6.4 Discrete Chebyshev transform and applications

non-periodic ft or non-uniform mesh ???

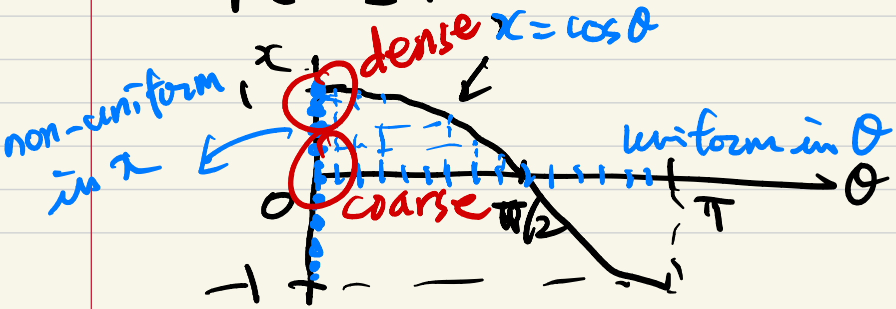
- Chebyshev transform

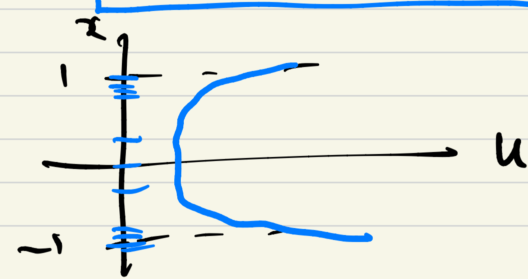
$u(x)$ defined in $-1 \le x \le 1$ $\qquad a \le x \le b \longrightarrow -1 \le \xi \le 1$

$$u(x) = \sum_{n=0}^{N} a_n T_n(x) \qquad T_n(x): \text{Chebyshev polynomials.}$$

$$\left(\begin{array}{l} \dfrac{d}{dx}\left[\sqrt{1-x^2}\,\dfrac{dT_n}{dx}\right] + \dfrac{\lambda_n}{\sqrt{1-x^2}} T_n = 0, \quad \lambda_n = n^2 \\ T_0 = 1,\ T_1 = x,\ T_2 = 2x^2 - 1,\ T_3 = 4x^3 - 3x, \cdots \end{array} \right)$$

$$-1 \le x \le 1 \longrightarrow 0 \le \theta \le \pi \Rightarrow \boxed{T_n(\cos\theta) = \cos n\theta}$$





recursive relation : $T_{n+1}(x) + T_{n-1}(x) = 2x T_n(x) \quad (n \geq 1)$

cosine mesh $\boxed{x_j = \cos \frac{\pi j}{N}}$, $j = N, N-1, \cdots, 1, 0$

$\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad \downarrow \quad\quad\quad\quad\quad \downarrow$

$\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad x=-1 \quad\quad\quad x=1$

Discrete chebyshev transform

$$u_j = \sum_{n=0}^{N} a_n T_n(x_j) = \boxed{\sum_{n=0}^{N} a_n \cos \frac{n\pi}{N} j} , \quad j = 0, 1, 2, \cdots, N$$

$$a_n = \frac{2}{c_n N} \sum_{j=0}^{N} \frac{1}{c_j} u_j T_n(x_j) \boxed{= \frac{2}{c_n N} \sum_{j=0}^{N} \frac{1}{c_j} u_j \cos \frac{n\pi}{N} j} , \quad n = 0, 1, 2, \cdots, N$$

The Chebyshev coeffs. for any ft. u in $-1 \leq x \leq 1$ are exactly the coeff. of the cosine transform obtained using the values of u at the cosine mesh, i.e., $u_j = u(\cos \pi j / N)$.

6.4.1 Numerical differentiation using Chebyshev transf.

$$T_n(x) = \cos n\theta, \quad x = \cos\theta$$

$$\rightarrow \frac{dT_n}{dx} = \frac{d\cos n\theta}{d\theta}\frac{d\theta}{dx} = \frac{n\sin n\theta}{\sin\theta}$$

$$\sin(n+1)\theta - \sin(n-1)\theta = 2\sin\theta\cos n\theta$$

$$\rightarrow 2T_n(x) = \frac{1}{n+1}T'_{n+1} - \frac{1}{n-1}T'_{n-1} \quad (n>1)$$

$$u(x) = \sum_{n=0}^{N} a_n T_n \longrightarrow u'(x) = \sum_{n=0}^{N} b_n T_n$$

coeff. of u'

d

$$u'(x) = \sum_{n=0}^{N} a_n T_n'$$

$$\boxed{b_n = \frac{2}{c_n}\sum_{\substack{p=n+1\\ p+n \text{ odd}}}^{N} p\, a_p}$$

$$u \xrightarrow{CT} a_n \longrightarrow b_n \xrightarrow{ICT} u'$$

6.4.2 Quadrature using Chebyshev transf.

$$2T_n(x) = \frac{1}{n+1}T_{n+1}' - \frac{1}{n-1}T_{n-1}' \quad (n>1)$$

Integrate both sides

$$\int T_n(x)\,dx = \begin{cases} T_1 + a_0 & (n=0) \\ \frac{1}{4}(T_0+T_2)+a_1 & (n=1) \\ \frac{1}{2}\left[\frac{1}{n+1}T_{n+1} - \frac{1}{n-1}T_{n-1}\right] + a_n & \text{otherwise} \end{cases}$$

$$g(x) = \int_{-1}^{x} u(\xi)\,d\xi = \sum_{n=0}^{N+1} d_n T_n$$ coeff. of $\int u$

$$= \sum_{n=0}^{N} a_n \int T_n(x)\,dx$$

$$\rightarrow \begin{cases} d_n = \frac{1}{2n}(c_{n-1}a_{n-1} - a_{n+1}) \\ \quad \text{with } a_{N+1} = a_{N+2} = 0 \\ \quad \text{for } n = 1,2,\dots,N+1 \\ d_0 = d_1 - d_2 + d_3 + \dots + (-1)^{N+2} d_{N+1} \end{cases}$$

$$u \xrightarrow{CT} a_n \longrightarrow d_n \xrightarrow{ICT} \int u\,dx$$

6.5 Method of weighted residual (MWR)

$$\mathcal{L}(u) = f(x,t) \quad \text{for } x \in D$$

with $B(u) = g(x,t)$ on ∂D.

$\tilde{u}$: approx. sol.

$$\tilde{u} = \sum_{n=0}^{N} c_n(t)\, \phi_n(x)$$

basis ft.
test ft.

Spectral method : $\phi = e^{ikx}$
e.g. Fourier

Residual : $R = \mathcal{L}(\tilde{u}) - f$

MWR aims to find sol. $\tilde{u}$ which minimizes the residual R in the weighted integral sense

$$\int_D w_i R \, dx = 0 \qquad i = 0, 1, \cdots, N.$$

weight ft.

$$\rightarrow \int_D w_i [\mathcal{L}(\tilde{u}) - f] \, dx = 0$$

$$\rightarrow \boxed{\int_D w_i \left[\mathcal{L}\left(\sum_{n=0}^{N} c_n \phi_n\right) - f\right] dx = 0}$$

weak form of original eq.

Galerkin method : $w_i = \phi_i \Rightarrow \int_D \phi_i \left[\mathcal{L}\left(\sum_{n=0}^{N} c_n \phi_n\right) - f \right] dx = 0$

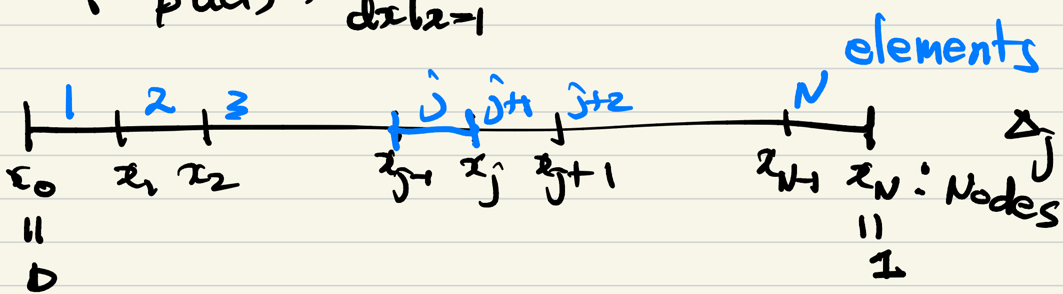
unknown

$i = 0, 1, \cdots, N$

6.6 Finite Element Method (FEM)

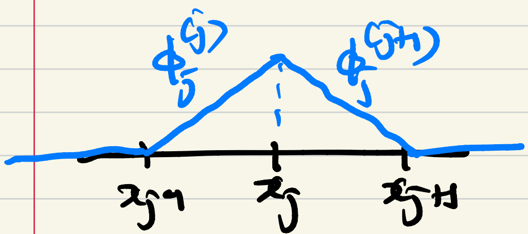
Consider $\dfrac{d^2u}{dx^2} + u = f \qquad 0 \le x \le 1$

$$\begin{cases} \alpha u(0) + \left.\dfrac{du}{dx}\right|_{x=0} = A \\ \beta u(1) + \left.\dfrac{du}{dx}\right|_{x=1} = B \end{cases}$$

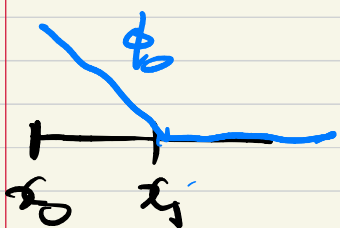


$\Delta_j = x_j - x_{j-1}$

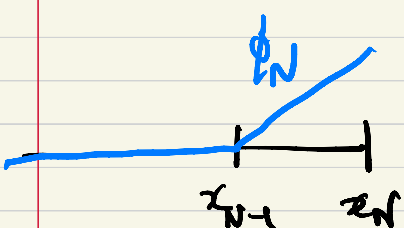
Choose a basis ft ϕ s.t. (piecewise linear ft.)



$$\phi_j(x) = \begin{cases} 0 & x < x_{j-1} \\ \dfrac{x - x_{j-1}}{x_j - x_{j-1}} \equiv \phi_j^{(j)} & x_{j-1} \le x < x_j \\ \dfrac{x - x_{j+1}}{x_j - x_{j+1}} \equiv \phi_j^{(j+1)} & x_j \le x < x_{j+1} \\ 0 & x \ge x_{j+1} \end{cases} \qquad j = 1, 2, \cdots, N-1$$



$$\phi_0 = \begin{cases} \dfrac{x - x_1}{x_0 - x_1} & x_0 \le x < x_1 \\ 0 & x \ge x_1 \end{cases}$$



$$\phi_N = \begin{cases} 0 & x < x_{N-1} \\ \dfrac{x - x_{N-1}}{x_N - x_{N-1}} & x_{N-1} \le x \le x_N \end{cases}$$

higher-order polynomials for ϕ_j can also be considered.

$$u(x) \simeq \tilde{u} = \sum_{j=0}^{N} u_j \phi_j(x)$$

$\hookrightarrow$ unknown

$$\frac{d^2u}{dx^2} + u = f$$

$$\text{MWR} \rightarrow \int_0^1 \left(\frac{d^2\tilde{u}}{dx^2} + \tilde{u} - f \right) w_i \, dx = 0 \qquad i = 0, 1, \cdots, N$$

$$\frac{d\tilde{u}}{dx} w_i \Big|_0^1 - \int_0^1 \frac{d\tilde{u}}{dx} \frac{dw_i}{dx} dx + \int_0^1 \tilde{u} w_i \, dx - \int_0^1 f w_i \, dx = 0$$

$$\rightarrow \frac{d\tilde{u}}{dx} w_i \Big|_0^1 - \int_0^1 \frac{d}{dx}\left(\sum_{j=0}^{N} u_j \phi_j \right) \frac{dw_i}{dx} dx + \int_0^1 \left(\sum_{j=0}^{N} u_j \phi_j \right) w_i \, dx - \int_0^1 f w_i \, dx = 0$$

Galerkin method : $w_i = \phi_i$

$$\rightarrow \begin{cases} \dfrac{d\tilde{u}}{dx} \phi_i \Big|_0^1 - \sum_{j=0}^{N} u_j \int_0^1 \dfrac{d\phi_j}{dx} \dfrac{d\phi_i}{dx} dx + \sum_{j=0}^{N} u_j \int_0^1 \phi_j \phi_i \, dx = \int_0^1 f \phi_i \, dx & i = 0, 1, 2, \cdots, N \\ 2 \text{ bc's} & \end{cases}$$

N+3 eqs.

unknowns : $\underbrace{N+1 \ u_j\text{'s and } \frac{du}{dx} \text{ @ boundaries}}_{N+3 \text{ unknowns.}}$

$\therefore$ So, closed.

$$\frac{d\tilde{u}}{dx}\phi_i\Big|_0^1 = \frac{d\tilde{u}}{dx}(1)\phi_i(1) - \frac{d\tilde{u}}{dx}(0)\phi_i(0)$$

$$= (B-\beta u_N)\underbrace{\phi_i(1)}_{=\delta_{iN}} - (A-\alpha u_0)\underbrace{\phi_i(0)}_{=\delta_{i0}}$$

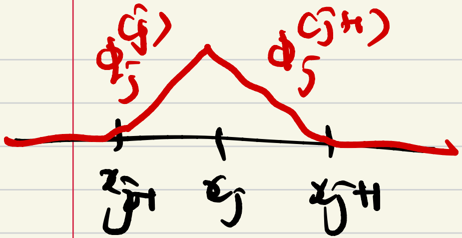
For simplicity, consider homo. Dirichlet b.c. : $u_0 = u_N = 0$,

Then, $-\sum_{j=1}^{N-1} u_j \int_0^1 \frac{d\phi_j}{dx}\frac{d\phi_i}{dx}dx + \sum_{j=1}^{N-1} u_j \int_0^1 \phi_j\phi_i dx = \int_0^1 f\phi_i dx$.

$\left(f(x) = \sum_{j=0}^{N} f_j\phi_j(x) \right)$

$$\Rightarrow \boxed{\sum_{j=1}^{N-1}(-D_{ij}+C_{ij})u_j = \sum_{j=0}^{N-1} C_{ij} f_j} \qquad i = 1, 2, \cdots, N-1$$

$$D_{ij} = \int_0^1 \frac{d\phi_i}{dx}\frac{d\phi_j}{dx}dx, \quad C_{ij} = \int_0^1 \phi_i\phi_j dx$$



Non-zero elts of D_{ij} are

$$D_{i-1,i} = -\frac{1}{\Delta_i}, \quad D_{i,i} = \frac{1}{\Delta_i} + \frac{1}{\Delta_{i+1}}, \quad D_{i+1,i} = -\frac{1}{\Delta_{i+1}}$$

$$C_{i-1,i} = \frac{\Delta_i}{6}, \quad C_{i,i} = \frac{\Delta_i}{3} + \frac{\Delta_{i+1}}{3}, \quad C_{i+1,i} = \frac{\Delta_{i+1}}{6}$$

$$\boxed{Au = b}$$

$$A_{ij} = C_{ij} - D_{ij}$$

tri-diagonal system

$$\begin{cases} A_{j,j-1} = \dfrac{1}{\Delta_j} + \dfrac{\Delta_j}{6} \\ A_{j,j} = -\left(\dfrac{1}{\Delta_j} + \dfrac{1}{\Delta_{j+1}}\right) + \dfrac{\Delta_j}{3} + \dfrac{\Delta_{j+1}}{3} \\ A_{j,j+1} = \dfrac{1}{\Delta_{j+1}} + \dfrac{\Delta_{j+1}}{6} \end{cases}$$

$$b_j = \frac{\Delta_j}{6} f_{j-1} + \left(\frac{\Delta_j}{3} + \frac{\Delta_{j+1}}{3}\right) f_j + \frac{\Delta_{j+1}}{6} f_{j+1}$$

For uniform mesh, $\Delta_j = \Delta$

$$\left(\frac{1}{\Delta}+\frac{\Delta}{6}\right)u_{j-1}+\left(-\frac{2}{\Delta}+\frac{2\Delta}{3}\right)u_j+\left(\frac{1}{\Delta}+\frac{\Delta}{6}\right)u_{j+1}$$

$$=\frac{\Delta}{6}f_{j-1}+\frac{2\Delta}{3}f_j+\frac{\Delta}{6}f_{j+1}$$

$$\Rightarrow \boxed{\underbrace{\frac{u_{j+1}-2u_j+u_{j-1}}{\Delta^2}}_{\text{2nd order CD}}+\underbrace{\left(\frac{1}{6}u_{j+1}+\frac{2}{3}u_j+\frac{1}{6}u_{j-1}\right)}_{\text{Simpson}}=\underbrace{\frac{1}{6}f_{j+1}+\frac{2}{3}f_j+\frac{1}{6}f_{j-1}}_{\text{Simpson}}}$$

$$\frac{d^2u}{dx^2}+u=f$$

FDM: $\dfrac{u_{j+1}-2u_j+u_{j-1}}{\Delta^2}+u_j=f_j$