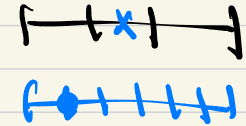
- What should we do for fg?

① $f_j \xrightarrow{FT} \hat{f}_k$, $g_j \xrightarrow{FT} \hat{g}_k$ $\Big\} \sum_m \hat{f}_k \hat{g}_{m-k} \xrightarrow{IFT} H_j = f_j g_j$

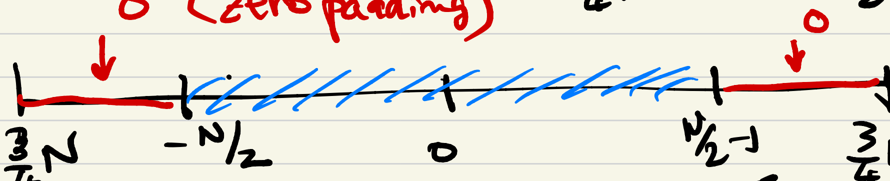
$O(N\log_2 N)$ $\qquad O(N^2)$ ← most expensive $\qquad O(N\log_2 N)$



② $f_j \xrightarrow{FT} \hat{f}_k$, $g_j \xrightarrow{FT} \hat{g}_k$ $\Big\}$ add zeros to $\hat{f}_k$ and $\hat{g}_k$ on $\frac{N}{2}-1 \le k \le \frac{3}{4}N-1$, $-\frac{3}{4}N \le k \le -\frac{N}{2}$

$O(N\log_2 N)$

0 (zero padding)



$\underbrace{\qquad\qquad\qquad}_{3/2\,N}$

$\downarrow$ IFT $O(\frac{3}{2}N\log_2 \frac{3}{2}N)$

f_j and g_j on $\frac{3}{2}N$ grid pts

spectral interpolation

$H_j = f_j g_j$ on $\frac{3}{2}N$ ← $O(\frac{3}{2}N)$

$\xrightarrow{FT}$ $\hat{H}_m$ $O(\frac{3}{2}N\log_2\frac{3}{2}N)$ $\quad \frac{3}{2}N$

remove values on $\frac{N}{2}-1 \le k \le \frac{3}{4}N-1$ & $-\frac{3}{4}N \le k \le -\frac{N}{2}$ $\quad N$

$\downarrow$ IFT $O(N\log_2 N)$

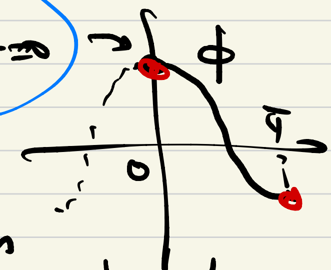
H_j on N grid pts.

② is cheaper than ① and has no aliasing error.

→ called "aliasing control" ($N_H = \frac{3}{2}N_f$ or $\frac{3}{2}N_g$)

6.1.6 Discrete sine and cosine transforms

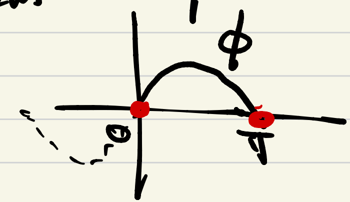
$\frac{\partial \phi}{\partial x} = 0$



f is not periodic → cannot use FT

f is an even ft. : $f(x) = f(-x) \Rightarrow$ cosine transform

f is an odd ft : $f(x) = -f(-x) \Rightarrow$ sine transform



f_j : $N+1$ pts on $0 \le x \le \pi$, $x_j = hj$, $h = \frac{\pi}{N}$

* Cosine transform

$$\begin{cases} f_j = \sum_{k=0}^{N} a_k \cos kx_j , & j = 0, 1, 2, \cdots, N \\ a_k = \frac{2}{c_k N} \sum_{j=0}^{N} \frac{1}{c_j} f_j \cos kx_j , & k = 0, 1, 2, \cdots, N \end{cases}$$

where

$$c_j = \begin{cases} 2 & \text{if } j = 0, N \\ 1 & \text{otherwise} \end{cases}$$

orthogonality $\sum_{j=0}^{N} \frac{1}{c_j} \cos kx_j \cos k'x_j = \begin{cases} 0 & \text{for } k \ne k' \\ \frac{1}{2} c_k N & \text{for } k = k' \end{cases}$

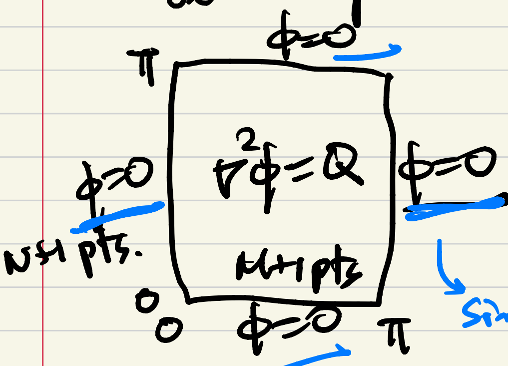
* sine transform

$$\begin{cases} f_j = \sum_{k=0}^{N} b_k \sin k x_j, & j = 0, 1, 2, \cdots, N \\ b_k = \frac{2}{N} \sum_{j=0}^{N} f_j \sin k x_j, & k = 0, 1, 2, \cdots, N \end{cases}$$

6.2 Application of discrete Fourier series

6.2.1 Direct sol. of finite differenced elliptic eqs.

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = Q(x, y) \quad \text{with } \phi = 0 \text{ on boundaries}$$



$\Delta x = \frac{\pi}{M}$ CD2

$\Delta y = \frac{\pi}{N}$

sine transf.

$$\left[\frac{\phi_{i+1,j} - 2\phi_{i,j} + \phi_{i-1,j}}{\Delta x^2} + \frac{\phi_{i,j+1} - 2\phi_{i,j} + \phi_{i,j-1}}{\Delta y^2} = Q_{i,j} \right] \tag{*}$$

$i = 1, 2, \cdots, M-1$

$j = 1, 2, \cdots, N-1$.

$\Rightarrow$ System of linear algebraic eqs. $\rightarrow$ expensive to solve.

Use Fourier sine series in x.

Assume $\phi_{ij} = \sum_{k=1}^{M-1} \hat{\phi}_{kj} \sin kx_i \qquad x_i = h_x i = \frac{\pi}{M} i$

$$Q_{ij} = \sum_{k=1}^{M-1} \hat{Q}_{kj} \sin kx_i$$

$\sin kx_i \left(2\cos\frac{\pi k}{M} - 2\right)$

$$\circledast : \sum_{k=1}^{M-1} \hat{\phi}_{kj} \boxed{(\sin kx_{i+1} - 2\sin kx_i + \sin kx_{i-1})} / \Delta x^2$$

$$+ \sum_{k=1}^{M-1} (\hat{\phi}_{k,j+1} - 2\hat{\phi}_{kj} + \hat{\phi}_{k,j-1}) \sin kx_i / \Delta y^2 = \sum_{k=1}^{M-1} \hat{Q}_{kj} \sin kx_i$$

Equating the coeff of $\sin kx_i$ gives

$$\boxed{\hat{\phi}_{k,j+1} + \left[\frac{\Delta y^2}{\Delta x^2}\left(2\cos\frac{\pi k}{M} - 2\right) - 2\right] \hat{\phi}_{k,j} + \hat{\phi}_{k,j-1} = \Delta y^2 \hat{Q}_{k,j}}$$

for $k = 1, 2, \cdots, M-1$.

TDMA

For each k, tri-diagonal sys. of eqs $\rightarrow$ easy to solve.

sol. procedure

$$Q \xrightarrow[\text{in } x]{FT} \hat{Q} \xrightarrow[\text{for each } k]{\text{solve TDMA}} \hat{\phi} \xrightarrow[\text{in } x]{IFT} \phi$$

$N\,O(M\log_2 M)$ $\quad$ $M\,O(N)$ $\quad$ $N\,O(M\log_2 M) \Rightarrow O(MN\log_2 M)$ operations

direct and low cost method.

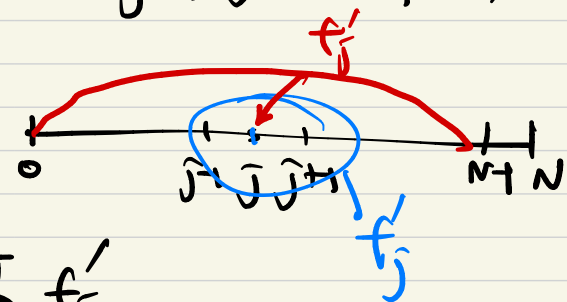
constraints
- uniform grids in one direction
- coeff. of PDE should not be a ft. of transform direction $\frac{\partial}{\partial x}\left(\mu(x)\frac{\partial\phi}{\partial x}\right)$ X

$\frac{\partial\phi}{\partial x} = 0$ on boundary $\Rightarrow$ use cosine transform.

6.2.2 Differentiation of periodic ft. using Fourier spectral method.

$f(x)$: periodic ft.

N equally spaced grid pts. $x_j = \Delta x \cdot j$. $j = 0, 1, 2, \cdots, N-1$



$$f_j = \sum_{k=-N/2}^{N/2-1} \hat{f}_k e^{ikx_j}$$

$$\frac{\partial f}{\partial x_j} = \sum \hat{f}_k \, ik \, e^{ikx_j}$$

$$f_j \xrightarrow{FT} \hat{f}_k \longrightarrow ik\hat{f}_k \xrightarrow{IFT} f_j'$$

$O(N\log_2 N)$ $\qquad$ $O(N\log_2 N)$

$$k = -\frac{N}{2}, -\frac{N}{2}+1, \cdots, 0, \cdots, \frac{N}{2}-1$$

oddball

$|k\hat{f}_k| \gg \varepsilon \Rightarrow ik\hat{f}_k\big|_{k=-N/2} = 0$ before IFT.

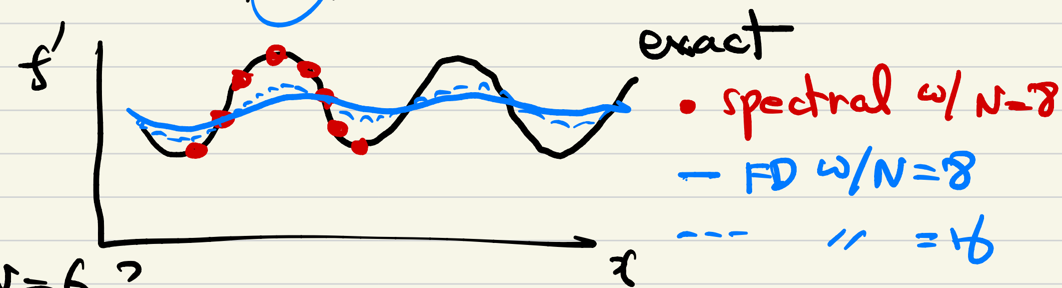
$\Rightarrow$ ensures that the derivative remains real in physical space.

this spectral derivative provides exact derivative of the harmonic ft. $f(x) = e^{ikx}$ at the grid pts. if $|k| \le \frac{N}{2} - 1$.

Spectral derivative is more accurate than any finite difference scheme for periodic ft, but cost is higher due to FFT

- $f = \cos 3x$ (k=3) $k = 0, \pm 1, \pm 2, \pm 3, -4$: $N = 8$

  $f' = -3 \sin 3x$

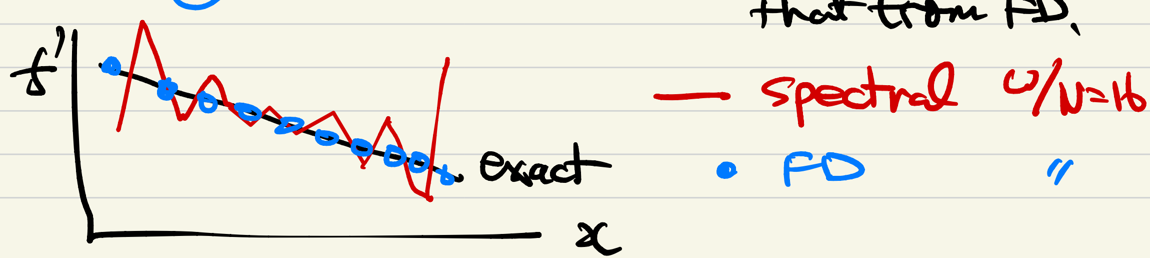


How about $N=6$?

$k = 0, \pm 1, \pm 2, -3 \Rightarrow$ spectral sol. is worse than that from FD.

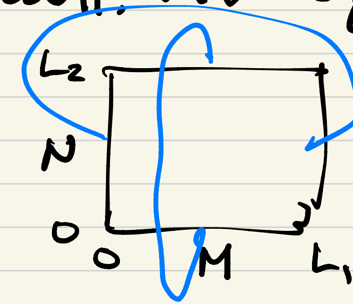
- $f = 2\pi x - x^2$

  $f' = 2\pi - 2x$



6.2.3 Numerical sol. of linear const. coeff. diff'l eq. w/ periodic b.c.'s.

$$\frac{\partial^2 p}{\partial x^2} + \frac{\partial^2 p}{\partial y^2} = Q(x,y) \quad — (*)$$



$k_1 = \frac{2\pi}{L_1} n_1$

$k_2 = \frac{2\pi}{L_2} n_2$

$$p = \sum_{k_1} \sum_{k_2} \hat{p}(k_1,k_2)\, e^{ik_1 x} e^{ik_2 y}$$

$Q = $ " $\hat{Q}$ " " "

$(*)$: $-k_1^2 \hat{p}_{k_1,k_2} - k_2^2 \hat{p}_{k_1,k_2} = \hat{Q}_{k_1,k_2} \Rightarrow \boxed{\hat{p}_{k_1,k_2} = \frac{-\hat{Q}_{k_1,k_2}}{k_1^2 + k_2^2}}$

for $k_1 = k_2 \neq 0$

$k_1 = k_2 = 0$?

$$\hat{p}_{k_1,k_2} = \frac{1}{M}\frac{1}{N}\sum_{l=0}^{M-1}\sum_{j=0}^{N-1} p_{l,j}\, e^{-ik_1 x_l} e^{-ik_2 y_j}$$

$\rightarrow \hat{p}_{0,0} = \frac{1}{M}\frac{1}{N}\sum_{l}\sum_{j} p_{l,j}$: average of p over the domain

p is sol. $\rightsquigarrow$ $p+c$ is also sol. $\Rightarrow$ set $\hat{p}_{0,0} = c$ (e.g. $c=0$)

$$Q_{l,j} \xrightarrow{FT} \hat{Q}_{k_1,k_2} \longrightarrow \frac{-\hat{Q}_{k_1,k_2}}{k_1^2+k_2^2} = \hat{P}_{k_1,k_2} \xrightarrow{IFT} P_{l,j}$$

direct sol.

- $\dfrac{\partial u}{\partial t} + \dfrac{\partial u}{\partial x} = \nu \dfrac{\partial^2 u}{\partial x^2} + f$

$u_j = \sum \hat{u}_k e^{ikx_j}$

u : periodic in x → FT

$$\frac{d\hat{u}_k}{dt} + ik\hat{u}_k = \nu(-k^2)\hat{u}_k + \hat{f}_k \qquad k = -\frac{N}{2}, \cdots, 0, \cdots, \frac{N}{2}-1.$$

$$\rightarrow \frac{d\hat{u}_k}{dt} = \underbrace{(-\nu k^2 - ik)}_{\lambda\,:\,\text{complex number}}\hat{u}_k + \hat{f}_k$$

apply a numerical method for time integration for each k,
to get $\hat{u}_k$ for $k = -\frac{N}{2}, \cdots, \frac{N}{2}-1$.

do IFT of $\hat{u}_k$ to obtain u.

FFT