Chapter 19

Fermi-Dirac Gases

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19.1 The Fermi Energy

- Fermions

  Fermi-Dirac statistics governs the behavior of indistinguishable particles of half-integer spin called fermions. Fermions obey the Pauli exclusion principle.

- Fermi-Dirac distribution

$$f_j = \frac{N_j}{g_j} = \frac{1}{e^{(\varepsilon_j - \mu)/kT} + 1}$$

$$\boldsymbol{f(\varepsilon) = \frac{N(\varepsilon)}{g(\varepsilon)} = \frac{1}{e^{(\varepsilon - \mu)/kT} + 1}} \quad \textbf{\textit{Fermi function}} \quad 0 \le f(\varepsilon) \le 1$$

19.1 The Fermi Energy

Fermi energy, $\mu(0)$: chemical potential where absolute temperature is 0

Denominator of Fermi-Dirac distribution

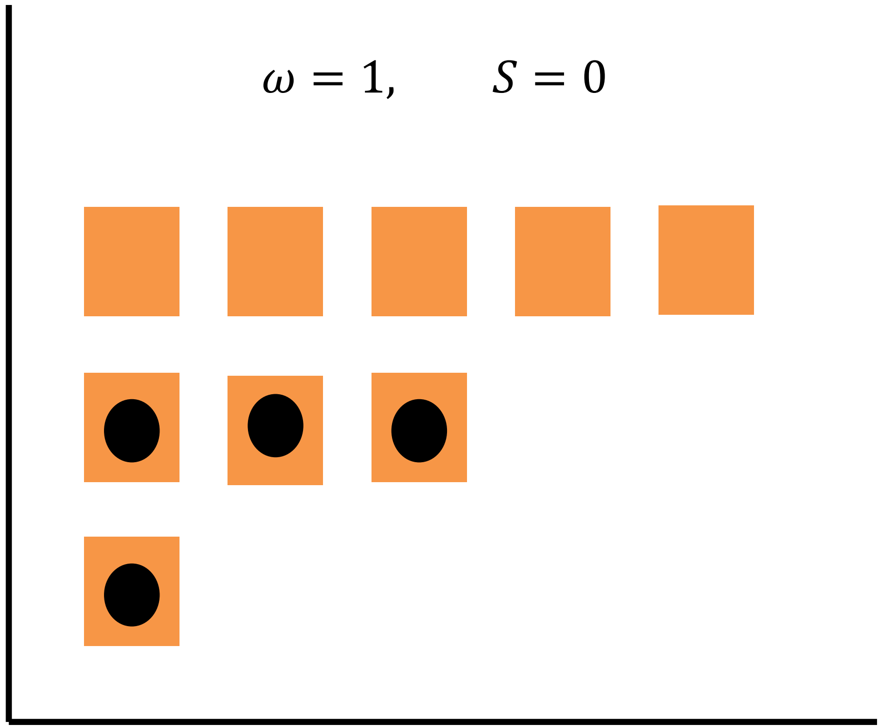
$$\{\varepsilon - \mu(0)\}/kT = \begin{cases} -\infty & if \;\; \varepsilon < \mu(0) \\ \infty & if \;\; \varepsilon > \mu(0) \end{cases}$$

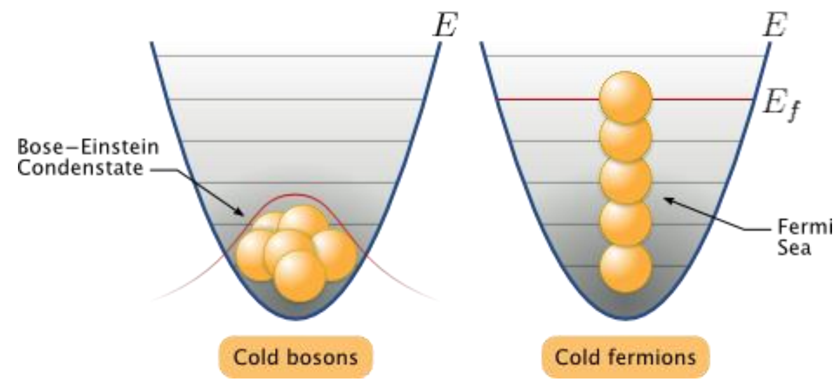
Correspondingly

$$f(\varepsilon) = \begin{cases} 1 & if \;\; \varepsilon < \mu(0) \\ 0 & if \;\; \varepsilon > \mu(0) \end{cases}$$

At $T = 0$, all states with energy $\boldsymbol{\varepsilon} < \boldsymbol{\mu}(\mathbf{0})$ are occupied, whereas all states $\boldsymbol{\varepsilon} > \boldsymbol{\mu}(\mathbf{0})$ are unoccupied.

19.1 The Fermi Energy





At absolute zero fermions will occupy from the lowest energy states available.

- For particles of spin 1/2, e.g. electrons, the spin factor γ_s is 2, so

$$g(\varepsilon)d\varepsilon = \gamma_s \frac{4\sqrt{2}\pi V}{h^3} m^{\frac{3}{2}} \varepsilon^{\frac{1}{2}} d\varepsilon \qquad (from\ Chap. 12)$$

$$= \frac{8\sqrt{2}\pi V}{h^3} m^{\frac{3}{2}} \varepsilon^{\frac{1}{2}} d\varepsilon$$

$$= 4\pi V \left(\frac{2m}{h^2}\right)^{3/2} \varepsilon^{1/2} d\varepsilon$$

- For conservation of particles, $\sum_{j=1} N_j = N$

$$\int_0^\infty N(\varepsilon)d\varepsilon = \int_0^\infty f(\varepsilon)g(\varepsilon)d\varepsilon = N$$

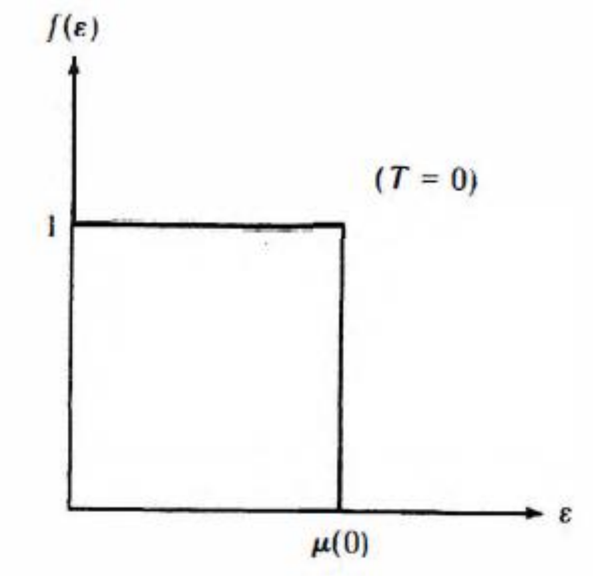


Fig. The Fermi function at T=0

- At $T = 0$, $f(\varepsilon) = 1$ until energy state reach fermi energy

$$N = \int_0^{\mu(0)} g(\varepsilon)d\varepsilon = 4\pi V\left(\frac{2m}{h^2}\right)^{\frac{3}{2}} \int_0^{\mu(0)} \varepsilon^{\frac{1}{2}} d\varepsilon = \frac{8\pi V}{3}\left(\frac{2m}{h^2}\right)^{\frac{3}{2}} \{\mu(0)\}^{\frac{3}{2}}$$

Hence, $\therefore \mu(0) = \frac{h^2}{2m}\left(\frac{3N}{8\pi V}\right)^{2/3}$

- Fermi Temperature: T_F such that $\mu(0) \equiv kT_F$

$$\boldsymbol{T_F = \frac{h^2}{2\pi mk}\left(\frac{N}{1.504V}\right)^{\frac{2}{3}}}$$

$$cf. Bose\ Temperature \quad T_B = \frac{h^2}{2\pi mk}\left(\frac{N}{2.612V}\right)^{\frac{2}{3}}$$

- The Calculation of $\mu(T)$

$$N = \int_0^\infty f(\varepsilon)g(\varepsilon)d\varepsilon = 4\pi V\left(\frac{2m}{h^2}\right)^{\frac{3}{2}}\int_0^\infty \frac{\varepsilon^{1/2}d\varepsilon}{e^{(\varepsilon-\mu)/kT}+1}$$

Then the integral

$$I = \int_0^\infty \frac{\varepsilon^{1/2}d\varepsilon}{e^{(\varepsilon-\mu)/kT}+1} = \int_0^{\mu(0)} \varepsilon^{\frac{1}{2}}d\varepsilon = \frac{2}{3}\{\mu(0)\}^{\frac{3}{2}}$$

Then, divide the inside of the integral into two parts

$$I = \int_0^\infty \varepsilon^{1/2} \frac{1}{e^{(\varepsilon-\mu)/kT}+1} d\varepsilon = \int_0^\infty \left(\frac{dF(\varepsilon)}{d\varepsilon}\right) f(\varepsilon) d\varepsilon \quad \text{where} \quad F(\varepsilon) = \frac{2}{3}\varepsilon^{3/2}, \quad f(\varepsilon) = \frac{1}{e^{(\varepsilon-\mu)/kT}+1}$$

$$I = f(\varepsilon)F(\varepsilon)\Big|_0^\infty - \int_0^\infty F(\varepsilon)\frac{df(\varepsilon)}{d\varepsilon} d\varepsilon = \frac{1}{kT}\int_0^\infty F(\varepsilon)\frac{e^{(\varepsilon-\mu)/kT}}{[e^{(\varepsilon-\mu)/kT}+1]^2} d\varepsilon$$

$\because f(\infty) = 0 \quad F(0) = 0$

Taylor series about μ

$$F(\varepsilon) = \sum_{n=0}^{\infty} \frac{F^{(n)}(\mu)}{n!} = F(\mu) + F'(\mu)(\varepsilon-\mu) + \frac{1}{2!}F''(\mu)(\varepsilon-\mu)^2 + \cdots$$

$$= \frac{2}{3}\mu^{\frac{3}{2}} + \mu^{\frac{1}{2}}(\varepsilon-\mu) + \frac{1}{4}\mu^{-\frac{1}{2}}(\varepsilon-\mu)^2 + \cdots$$

$$\therefore F(\varepsilon) \approx \frac{2}{3}\mu^{\frac{3}{2}} + \mu^{\frac{1}{2}}(\varepsilon-\mu) + \frac{1}{4}\mu^{-\frac{1}{2}}(\varepsilon-\mu)^2$$

19.2 The Calculation of $\mu(T)$

- Set $y = (\varepsilon - \mu)/kT$ $\qquad \varepsilon = 0 \rightarrow y = -\mu/kT \qquad dy = \frac{d\varepsilon}{kT}$

$$I = \frac{1}{kT}\int_0^\infty F(\varepsilon)\frac{e^{(\varepsilon-\mu)/kT}}{[e^{(\varepsilon-\mu)/kT}+1]^2}d\varepsilon = \int_{-\mu/kT}^\infty \frac{F(y)e^y dy}{(e^y+1)^2}$$

Substitute $F(y)$

$$I = \int_{-\frac{\mu}{kT}}^\infty \left\{\frac{2}{3}\mu^{\frac{3}{2}} + \mu^{\frac{1}{2}}(kT)y + \frac{(kT)^2}{4\mu^{1/2}}y^2\right\}\frac{e^y}{(e^y+1)^2}dy$$

$\rightarrow -\infty$ For covering the region of concern

$$= \frac{2}{3}\mu^{\frac{3}{2}} + 0 + \frac{\pi^2}{12}\frac{(kT)^2}{\mu^{1/2}} = \frac{2}{3}\{\mu(0)\}^{\frac{3}{2}}$$

$$\frac{2}{3}\mu^{\frac{3}{2}}\left[1 + \frac{\pi^2}{8}\left(\frac{kT}{\mu}\right)^2\right] = \frac{2}{3}\{\mu(0)\}^{\frac{3}{2}}$$

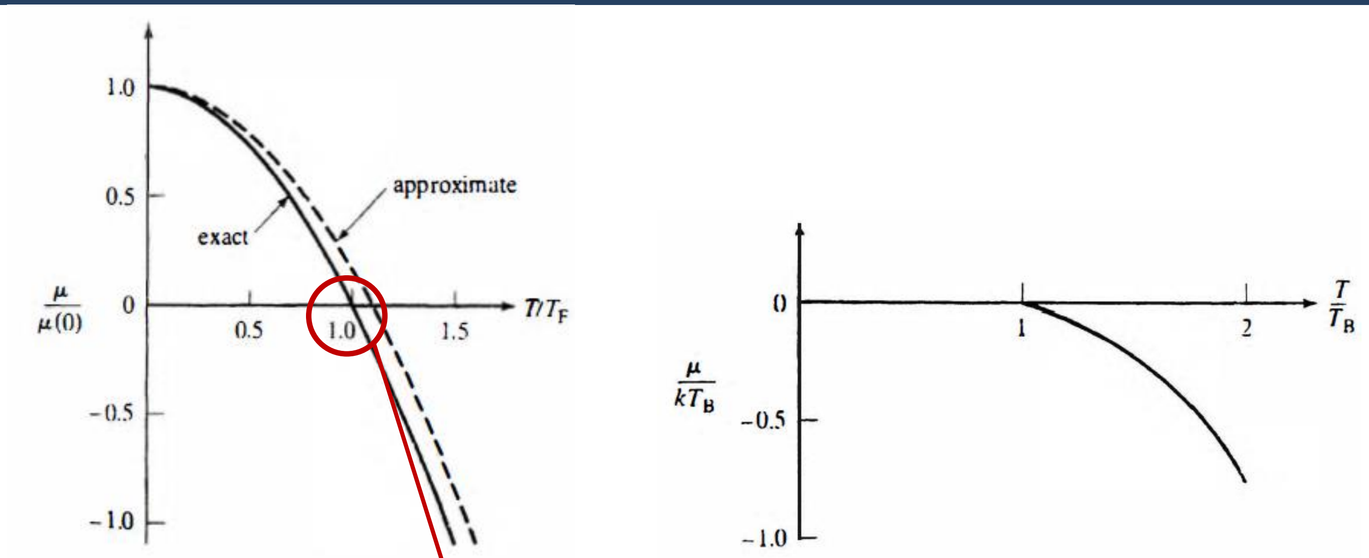
19.2 The Calculation of $\mu(T)$

$$\frac{2}{3}\mu^{\frac{3}{2}}\left[1+\frac{\pi^2}{8}\left(\frac{kT}{\mu}\right)^2\right]=\frac{2}{3}\{\mu(0)\}^{\frac{3}{2}}$$

$$\mu=\mu(0)\left[1+\frac{\pi^2}{8}\left(\frac{kT}{\mu}\right)^2\right]^{-2/3}\approx\mu(0)\left[1-\frac{\pi^2}{12}\left(\frac{kT}{\mu}\right)^2\right]$$

Replace μ in the corrected term by $\mu(0)=kT_F\ (approximation)$

$$\therefore\ \boldsymbol{\mu}=\boldsymbol{\mu}(\mathbf{0})\left[\mathbf{1}-\frac{\boldsymbol{\pi}^{\mathbf{2}}}{\mathbf{12}}\left(\frac{\boldsymbol{T}}{\boldsymbol{T}_{\boldsymbol{F}}}\right)^{\mathbf{2}}\right]\qquad \boldsymbol{for}\ \boldsymbol{T}\ll\boldsymbol{T}_{\boldsymbol{F}}$$



<Fermion> <Boson>

Fig. Exact and approximate calculations of μ / $\mu(0)$ versus T/T_F.

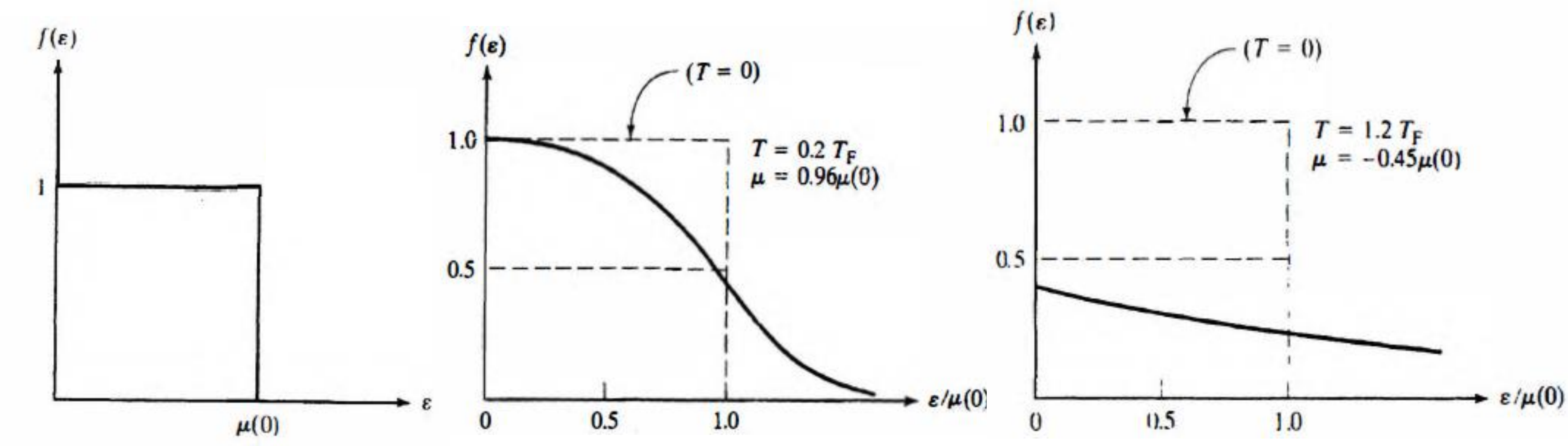
$$T < T_F \rightarrow \mu > 0 \;\&\; T > T_F \rightarrow \mu < 0$$

As the temperature increases above T_F, more of the fermions are in the excited states and the mean occupancy of the ground state falls below $1/2$. In this region,

$$f(0) = \frac{1}{e^{-\mu/kT} + 1} < \frac{1}{2} \qquad \frac{\mu}{kT} < 0$$

19.2 The Calculation of $\mu(T)$

Fermi function $f(\varepsilon) = \dfrac{1}{e^{(\varepsilon-\mu)/kT} + 1}$



$T = 0$ $0 < T < T_F$ $T_F < T$

Fig. Fermi function at the different temperature range

19.4 Properties of a Fermion Gas

The number of fermions in single particle energy range $\varepsilon < < d\varepsilon$

$$N(\varepsilon)d\varepsilon = f(\varepsilon)g(\varepsilon)d\varepsilon$$

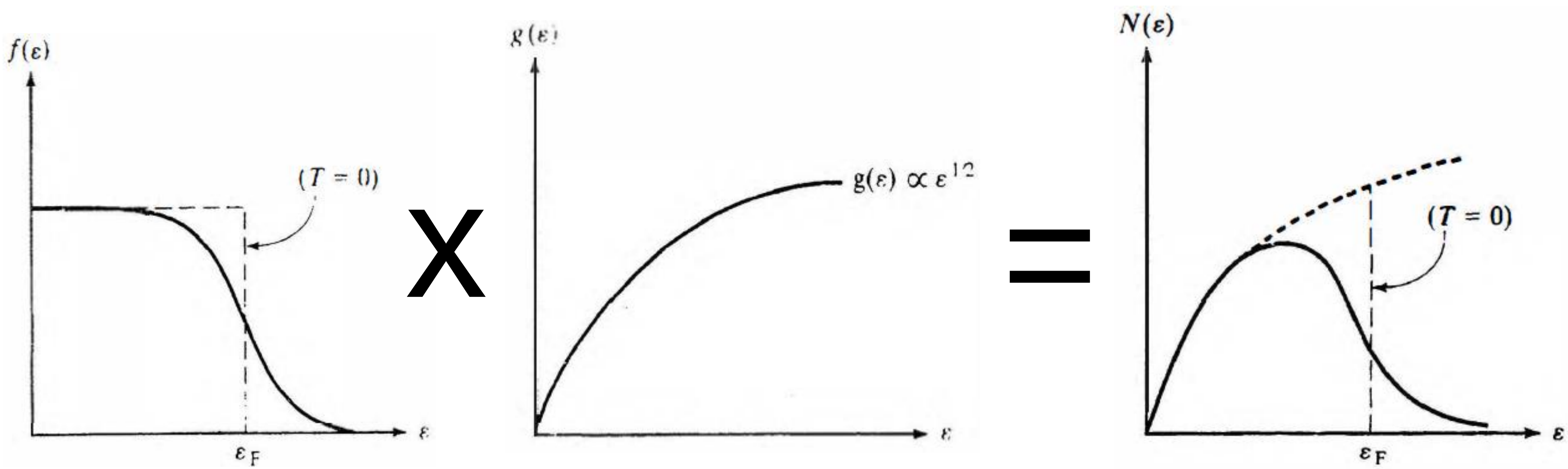


Fig. Variation of the Fermi function, degeneracy and fermions for $0 < T \ll T_F$

The number of fermions in single particle energy range $\varepsilon << d\varepsilon$

$$N(\varepsilon)d\varepsilon = f(\varepsilon)\mathrm{g}(\varepsilon)d\varepsilon$$

The internal energy of a gas of N fermions

$$U = \int_0^\infty \varepsilon N(\varepsilon)d\varepsilon = \int_0^\infty \varepsilon f(\varepsilon)g(\varepsilon)d\varepsilon = 4\pi V\left(\frac{2m}{h^2}\right)^{3/2}\int_0^\infty \frac{\varepsilon^{3/2}d\varepsilon}{e^{(\varepsilon-\mu)/kT}+1}$$

$$\approx \frac{3}{5}N\varepsilon_F\left[1+\frac{5\pi^2}{12}\left(\frac{T}{T_F}\right)^2-\frac{\pi^4}{16}\left(\frac{T}{T_F}\right)^4+\cdots\right]$$ By using algebra for approximation

The electronic heat capacity C_e

$$C_e = \frac{dU}{dT} = \frac{\pi^2}{2}Nk\left[\left(\frac{T}{T_F}\right)-\frac{3\pi^2}{10}\left(\frac{T}{T_F}\right)^3+\cdots\right] \qquad (\varepsilon_F = kT_F)$$