac{\partial \phi}{\partial t} + c\frac{\partial \phi}{\partial x} = 0 \rightarrow \left(\begin{array}{ll} \phi_j^* = \phi_j^n - \frac{c\Delta t}{\Delta x}(\phi_{j+1}^n - \phi_j^n) & \text{: predictor} \\ \phi_j^{n+1} = \frac{1}{2}(\phi_j^* + \phi_j^n) - \frac{c\Delta t}{\Delta x}(\phi_j^* - \phi_{j-1}^*) & \text{: corrector} \end{array} \right.$$

upwind

explicit
equivalent to Lax-Wendroff scheme for linear prob.
$O(\Delta t^2)$, $O(\Delta x^2)$
stable for CFL<1
readily extended to 2D & 3D.
easier to program
desirable nonlinear properties

Ch. 6. Discrete Transform methods : spectral method.