· DFT of a product of two fts.

Let $H(x) = f(x)\, g(x)$

$$\hat{H}_m = \widehat{(fg)}_m = \frac{1}{N}\sum_{j=0}^{N-1} f_j\, g_j\, e^{-imx_j}, \quad m = -\frac{N}{2}, \cdots, 0, \cdots, \frac{N}{2}-1$$

$$= \frac{1}{N}\sum_{j=0}^{N-1}\left(\sum_k \hat{f}_k e^{ikx_j}\right)\left(\sum_{k'} \hat{g}_{k'} e^{ik'x_j}\right) e^{-imx_j}$$

$$= \frac{1}{N}\sum_{j=0}^{N-1}\sum_k\sum_{k'} \hat{f}_k\, \hat{g}_{k'}\, e^{i(k+k'-m)x_j}$$

(orthogonality

$$\frac{1}{N}\sum_{j=0}^{N-1} e^{i(k+k'-m)x_j} = \begin{cases} 1 & \text{if } k+k'-m = \pm\alpha N \\ & (\alpha = 0, 1, 2, \cdots) \\ 0 & \text{otherwise} \end{cases}$$
)

Sum over j is non-zero only if $k+k'=m$ or $m \pm N$

because $-N/2 \le k, k', m \le \frac{N}{2}-1$.

$k' = m - k$

$$\Rightarrow \boxed{\hat{H}_m = \sum_{k=-\frac{N}{2}}^{\frac{N}{2}-1} \hat{f}_k \, \hat{g}_{m-k}}$$

convolution sum of $\hat{f}$ and $\hat{g}$

↳ $O(N^2)$ operation

expensive

the part of summation corresponding to $k+k' = m \pm N$ is known as the aliasing error and should be discarded because Fourier exponentials corresponding to these wavenumbers cannot be resolved on the grid size N.

If we simply multiply f and g at each grid point, the resulting discrete FT. will be contaminated by the aliasing error and not be equal to the inverse FT of $\hat{H}_m$.

$$\left.\begin{matrix} f_j \\ g_j \end{matrix}\right\rangle f_j g_j \xrightarrow{FT} \widehat{(f_j g_j)}$$

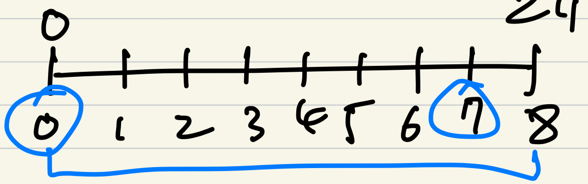
?

ex) $f(x) = \sin 2x$, $g(x) = \sin 3x$ $\quad 0 \le x \le 2\pi$.

$k=2$ $\qquad$ $k=3$

$x_j = \frac{2\pi}{N} j$, $j = 0, 1, \cdots, N-1$ $\qquad \hat{f}_k = \frac{1}{N} \sum_{j=0}^{N-1} f_j e^{-ikx_j}$

For $N = 8$



$\hookrightarrow k = 0, \pm 1, \pm 2, \pm 3, -4$

$$\hat{f}_k = \begin{cases} \pm i \frac{1}{2} & \text{for } k = \pm 2 \\ 0 & \text{otherwise} \end{cases}$$

$0, \pm 1, \pm 2, \pm 3, -4$

$N = 6$

$$\hat{g}_k = \begin{cases} \pm i \frac{1}{2} & \text{for } k = \pm 3 \\ 0 & \text{otherwise} \end{cases}$$

$0, \pm 1, \pm 2, \pm 3, -4$

$N = 8$

$$H(x) = f(x) g(x) = \sin 2x \sin 3x = \frac{1}{2}(\cos x - \cos 5x)$$

$\hookrightarrow k = \pm 1$ $\qquad \hookrightarrow k = \pm 5$

$$\hat{H}_k = \begin{cases} \frac{1}{4} & \text{for } k = \pm 1 \\ -\frac{1}{4} & \text{for } k = \pm 5 \\ 0 & \text{otherwise} \end{cases}$$

exact sol.

$\rightarrow$ We need $k = 0, \pm 1, \pm 2, \pm 3, \pm 4, \pm 5, -6$

we need $\boxed{N = 12}$ to get exact sol.

$N_H \geq \frac{3}{2} N_{f \text{ or } g}$ $\left(\frac{3}{2} \times 8 = 12 \right)$

$\boxed{\text{With } N = 16}$ $(k = 0, \pm 1, \pm 2, \pm 3, \pm 4, \pm 5, \pm 6, \pm 7, -8)$

$$\hat{H}_m = \sum_{k=-8}^{7} \hat{f}_k \hat{g}_{m-k}$$

$$\hat{H}_5 = \sum_{k=-8}^{7} \hat{f}_k \hat{g}_{5-k} = \hat{f}_2 \hat{g}_3 = -\frac{1}{4}$$

$$\hat{H}_1 = \sum \hat{f}_k \hat{g}_{1-k} = \hat{f}_{-2} \hat{g}_3 = \frac{1}{4}$$

$\vdots$

$$\Rightarrow \hat{H}_m = \begin{cases} \frac{1}{4} & \text{for } m = \pm 1 \\ -\frac{1}{4} & \text{for } m = \pm 5 \\ 0 & \text{otherwise} \end{cases}$$

accurate

$\boxed{\text{with } N = 8}$ $(k = 0, \pm 1, \pm 2, \pm 3, -4)$

$$\hat{H}_m = \sum_{k=-4}^{3} \hat{f}_k \hat{g}_{m-k}$$

$$\hat{H}_1 = \sum_{k=-4}^{3} \hat{f}_k \hat{g}_{1-k} = \hat{f}_{-2} \hat{g}_3 = \frac{1}{4}$$

$\vdots$

$$\hat{H}_m = \begin{cases} \frac{1}{4} & \text{for } m = \pm 1 \\ 0 & \text{otherwise} \end{cases} \leftarrow$$

accurate for $m = \pm 1$.
but lose $\cos 5x$!

The result is ok within the number of grid pts.

Now, multiplying f and g at each grid pt. $(N=8)$ and FT. What happen?

$\cos 5x \in fg$

$$\hat{H}_m = \frac{1}{N}\sum_{j=0}^{N} f_j g_j e^{-imx_j}$$

↳ contains $-\frac{1}{2}\cos 5x_j \rightarrow -\frac{1}{2}e^{\pm i5x_j}$

orthogonality : $\pm 5 - m = \alpha N = 8\alpha$ $\quad N=8$

$\alpha = 0$: $m = \pm 5$ ✗ $(m = 0, \pm 1, \pm 2, \pm 3, -4)$

$\alpha = 1$: $m = \pm 5 - 8 = -3$ or $\cancel{-13}$

↳ $\hat{H}_{-3} = -\frac{1}{4}$

$\alpha = -1$: $m = \pm 5 + 8 = \cancel{+13}$ or 3

↳ $\hat{H}_3 = -\frac{1}{4}$

$$\Rightarrow \quad \hat{H}_m = \begin{cases} \frac{1}{4} & \text{for } m = \pm 1 \\ -\frac{1}{4} & \text{for } m = \pm 3 \\ 0 & \text{otherwise} \end{cases}$$

not good.

$0.5(\cos x - \cos 5x)$

$\Rightarrow 0.5(\cos x - \cos 3x)$

aliasing error

- what should we do for $f \times g$?

① $f_j \xrightarrow{FT} \hat{f}_k$, $g_j \xrightarrow{FT} \hat{g}_k$ $\Big\rangle$ $\sum_m \hat{f}_k \hat{g}_{m-k} \xrightarrow{IFT} H_j = f_j g_j$

$O(N \log_2 N)$ — $O(N^2)$ most expensive — $O(N \log_2 N)$

② $f_j \xrightarrow{FT} \hat{f}_k$, $g_j \xrightarrow{FT} \hat{g}_k$ $\Big\rangle$ add zeros to $\hat{f}_k$ and $\hat{g}_k$ on

N

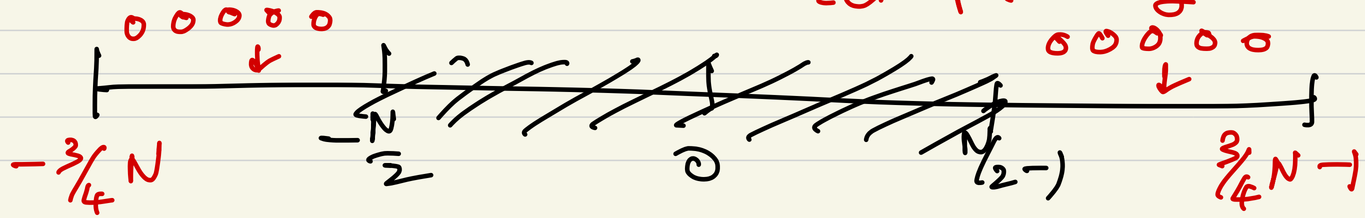
$O(N \log_2 N)$

$$\frac{N}{2} - 1 \leq k \leq \frac{3}{4}N - 1$$

$$-\frac{3}{4}N \leq k \leq -\frac{N}{2}$$

zero padding

$\frac{3}{2}N \Rightarrow$



$\downarrow$ IFT $O\left(\frac{3}{2}N \log_2\left(\frac{3}{2}N\right)\right)$

$O\left(\frac{3}{2}N \log_2\left(\frac{3}{2}N\right)\right)$ $\quad O\left(\frac{3}{2}N\right)$

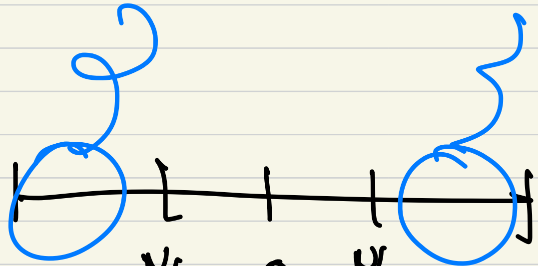
$$\hat{H}_m \xleftarrow{\text{FT}} H_j = f_j g_j \longleftarrow f_j \text{ and } g_j \text{ on } \tfrac{3}{2}N \text{ grid pts.}$$

$\frac{3}{2}N$ $\quad \frac{3}{2}N$ $\quad$ spectral interpolation

$\downarrow$

remove values on $\frac{N}{2}-1 \le k \le \frac{3}{4}N - 1$

$-\frac{3}{4}N \le k \le -\frac{N}{2}$



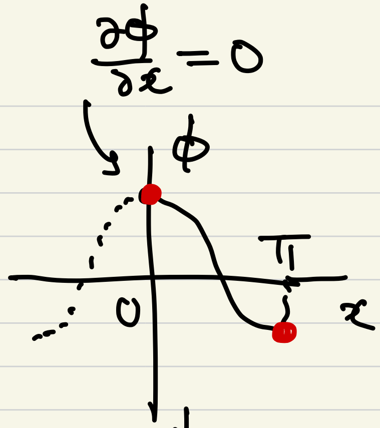
$\downarrow$ IFT $O(N \log_2 N)$

N $\quad H_j$

② is cheaper than ① and has no aliasing error

$\rightarrow$ called "aliasing control". $\left(N_H = \frac{3}{2}N_f \text{ or } \frac{3}{2}N_g\right)$

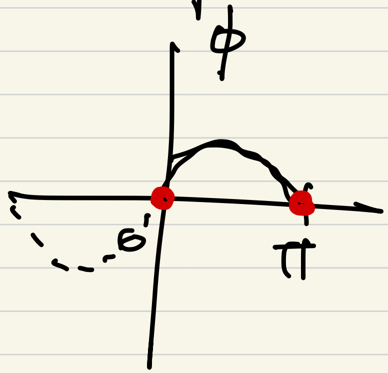
6.1.6 Discrete sine and cosine transforms



f is not periodic $\rightarrow$ cannot use FT.

f is an even ft. $\Rightarrow f(x)=f(-x)$: cosine transform

f is an odd ft $\Rightarrow f(x)=-f(-x)$: sine transform



f_j : N+1 pts on $0\le x\le\pi$, $x_j=hj$, $j=\frac{\pi}{N}$

* cosine transform

$$\begin{cases} f_j = \sum_{k=0}^{N} a_k \cos kx_j, & j=0,1,2,\cdots,N \\ a_k = \frac{2}{c_k N}\sum_{j=0}^{N}\frac{1}{c_j} f_j \cos kx_j, & k=0,1,2,\cdots,N \end{cases}$$

where $c_j = \begin{cases} 2 & \text{if } j=0,N \\ 1 & \text{otherwise} \end{cases}$

orthogonality $\sum_{j=0}^{N}\frac{1}{c_j}\cos kx_j \cos k'x_j = \begin{cases} 0 & \text{for } k\neq k' \\ \frac{1}{2}c_k N & \text{for } k=k' \end{cases}$

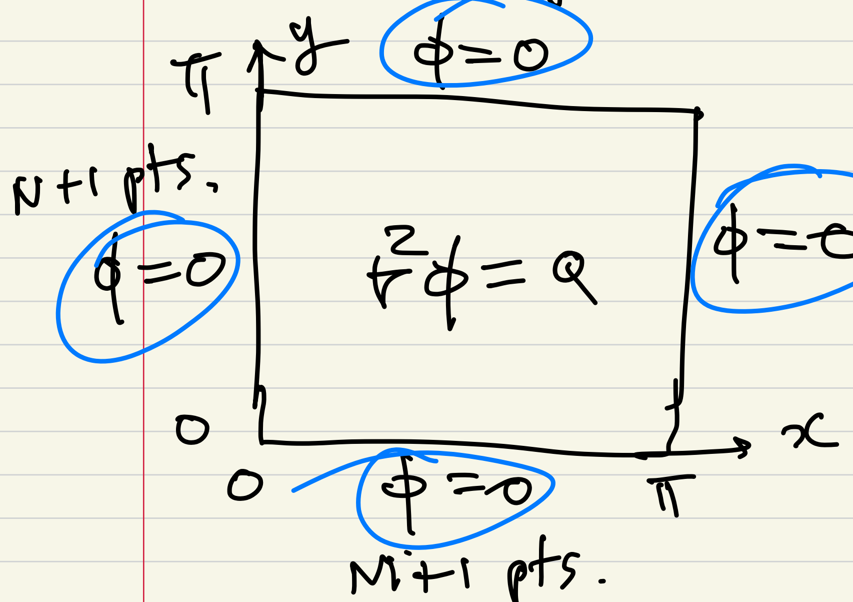
* sine transform

$$\begin{cases} f_j = \sum_{k=0}^{N} b_k \sin k x_j , & j=0,1,2,\cdots,N \\ b_k = \dfrac{2}{N} \sum_{j=0}^{N} f_j \sin k x_j , & k=0,1,2,\cdots,N \end{cases}$$

6.2 Application of discrete Fourier series

6.2.1 Direct sol. of finite differenced elliptic eqs.

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = Q(x,y)$$ with $\phi = 0$ on boundaries.



CD2

$$\Delta x = \frac{\pi}{M}, \quad \Delta y = \frac{\pi}{N}$$

$$\frac{\phi_{i+1,j} - 2\phi_{i,j} + \phi_{i-1,j}}{\Delta x^2} + \frac{\phi_{i,j+1} - 2\phi_{i,j} + \phi_{i,j-1}}{\Delta y^2} = Q_{i,j} \quad (*)$$

$i=1,2,\cdots,M-1, \quad j=1,2,\cdots,N-1$

$\Rightarrow$ system of linear algebraic eqs. $\rightarrow$ expensive to solve.

Use Fourier sine series in x.

Assume $\phi_{i,j} = \sum_{k=1}^{M-1} \hat{\phi}_{k,j} \sin kx_i$ $\quad x_i = h_x i = \frac{\pi}{M} i$

$Q_{i,j} = \quad " \quad \hat{Q}_{k,j} \sin kx_i$

$\sin kx_i \cdot (2\cos\frac{\pi k}{M} - 2)$

$$\rightarrow (*): \sum_{k=1}^{M-1} \hat{\phi}_{k,j} \left(\sin kx_{i+1} - 2\sin kx_i + \sin kx_{i-1} \right) / \Delta x^2$$

$$+ \sum_{k=1}^{M-1} \left(\hat{\phi}_{k,j+1} - 2\hat{\phi}_{k,j} + \hat{\phi}_{k,j-1} \right) \sin kx_i / \Delta y^2$$

$$= \sum_{k=1}^{M-1} \hat{Q}_{k,j} \sin kx_i$$

Equating the coeff of $\sin kx_i$ gives

$$\hat{\phi}_{k,j+1} + \left[\frac{\Delta y^2}{\Delta x^2} \left(2\cos\frac{\pi k}{M} - 2 \right) - 2 \right] \hat{\phi}_{k,j} + \hat{\phi}_{k,j-1} = \Delta y^2 \hat{Q}_{k,j}$$

$k = 1, 2, \cdots, M-1$

For each k, tri-diagonal sys. of eqs $\rightarrow$ easy to solve no iterative method required!

procedure

$$Q \xrightarrow[\text{in } x]{FT} \hat{Q} \xrightarrow[\text{for each } k]{\text{solve TDMA}} \hat{\phi} \xrightarrow[\text{in } x]{IFT} \phi$$

$N\,O(M\log_2 M)$ $\quad M\,O(N)$ $\quad N\,O(M\log_2 M)$

$\Rightarrow O(MN\log_2 M)$ operations

direct and low cost method!

constraints : uniform grids in one (x) direction

coeff. of PDE should not be a ft. of transform direction

$\frac{\partial}{\partial x}\left(\mu(x)\frac{\partial\phi}{\partial x}\right)$ X

$\frac{\partial\phi}{\partial x}=0$ Neumann b.c.

↳ use consine transform.

$f = \sin k_1 x$, $g = \sin k_2 x$ $\quad (k_1 \geq k_2)$

$$H = fg = -\frac{1}{2}\left[\cos(k_1+k_2)x - \cos(k_1-k_2)x\right]$$

# of grid pts to resolve f is $N_1 = 2(k_1+1)$

(e.g. $k_1 = 3$, $k = 0, \pm 1, \pm 2, \pm 3, -4 \Rightarrow N = 8$)

" " g is $N_2 = 2(k_2+1)$

$(N_1 \geq N_2)$

" " $H = fg$ is $\underline{N = 2(k_1+k_2+1)}$.

$$\hat{H}_m = \frac{1}{N}\sum_{j=0}^{N} \underbrace{f_j g_j}_{\pm(k_1+k_2)} e^{-imk_j}$$

$\pm(k_1+k_2) - m = \alpha N^*$

$\alpha = 0$ ok

$\alpha = \pm 1$ causes a prob.

# of grid pts. in use

If $N^* = N_1$, $m = \pm(k_1+k_2) - N_1$

$$= \pm\left(\frac{N_1}{2} - 1 + \frac{N_2}{2} - 1\right) - N_1$$

$$= \underbrace{-\frac{N_1}{2} + \frac{N_2}{2} - 2}_{\text{aliasing error}} \quad \text{or} \quad \underbrace{-\frac{3}{2}N_1 - \frac{N_2}{2} + 2}_{\text{outside } N}$$

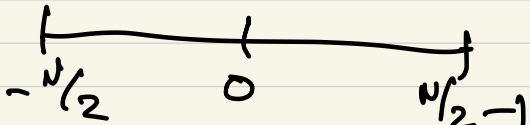
(if $N_1 = N_2$, $m = -2$)

If $N^* > N_1$, $\quad m = \pm(k_1 + k_2) - N^*$

$$= \pm\left(\frac{N_1}{2} - 1 + \frac{N_2}{2} - 1\right) - N^*$$

should be outside the wavenumbers resolved by N^* → no aliasing error.

$$\Rightarrow m = \frac{N_1}{2} - 1 + \frac{N_2}{2} - 1 - N^* < -\frac{N_1}{2}$$



$$\rightarrow N^* > \frac{N_1}{2} + \frac{N_2}{2} - 2 + \frac{N_1}{2} = N_1 + \frac{N_2}{2} - 2$$

for $N_1 = N_2$, $\quad N^* > \frac{3}{2}N_1 - 2 \quad \Rightarrow \quad N^* = \frac{3}{2}N_1$ ✗

even number

aliasing control

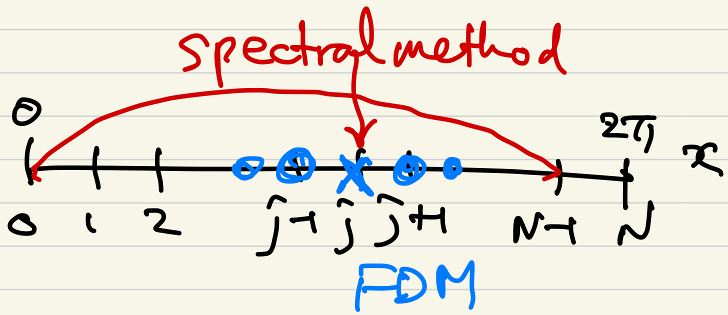
6.2.2 Differentiation of a periodic ft. using Fourier spectral method

$f(x)$: periodic ft.

N equally spaced grid pts. $x_j = \Delta x\, j$, $j = 0, 1, 2, \cdots, N-1$

$$f_j = \sum_{k=-N/2}^{N/2-1} \hat{f}_k e^{ikx_j}$$

$$\frac{\partial f}{\partial x_j} = \sum_{k=-N/2}^{N/2-1} \hat{f}_k\, ik\, e^{ikx_j}$$



To get spectral derivative of f,

$$f_j \xrightarrow{FT} \hat{f}_k \longrightarrow ik\hat{f}_k \xrightarrow{IFT} \frac{\partial f}{\partial x_j}$$

$O(N\log_2 N)$

$k = -\frac{N}{2}, -\frac{N}{2}+1, \cdots, 0, \cdots, \frac{N}{2}-1$

odd ball

$\hat{f}_{k=-\frac{N}{2}} \neq 0$
$= \varepsilon$

f' : $\underset{-N/2}{ik}\, \hat{f}_{k=-\frac{N}{2}} \gg \varepsilon$

$\Downarrow$

set $ik\hat{f}_{k=-\frac{N}{2}} \equiv 0$ before IFT

ensures that the derivative remains real in physical space.

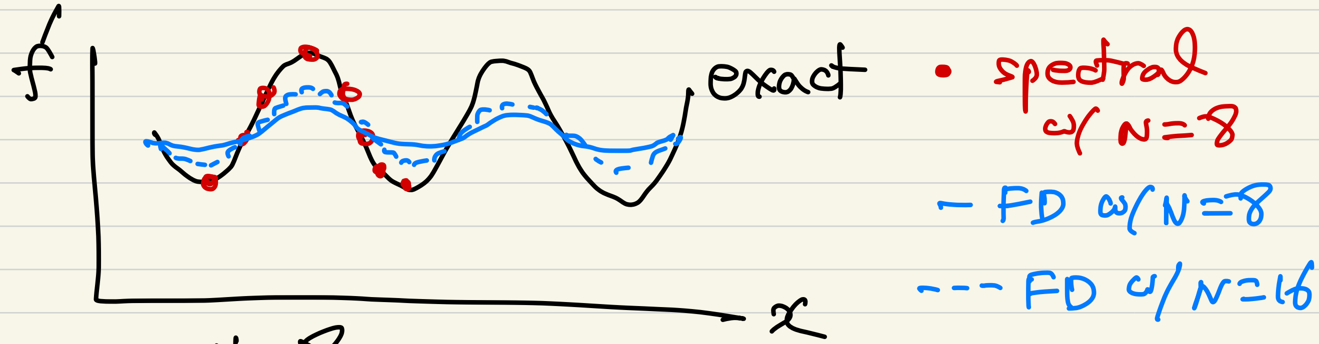
This spectral derivative provides exact derivative of the harmonic ft. $f(x)=e^{ikx}$ at the grid pts. if $|k| \le \frac{N}{2}-1$.

Spectral derivative is more accurate than any finite difference scheme for periodic ft., but cost is higher due to FFT.

- $f = \cos 3x$

  $f' = -3\sin 3x$

  $k = 0, \pm1, \pm2, \pm3, -4 \rightarrow N=8$



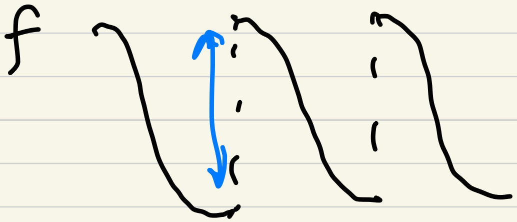
what if $N=6$?

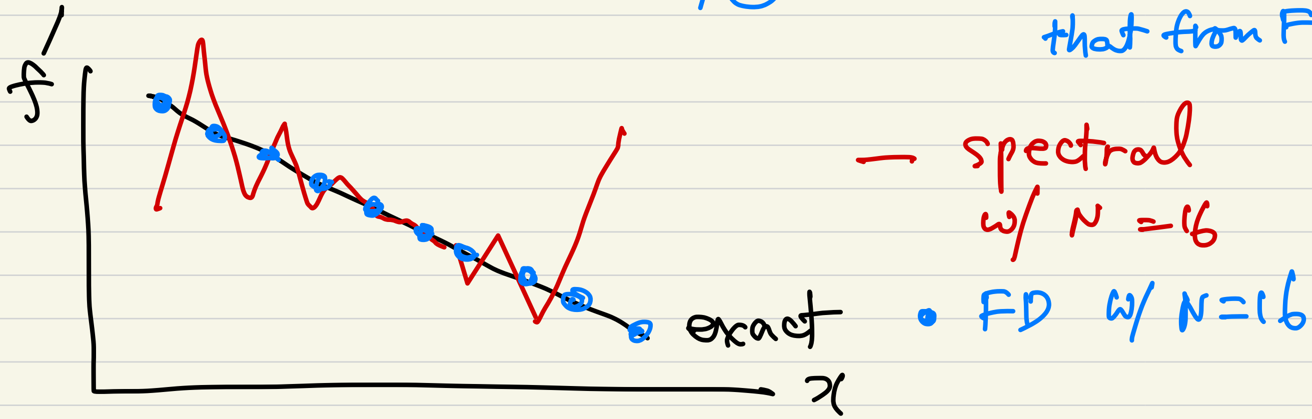
$k = 0, \pm1, \pm2, -3$

spectral sol. is worse than that from FD.

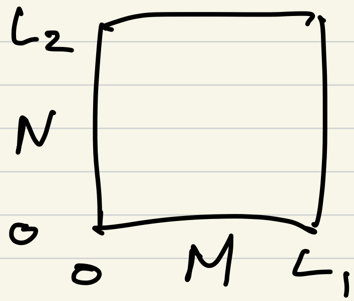
- $f = 2\pi x - x^2$

  $f' = 2\pi - 2x$





6.2.3 Numerical sol. of linear constant coefficient diff'l eq. w/ periodic b.c.'s.



$$k_1 = \frac{2\pi}{L_1} n_1$$

$$k_2 = \frac{2\pi}{L_2} n_2$$

$$\frac{\partial^2 p}{\partial x^2} + \frac{\partial^2 p}{\partial y^2} = Q(x,y)$$

$$p = \sum_{k_1} \sum_{k_2} \hat{p}(k_1,k_2)\, e^{ik_1 x} e^{ik_2 y}$$

$$Q = \sum_{k_1} \sum_{k_2} \hat{Q}(k_1,k_2)\, e^{ik_1 x} e^{ik_2 y}$$

$$-k_1^2 \hat{P}_{k_1,k_2} - k_2^2 \hat{P}_{k_1,k_2} = \hat{Q}_{k_1,k_2} \Rightarrow \boxed{\hat{P}_{k_1,k_2} = \frac{-\hat{Q}_{k_1,k_2}}{k_1^2+k_2^2}}$$

for $k_1 = k_2 \neq 0$

$k_1 = k_2 = 0$?

$$\hat{P}_{k_1,k_2} = \frac{1}{M}\frac{1}{N}\sum_{l=0}^{M-1}\sum_{j=0}^{N-1} P_{l,j}\, e^{-ik_1 x_l} e^{-ik_2 y_j}$$

$\rightarrow \hat{P}_{0,0} = \frac{1}{M}\frac{1}{N}\sum\sum P_{l,j}$: average of P over the domain.

Sol. of Poisson eq. w/ periodic b.c.'s is indeterminant to within an arbitrary constant.

Thus, set $\hat{P}_{0,0} = c$ (e.g. $c=0$)

$$Q_{l,j} \xrightarrow{FT} \hat{Q}_{k_1,k_2} \longrightarrow \frac{-\hat{Q}_{k_1,k_2}}{k_1^2+k_2^2} = \hat{P}_{k_1,k_2} \xrightarrow{IFT} P_{l,j}$$

direct sol.

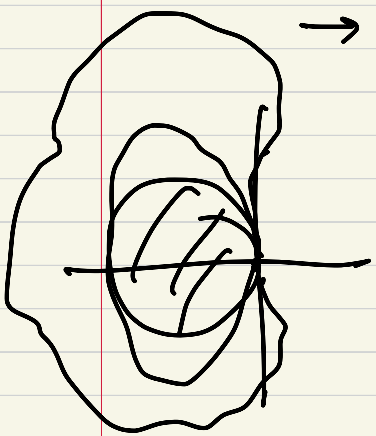
- $\dfrac{\partial u}{\partial t} + \dfrac{\partial u}{\partial x} = \nu \dfrac{\partial^2 u}{\partial x^2} + f$

$u_j = \sum \hat{u}_k e^{ikx_j}$

u : periodic in x $\rightarrow$ FT

$$\frac{d\hat{u}_k}{dt} + ik\hat{u}_k = \nu(-k^2)\hat{u}_k + \hat{f}_k \qquad k = -\frac{N}{2}, \cdots, 0, \cdots, \frac{N}{2}-1.$$

$$\rightarrow \frac{d\hat{u}_k}{dt} = \underbrace{(-ik-\nu k^2)}_{\lambda \text{ : complex number}} \hat{u}_k + \hat{f}_k$$



apply a numerical method for time integration for each k. to get $\hat{u}_k$

do IFT of $\hat{u}_k$ to obtain u.

6.4 Discrete Chebyshev transform and applications

non-periodic fct. or non-uniform mesh?

- Chebyshev transform

$a \le x \le b \;\to\; -1 \le \xi \le 1$

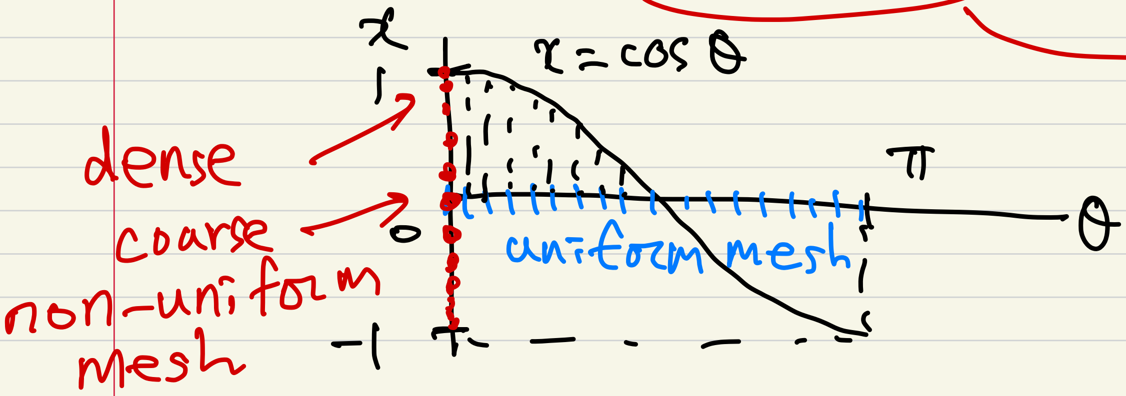
$u(x)$ defined in $-1 \le x \le 1$

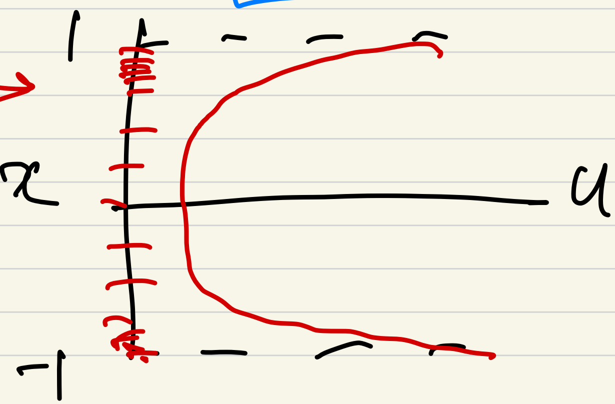
$$u(x) = \sum_{n=0}^{N} a_n T_n(x) \qquad T_n(x): \text{ Chebyshev polynomials.}$$

$$\left(\frac{d}{dx}\left[\sqrt{1-x^2}\,\frac{dT_n}{dx}\right] + \frac{\lambda_n}{\sqrt{1-x^2}} T_n = 0, \quad \lambda_n = n^2 \right.$$

$$\left. T_0 = 1,\; T_1 = x,\; T_2 = 2x^2-1,\; T_3 = 4x^3-3x,\; \cdots \right)$$

$$-1 \le x \le 1 \xrightarrow{x = \cos\theta} 0 \le \theta \le \pi \;\Rightarrow\; \boxed{T_n(\cos\theta) = \cos n\theta}$$





recursive relation $T_{n+1}(x) + T_{n-1}(x) = 2x T_n(x)$ $(n \geq 1)$

mesh (cosine mesh) $\boxed{x_j = \cos\frac{\pi j}{N}}$, $j = N, N-1, \cdots, 1, 0$

Discrete Chebyshev transform

$$\begin{cases} u_j = \sum_{n=0}^{N} a_n T_n(x_j) = \sum_{n=0}^{N} a_n \cos\frac{n\pi}{N} j, & j = 0, 1, 2, \cdots, N \\ a_n = \frac{2}{c_n N} \sum_{j=0}^{N} \frac{1}{c_j} u_j T_n(x_j) = \frac{2}{c_n N} \sum_{j=0}^{N} \frac{1}{c_j} u_j \cos\frac{n\pi}{N} j, & n = 0, 1, 2, \cdots, N \end{cases}$$

the Chebyshev coeffs. for any ft. u in $-1 \leq x \leq 1$ are exactly the coeff. of the cosine transform obtained using the values of u at the cosine mesh, i.e., $u_j = u(\cos \pi j / N)$.

6.4.1 Numerical differentiation using chebyshev. trasf.

$T_n(x) = \cos n\theta$, $x = \cos\theta$

$$\rightarrow \frac{dT_n}{dx} = \frac{d\cos n\theta}{d\theta}\frac{d\theta}{dx} = \frac{n \sin n\theta}{\sin\theta}$$

(trigonometric identity) $\sin(n+1)\theta - \sin(n-1)\theta = 2\sin\theta \cos n\theta$

$$\rightarrow \quad 2T_n(x) = \frac{1}{n+1} T_{n+1}' - \frac{1}{n-1} T_{n-1}' \quad (n > 1)$$

$$u(x) = \sum_{n=0}^{N} a_n T_n \longrightarrow u'(x) = \sum_{n=0}^{N} b_n T_n$$

$$\downarrow$$

$$u'(x) = \sum_{n=0}^{N} a_n T_n'$$

$$\boxed{b_n = \frac{2}{c_n} \sum_{\substack{p=n+1 \\ p+n \text{ odd}}}^{N} p a_p}$$

$$u \xrightarrow{CT} a_n \longrightarrow b_n \xrightarrow{ICT} u'$$

6.4.2 Quadrature using Chebyshev trans.

$$2T_n(x) = \frac{1}{n+1} T_{n+1}' - \frac{1}{n-1} T_{n-1}' \quad (n > 1)$$

Integrating both sides

$$\int T_n(x)\,dx = \begin{cases} T_1 + d_0 & (n=0) \\ \frac{1}{4}(T_0 + T_2) + d_1 & (n=1) \\ \frac{1}{2}\left[\frac{1}{n+1} T_{n+1} - \frac{1}{n-1} T_{n-1}\right] + d_n & \text{otherwise} \end{cases}$$

$$g(x) = \int_{-1}^{x} u(\xi)\,d\xi = \sum_{n=0}^{N+1} d_n T_n$$

$$\sum_{n=0}^{N} a_n \int T_n(x)\,dx$$

$$\begin{cases} d_n = \frac{1}{2n}\left(c_{n-1}a_{n-1} - a_{n+1}\right) & n = 1, 2, \cdots, N+1 \\ \text{w/ } a_{N+1} = a_{N+2} = 0 \\ d_0 = d_1 - d_2 + d_3 + \cdots + (-1)^{N+2} d_{N+1} \end{cases}$$

$$u \xrightarrow{CT} a_n \longrightarrow d_n \xrightarrow{ICT} \int u\,dx$$

6.5 Method of weighted residual (MWR)

$\mathcal{L}(u) = f(x,t)$ for $x \in \mathcal{D}$ w/ $B(u) = g(x,t)$ on $\partial\mathcal{D}$

$\tilde{u}$: approx. sol

$$\tilde{u} = \sum_{n=0}^{N} c_n(t)\,\phi_n(x)$$

basis ft. or test ft.

residual $R = \mathcal{L}(\tilde{u}) - f$

MWR aims to find the sol. $\tilde{u}$ which minimizes the residual R in the weighed integral sense

$$\int_{\mathcal{D}} \omega_i R\,dx = 0 \qquad i = 0, 1, \cdots, N$$

ω_i : weight ft $\omega_i(x)$

$$\rightarrow \int_D w_i [\mathcal{L}(\hat{u}) - f]\, dx = 0$$

$$\int_D w_i \left[\mathcal{L}\left(\sum_{n=0}^{N} c_n \phi_n\right) - f\right] dx = 0$$: weak form of original eq.

$\mathcal{L}(u) = f$: FDM

$w_i = 1$: FVM (finite volume method)

$w_i = \phi_i$: Galerkin method

6.6 FEM