Example of Impulse Function (1)

– Hammerblow response of a mass-spring system

$$y'' + 3y' + 2y = \delta(t-1)$$

– Laplace transform

$$\left(s^2 + 3s + 2\right)Y = e^{-s}$$

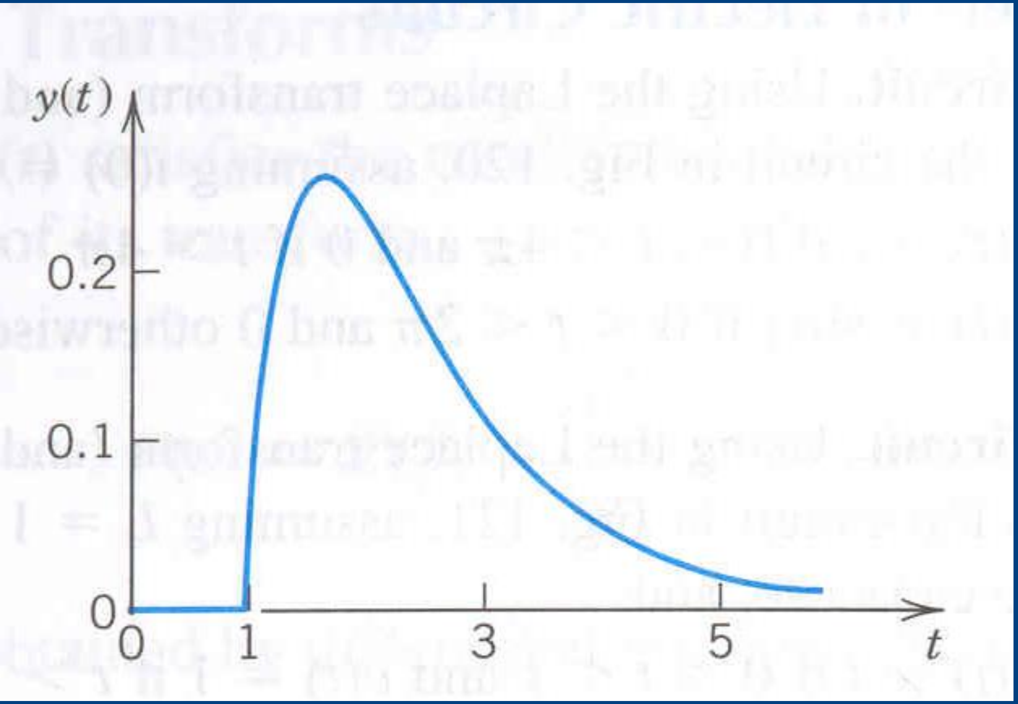
– Solving algebraically

$$Y(s) = \frac{e^{-s}}{(s+1)(s+2)} = \left(\frac{1}{s+1} - \frac{1}{s+2}\right)e^{-s}$$

– Inverse transform

$$y(t) = \mathcal{L}^{-1}(Y) = \begin{cases} 0 & \text{if } \quad 0 < t < 1 \\ e^{-(t-1)} - e^{-2(t-1)} & \text{if } \quad t > 1 \end{cases}$$

Example of Impulse Function (2)



Partial Fractions

- $Y(s)$ in a subsidiary eqn. --- usually a quotient of polynomials. Thus, a partial fraction representation leads to a sum of expressions whose inverses can be obtained easily.
- Unrepeated factor s-a --- $A/(s$-$a)$
- Repeated real factors $(s\text{-}a)^2$, $(s\text{-}a)^3$, …--- requires partial fractions

$$\frac{A_2}{(s-a)^2}+\frac{A_1}{s-a},\frac{A_3}{(s-a)^3}+\frac{A_2}{(s-a)^2}+\frac{A_1}{s-a},\text{etc.}$$

Inverses are $\left(A_2t+A_1\right)e^{at},\left(\frac{1}{2}A_3t^2+A_2t+A_1\right)e^{at},\text{etc.}$

- Unrepeated complex factors $(s-a)(s-\bar{a}),a=\alpha+i\beta,\bar{a}=\alpha-i\beta$ --- requires partial fractions $\left(As+B\right)\Big/\left[\left(s-\alpha\right)^2+\beta\right]$

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– Damped forced vibrations

$$y'' + 2y' + 2y = r(t)$$

$$r(t) = \begin{cases} 10\sin 2t & \text{if} \quad 0 < t < \pi \\ 0 & \text{if} \quad t > \pi \end{cases}$$

$$y(0) = 1, \; y'(0) = -5$$

– Subsidiary eqn.

$$\left(s^2 Y - s + 5\right) + 2\left(sY - 1\right) + 2Y = 10\frac{2}{s^2+4}\left(1 - e^{-\pi s}\right)$$

– Solving algebraically

$$Y(s) = \frac{20}{\left(s^2+4\right)\left(s^2+2s+2\right)} - \frac{20e^{-\pi s}}{\left(s^2+4\right)\left(s^2+2s+2\right)} + \frac{s-3}{s^2+2s+2}$$

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– Last fraction

$$\mathcal{L}^{-1}\left\{\frac{s+1-4}{(s+1)^2+1}\right\} = e^{-t}(\cos t - 4\sin t)$$

– First fraction --- unrepeated complex roots

$$\frac{20}{(s^2+4)(s^2+2s+2)} = \frac{As+B}{s^2+4} - \frac{Ms+N}{s^2+2s+2}$$

– Multiplication by the commen denominator

$$20 = (As+B)(s^2+2s+2) + (Ms+N)(s^2+4)$$

Example of Partial Fractions (3)

– Determine the unknown coeff.

$$\left[s^3\right]: 0 = A + M$$

$$\left[s^2\right]: 0 = 2A + B + N$$

$$\left[s\right]: 0 = 2A + 2B + 4M$$

$$\left[s^0\right]: 20 = 2B + 4N$$

$$A = -2, B = -2,$$
$$M = 2, N = 6$$

– Then, the first fraction

$$\frac{-2s+2}{s^2+4} + \frac{2(s+1)+6-2}{\left(s^2+1\right)^2+1}$$

– Its inverse transform

$$-2\cos 2t - \sin 2t + e^{-t}\left(2\cos t + 4\sin t\right)$$

Example of Partial Fractions (4)

– Solution for $0 < t < \pi$

$$y(t) = 3e^{-t}\cos t - 2\cos 2t - \sin 2t$$

– For the second fraction, its inverse transform

$$2\cos 2(t-\pi) + \sin 2(t-\pi) - e^{-(t-\pi)}\left[2\cos(t-\pi) + 4\sin(t-\pi)\right]$$

$$= 2\cos 2t + \sin 2t - e^{-(t-\pi)}(2\cos t + 4\sin t)$$

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$$y(t) = e^{-t}\left[\left(3 + 2e^{\pi}\right)\cos t + 4e^{\pi}\sin t\right]$$

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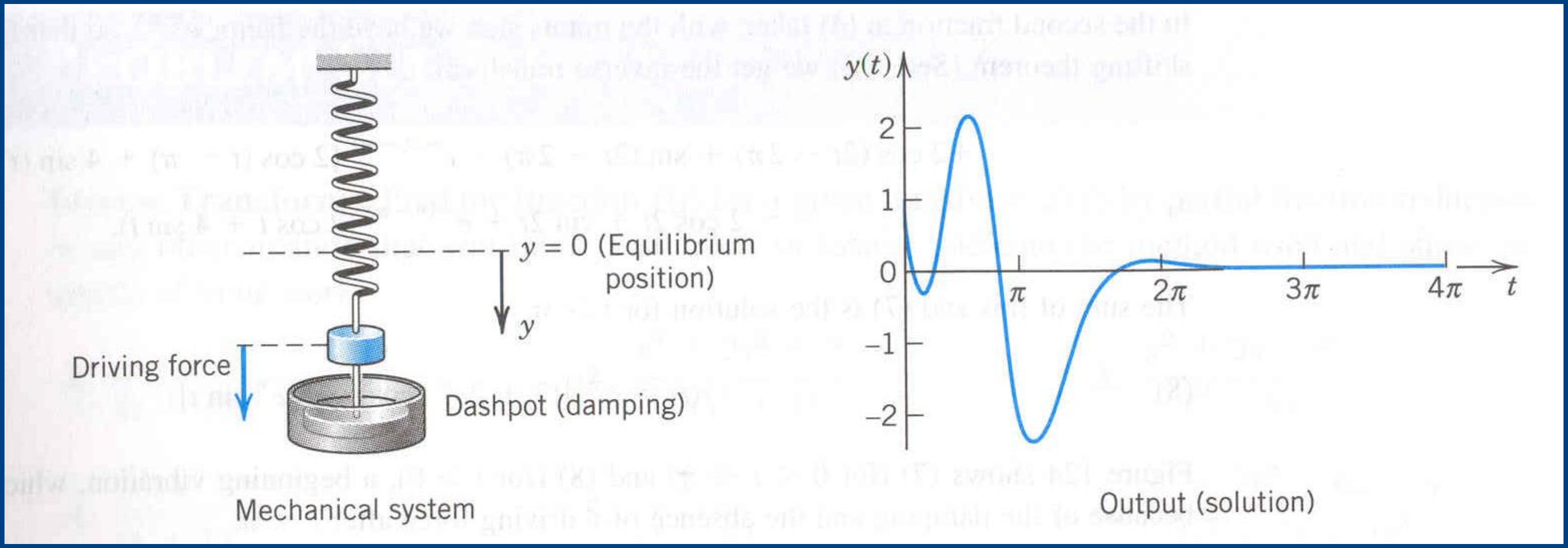


Convolution (1)

– In the Laplace domain $H(s) = F(s)G(s)$
  Then, in the time domain

$$h(t) = (f * g)(t) = \int_0^t f(\tau)\, f(t-\tau)\, d\tau$$

Convolution

– Example --- find the inverse

$$H(s) = \frac{1}{(s^2+1)^2} = \frac{1}{(s^2+1)} \cdot \frac{1}{(s^2+1)}$$

$$h(t) = \mathcal{L}^{-1}(H) = \sin t * \sin t = \int_0^t \sin\tau \sin(t-\tau)\, d\tau$$

$$= \frac{1}{2}\int_0^t -\cos t\, d\tau + \frac{1}{2}\int_0^t \cos(2t-\tau)\, d\tau = -\frac{1}{2} t\cos t + \frac{1}{2}\sin t$$

Convolution (2)

– Proof

$$e^{-s\tau}G(s)=\mathcal{L}\{g(t-\tau)u(t-\tau)\}=\int_0^{\infty}e^{-st}g(t-\tau)u(t-\tau)d\tau$$

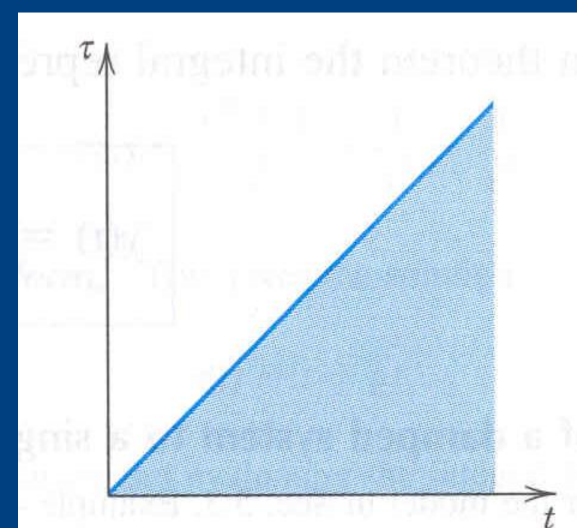
$$=\int_{\tau}^{\infty}e^{-st}g(t-\tau)d\tau$$

$$F(s)G(s)=\int_0^{\infty}e^{-st}f(\tau)G(s)d\tau=\int_0^{\infty}f(\tau)\int_{\tau}^{\infty}e^{-st}g(t-\tau)dtd\tau$$

– Switching the order of integration

$$F(s)G(s)=\int_0^{\infty}e^{-st}\int_{\tau}^{t}f(\tau)g(t-\tau)d\tau dt$$

$$=\int_0^{\infty}e^{-st}h(t)dt=\mathcal{L}(h)$$



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– Properties

$$f * g = g * f$$

Commutative law

$$f * (g_1 + g_2) = f * g_1 + f * g_2$$

Distributive law

$$(f * g) * v = f * (g * v)$$

Associative law

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– Solution of DE by convolution

$$y'' + ay' + by = r(t)$$

$$(s^2 + as + b)Y = (s + a)y(0) + y'(0) + \mathcal{L}(r)$$

$$Y(s) = [(s + a)y(0) + y'(0)]Q(s) + R(s)Q(s)$$

$$Q(s) = 1/(s^2 + as + b)$$

Transfer function

Solution of DE by Convolution (1)

– If zero IC's

$$Y(s) = R(s)Q(s)$$

$$y(t) = \int_0^t q(t-\tau)\, r(\tau)\, d\tau$$

– Example --- single square wave excitation to a damped system

$$y'' + 3y' + 2y = r(t)$$

$$r(t) = \begin{cases} 1 & \text{if} \quad 1 < t < 2 \\ 0 & \text{otherwise} \end{cases}$$

$$y(0) = 0,\ y'(0) = 0$$

Solution of DE by Convolution (2)

– Transfer function

$$Q(s)=\frac{1}{s^2+3s+2}=\frac{1}{s+1}-\frac{1}{s+2},\, q(t)=e^{-t}-e^{-2t}$$

– Convolution

$$\int\left[e^{-(t-\tau)}-e^{-2(t-\tau)}\right]d\tau = e^{-(t-\tau)}-\frac{1}{2}e^{-2(t-\tau)}$$

– If 1 < *t* < 2, need to integrate from 1 to *t*

$$e^{-0}-e^{-(t-1)}-\frac{1}{2}\left(e^{-0}-e^{-2(t-1)}\right)=\frac{1}{2}-e^{-(t-1)}-\frac{1}{2}e^{-2(t-1)}$$

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$$e^{-(t-2)}-e^{-(t-1)}-\frac{1}{2}\left[e^{-2(t-2)}-e^{-2(t-1)}\right]$$

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$$y(t) = t + \int_0^t y(\tau)\sin(t-\tau)\,d\tau$$

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$$Y(s) = \frac{1}{s^2} + Y(s)\frac{1}{s^2+1},\; Y(s) = \frac{s^2+1}{s^4} = \frac{1}{s^2} + \frac{1}{s^4}$$

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Differentiation of Transforms (1)

– Differentiation of transform --- differentiating *F*(*s*) under the integral sign with respect to *s*

$$F(s)=\int_0^{\infty} e^{-st} f(t)\,dt,\; F'(s)=-\int_0^{\infty} e^{-st} t f(t)\,dt$$

$$\mathcal{L}\{tf(t)\}=-F'(s)$$

$$\mathcal{L}^{-1}\{F'(s)\}=-tf(t)$$

$L(f)$	$f(t)$	$L(f)$	$f(t)$
$\dfrac{1}{\left(s^2+\beta^2\right)^2}$	$\dfrac{1}{2\beta^3}\left(\sin\beta t-\beta t\cos\beta t\right)$	$\dfrac{s}{\left(s^2+\beta^2\right)^2}$	$\dfrac{1}{2\beta}\sin\beta t$
$\dfrac{s^2}{\left(s^2+\beta^2\right)^2}$	$\dfrac{1}{2\beta}\left(\sin\beta t+\beta t\cos\beta t\right)$		

Differentiation of Transforms (2)

– Proof of the second formula

$$\mathcal{L}\{\sin \beta t\} = \frac{\beta}{s^2+\beta^2} \qquad \mathcal{L}\{t \sin \beta t\} = \frac{2\beta s}{\left(s^2+\beta^2\right)^2}$$

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Integration of Transforms

$$\mathcal{L}\left\{\frac{f(t)}{t}\right\} = \int_s^\infty F(\tilde{s})\,d\tilde{s}$$

$$\mathcal{L}^{-1}\left\{\int_s^\infty F(\tilde{s})\,d\tilde{s}\right\} = \frac{f(t)}{t}$$

– Proof

$$\int_s^\infty F(\tilde{s})\,d\tilde{s} = \int_s^\infty \left[\int_0^\infty e^{-\tilde{s}t} f(t)\,dt\right] d\tilde{s}$$

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Special Linear ODE w/ Variable Coeff. (1)

– Certain ODEs with variable coeff.

$$\mathcal{L}\left(y'\right)=sY-y\left(0\right)$$

$$\mathcal{L}\left(ty'\right)=-\frac{d}{ds}\left[sY-y\left(0\right)\right]=-Y-s\frac{dY}{ds}$$

$$\mathcal{L}\left(y''\right)=s^2Y-sy\left(0\right)-y'\left(0\right)$$

$$\mathcal{L}\left(ty''\right)=-\frac{d}{ds}\left[s^2Y-sy\left(0\right)-y'\left(0\right)\right]=-2sY-s^2\frac{dY}{ds}+y\left(0\right)$$

– Laquerre's eqn.

$$ty''+\left(1-t\right)y'+ny=0$$

– Subsidiary eqn.

$$\left[-2sY-s^2\frac{dY}{ds}+y\left(0\right)\right]+sY-y\left(0\right)-\left(-Y-s\frac{dY}{ds}\right)+nY=0$$

Special Linear ODE w/ Variable Coeff. (2)

– Simplifying

$$\left(s-s^2\right)\frac{dY}{ds}+\left(n+1-s\right)Y=0$$

– Separating, partial fractions, integrating, taking exponentials

$$\frac{dY}{Y}=-\frac{n+1-s}{s-s^2}ds=\left(\frac{n}{s-1}-\frac{n+1}{s}\right)ds,\; Y=\frac{\left(s-1\right)^n}{s^{n+1}}$$

– Inverse transform --- Rodrigues's formula, Laguerre's polynomial

$$l_n=\mathcal{L}^{-1}\left(Y\right)$$

$$l_0=1, l_n\left(t\right)=\frac{e^t}{n!}\frac{d^n}{dt^n}\left(t^n e^{-t}\right) \qquad n=1,2,\cdots$$

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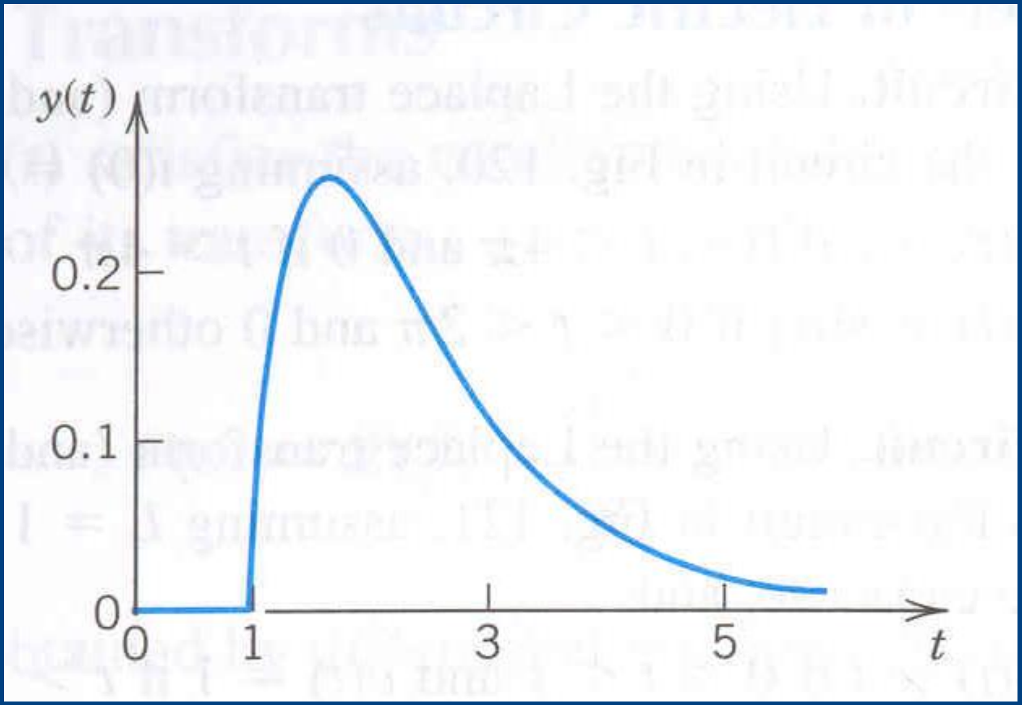

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- Solution for $0 < t < \pi$

$$y(t) = 3e^{-t}\cos t - 2\cos 2t - \sin 2t$$

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$$2\cos 2(t-\pi) + \sin 2(t-\pi) - e^{-(t-\pi)}\left[2\cos(t-\pi) + 4\sin(t-\pi)\right]$$

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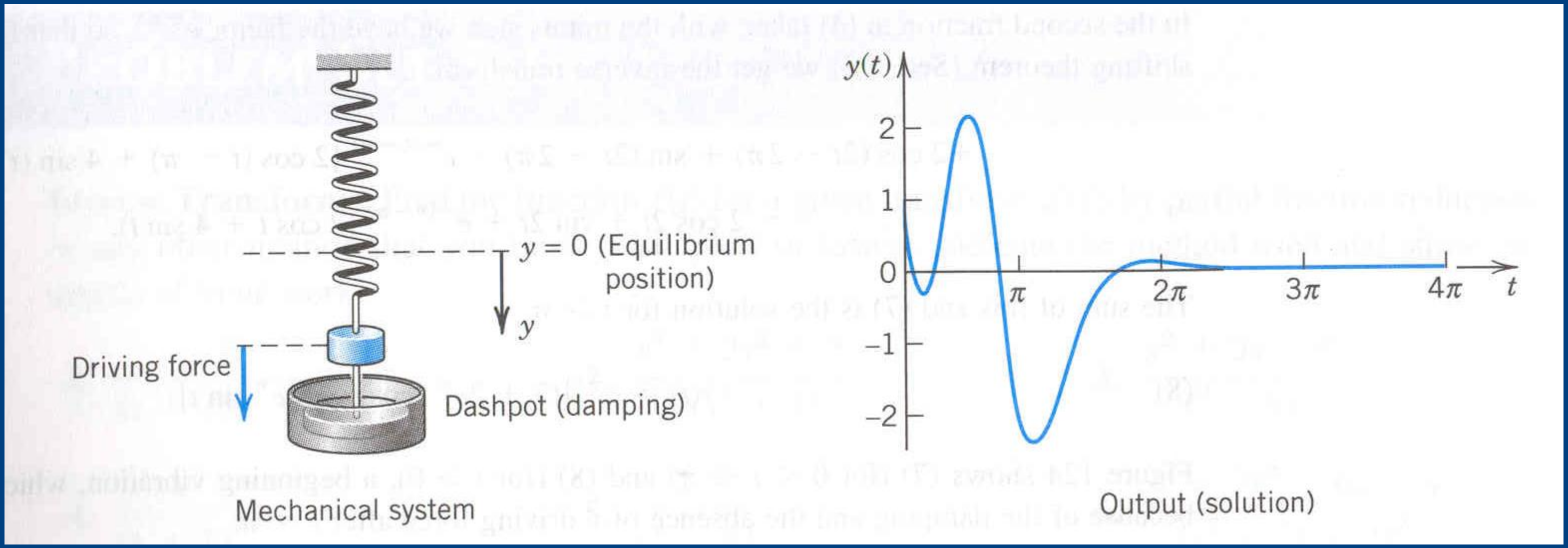

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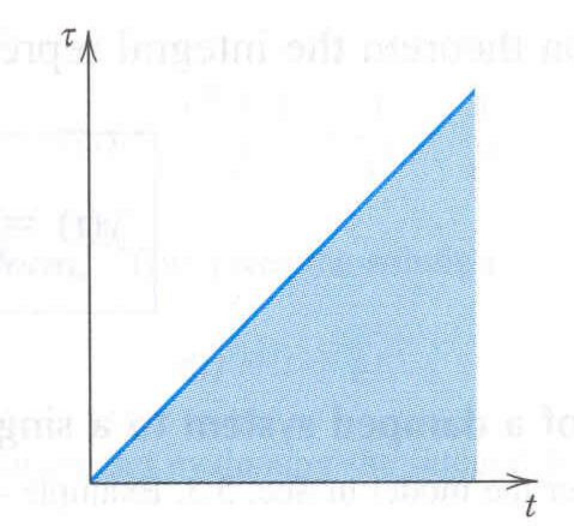

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$$Y(s) = [(s + a)y(0) + y'(0)]Q(s) + R(s)Q(s)$$

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Transfer function

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– If zero IC's

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Solution of DE by Convolution (2)

- Transfer function

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- Convolution

$$\int\left[e^{-(t-\tau)}-e^{-2(t-\tau)}\right]d\tau = e^{-(t-\tau)}-\frac{1}{2}e^{-2(t-\tau)}$$

- If 1 < *t* < 2, need to integrate from 1 to *t*

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- If *t* > 2, need to integrate from 1 to 2

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– Integral eqn.

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$L(f)$	$f(t)$	$L(f)$	$f(t)$
$\dfrac{1}{(s^2+\beta^2)^2}$	$\dfrac{1}{2\beta^3}(\sin\beta t-\beta t\cos\beta t)$	$\dfrac{s}{(s^2+\beta^2)^2}$	$\dfrac{1}{2\beta}\sin\beta t$
$\dfrac{s^2}{(s^2+\beta^2)^2}$	$\dfrac{1}{2\beta}(\sin\beta t+\beta t\cos\beta t)$		

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– Proof of the second formula

$$\mathcal{L}\{\sin \beta t\} = \frac{\beta}{s^2+\beta^2} \qquad \mathcal{L}\{t \sin \beta t\} = \frac{2\beta s}{\left(s^2+\beta^2\right)^2}$$

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$$\int_s^\infty F(\tilde{s})\,d\tilde{s} = \int_s^\infty \left[\int_0^\infty e^{-\tilde{s}t} f(t)\,dt\right] d\tilde{s}$$

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$$\mathcal{L}(ty') = -\frac{d}{ds}\left[sY - y(0)\right] = -Y - s\frac{dY}{ds}$$

$$\mathcal{L}(y'') = s^2Y - sy(0) - y'(0)$$

$$\mathcal{L}(ty'') = -\frac{d}{ds}\left[s^2Y - sy(0) - y'(0)\right] = -2sY - s^2\frac{dY}{ds} + y(0)$$

– Laquerre's eqn.

$$ty'' + (1-t)y' + ny = 0$$

– Subsidiary eqn.

$$\left[-2sY - s^2\frac{dY}{ds} + y(0)\right] + sY - y(0) - \left(-Y - s\frac{dY}{ds}\right) + nY = 0$$

Special Linear ODE w/ Variable Coeff. (2)

– Simplifying

$$\left(s-s^2\right)\frac{dY}{ds}+\left(n+1-s\right)Y=0$$

– Separating, partial fractions, integrating, taking exponentials

$$\frac{dY}{Y}=-\frac{n+1-s}{s-s^2}ds=\left(\frac{n}{s-1}-\frac{n+1}{s}\right)ds, Y=\frac{\left(s-1\right)^n}{s^{n+1}}$$

– Inverse transform --- Rodrigues's formula, Laguerre's polynomial

$$l_n=\mathcal{L}^{-1}\left(Y\right)$$

$$l_0=1, l_n\left(t\right)=\frac{e^t}{n!}\frac{d^n}{dt^n}\left(t^n e^{-t}\right) \qquad n=1,2,\cdots$$

Systems of ODEs

– First-order linear system with constant coefficients

$$y_1' = a_{11}y_1 + a_{12}y_2 + g_1(t)$$
$$y_2' = a_{21}y_1 + a_{22}y_2 + g_2(t)$$

⇨

In vector form, $\mathbf{y}' = \mathbf{A}\mathbf{y} + \mathbf{g}$

$\mathbf{y} = [y_1 \;\; y_2]^{\mathrm{T}}$, $\mathbf{A} = [a_{jk}]$, $\mathbf{g} = [g_1 \;\; g_2]^{\mathrm{T}}$

– By writing $Y_1 = \mathcal{L}(y_1), Y_2 = \mathcal{L}(y_2), G_1 = \mathcal{L}(g_1), G_2 = \mathcal{L}(g_2)$,

$$sY_1 - y_1(0) = a_{11}Y_1 + a_{12}Y_2 + G_1(s)$$
$$sY_2 - y_2(0) = a_{21}Y_1 + a_{22}Y_2 + G_2(s)$$

⇨

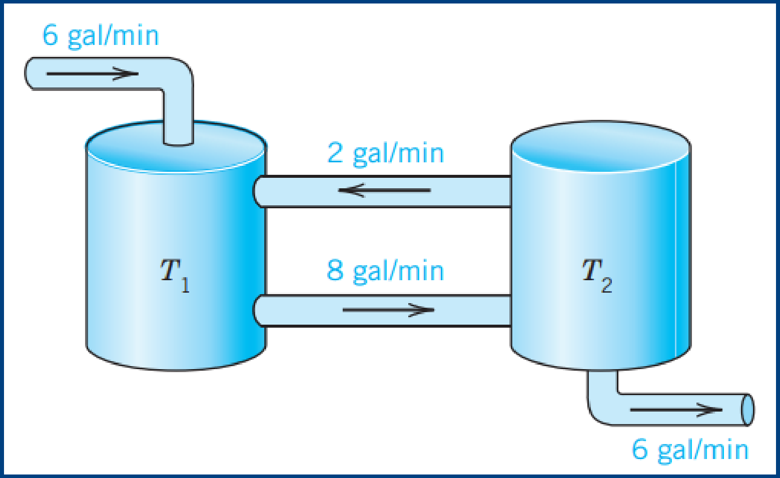
In vector form, $(\mathbf{A} - s\mathbf{I})\mathbf{Y} = -\mathbf{y}(\mathbf{0}) - \mathbf{G}$

$\mathbf{Y} = [Y_1 \;\; Y_2]^{\mathrm{T}}$, $\mathbf{A} = [a_{jk}]$, $\mathbf{G} = [G_1 \;\; G_2]^{\mathrm{T}}$

– By solving this system algebraically for $Y_1(s)$, $Y_2(s)$, we can obtain the solution $y_1 = \mathcal{L}^{-1}(Y_1), y_2 = \mathcal{L}^{-1}(Y_2)$

Systems of ODEs : example 1

- Mixing problem
  - T_1 with 100 gal of pure water, T_2 with 100 gal of water with 150 lb of salt dissolved
  - Inflow into T_1 : 2 gal/min from T_2 and 6 gal/min with 6 lb of salt from outside
  - Inflow into T_2 : 8 gal/min from T_1, outflow from T_2 : 8 gal/min



Systems of ODEs : example 1

- Salt contents : $y_1(t)$, $y_2(t)$
- Modeling

$$y_1' = -\frac{8}{100}y_1 + \frac{2}{100}y_2 + 6$$
$$y_2' = \frac{8}{100}y_1 - \frac{8}{100}y_2$$
with $y_1(0) = 0, y_2(0) = 150$

- By taking Laplace transform, ($Y = \mathcal{L}(y)$)

$$(-0.08 - s)Y_1 + 0.02Y_2 = -\frac{6}{s}$$
$$0.08Y_1 + (-0.08 - s)Y_2 = -150$$

$\Rightarrow$

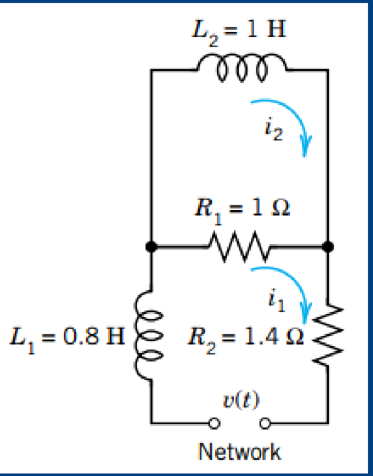
$$Y_1 = \frac{100}{s} - \frac{62.5}{s+0.12} - \frac{37.5}{s+0.04}$$
$$Y_2 = \frac{100}{s} + \frac{125}{s+0.12} - \frac{75}{s+0.04}$$

- By taking the inverse transform,

$$y_1 = 100 - 62.5e^{-0.12t} - 37.5e^{-0.04t}$$
$$y_2 = 100 + 125e^{-0.12t} - 75e^{-0.04t}$$

Systems of ODEs : example 2

- Electrical network
  - $v(t) = 100$ volts if $0 \leq t \leq 0.5$, $i(0) = 0$, $i'(0) = 0$
  - Modeling



$$0.8i_1' + 1(i_1 - i_2) + 1.4i_1 = 100\left[1 - u\left(t - \frac{1}{2}\right)\right]$$

$$1i_2' + 1(i_2 - i_1) = 0$$

  - By taking Laplace transform, $(I = \mathcal{L}(i))$

$$(s+3)I_1 - 1.25I_2 = 125\left(\frac{1}{s} - \frac{e^{-s/2}}{s}\right)$$

$$-I_1 + (s+1)I_2 = 0$$

$\Rightarrow$

$$I_1 = \frac{125(s+1)}{s\left(s+\frac{1}{2}\right)\left(s+\frac{7}{2}\right)}(1 - e^{-s/2}), \quad I_2 = \frac{125}{s\left(s+\frac{1}{2}\right)\left(s+\frac{7}{2}\right)}(1 - e^{-s/2})$$

Systems of ODEs : example 2

– By taking the inverse transform,

$$i_1(t) = -\frac{125}{3}e^{-\frac{t}{2}} - \frac{625}{21}e^{-\frac{7t}{2}} + \frac{500}{7}$$
$$i_2(t) = -\frac{250}{3}e^{-\frac{t}{2}} + \frac{250}{21}e^{-\frac{7t}{2}} + \frac{500}{7}$$
for $0 \le t \le \frac{1}{2}$

– According to the second shifting theorem, the solution for $t > \frac{1}{2}$ is

$$i_1(t) - i_1\left(t - \frac{1}{2}\right), \quad i_2(t) - i_2\left(t - \frac{1}{2}\right)$$

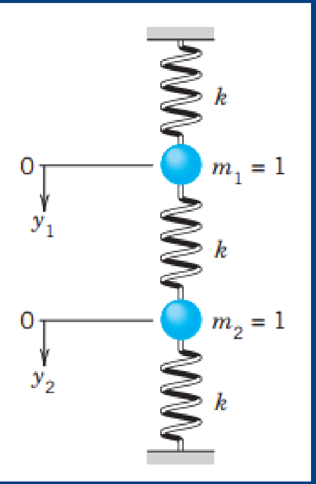
– that is,

$$i_1(t) = -\frac{125}{3}(1 - e^{\frac{1}{4}})e^{-\frac{t}{2}} - \frac{625}{21}(1 - e^{\frac{7}{4}})e^{-\frac{7t}{2}}$$
$$i_2(t) = -\frac{250}{3}(1 - e^{\frac{1}{4}})e^{-\frac{t}{2}} + \frac{250}{21}(1 - e^{\frac{7}{4}})e^{-\frac{7t}{2}}$$
for $t > \frac{1}{2}$

Systems of ODEs : example 3

- Model of two masses on springs
  - y_1, y_2 : displacements from static equilibrium positions
  - Modeling

$$\begin{aligned} y_1'' &= -ky_1 + k(y_2 - y_1) \\ y_2'' &= -k(y_2 - y_1) - ky_2 \end{aligned} \quad \text{with} \quad \begin{aligned} y_1(0) = y_2(0) = 1 \\ y_1'(0) = \sqrt{3k}, y_2'(0) = -\sqrt{3k} \end{aligned}$$



  - By taking Laplace transform, ($Y = \mathcal{L}(y)$)

$$\begin{aligned} s^2Y_1 - s - \sqrt{3k} &= -kY_1 + k(Y_2 - Y_1) \\ s^2Y_2 - s + \sqrt{3k} &= -k(Y_2 - Y_1) - kY_2 \end{aligned}$$

$$\Rightarrow \quad Y_1 = \frac{s}{s^2+k} + \frac{\sqrt{3k}}{s^2+3k}, \quad Y_2 = \frac{s}{s^2+k} - \frac{\sqrt{3k}}{s^2+3k}$$

Systems of ODEs : example 3

– By taking the inverse transform,

$$y_1(t) = \mathcal{L}^{-1}(Y_1) = cos\sqrt{k}t + sin\sqrt{3k}t$$
$$y_2(t) = \mathcal{L}^{-1}(Y_2) = cos\sqrt{k}t - sin\sqrt{3k}t$$

– The motion of each mass is harmonic, being the superposition of a ‘slow’ oscillation and a ‘rapid’ oscillation.