Differential Equations

- Ordinary Differential Equation
  - Contains one or several derivatives of unknown functions of a single variable

$$y' = \cos x,$$

$$y'' + 4y = 0,$$

$$x^2 y''' y' + 2e^x y'' = \left(x^2 + 2\right) y^2$$

- Partial Differential Equation
  - Contains one or several derivatives of unknown functions of two or more variables

Examples of ODE's (Modeling)

- Population growth rate

$$y' = y$$

$$\Rightarrow \quad y = e^x \left(y = ce^x \right)$$

arbitrary constant

- Falling stone

$$y'' = g$$

$$\Rightarrow \quad y' = dy/dx = gx + v_o$$

Initial velocity

$$\Rightarrow \quad y = \frac{1}{2} gx^2 + v_o x + y_o$$

Initial distance

First Order DE

- Order of DE
  - Order of the highest derivative that appears in the equation

- First order DE

$$F\left(x, y, y'\right) = 0$$
$$y' = f\left(x, y\right)$$

Solutions of DE (1)

- Explicit Solution

$$xy' = 2x$$

$$\implies \quad y = x^2$$

- Implicit Solution

$$yy' = -x$$

$$\implies \quad x^2 + y^2 - 1 = 0 \,(y > 0)$$

on the $-1 < x < 1$

Solutions of DE (2)

- General Solution

$$y' = \cos x$$

$$\Longrightarrow \quad y = \sin x + c \qquad \boxed{\textit{arbitrary constant}}$$

- Particular Solution

$$y' = \cos x$$

$$y = \sin x,$$

$$y = \sin x - 2,$$

$$\Longrightarrow \quad y = \sin x + 0.75,$$

$$\vdots$$

Solutions of DE (3)

- Singular Solution
  - An additional solution that cannot be obtained from general solution

$$y'^2 - xy' + y = 0$$

$$y = cx - c^2 \quad \longleftarrow \quad \text{general solution}$$

$$y = x^2 / 4 \quad \longleftarrow \quad \text{singular solution}$$

- No Solution at all

$$y'^2 = 1$$

Solutions of DE (4)

- No General Solution

$$|y'| + |y| = 1$$

$$\implies \quad y \equiv 0 \quad \longleftarrow \boxed{\textit{trivial solution}}$$

Modeling

- Modeling

| Physical system | → | Mathematical model |

- Modeling and Solution (Radioactivity)

$$\frac{dy}{dt} = ky$$

$$\Longrightarrow \quad y(t) = ce^{kt} \qquad \textit{general solution}$$

- Initial Condition

$$y(0) = 2 \quad \Longrightarrow \quad y(t) = 2e^{kt} \qquad \textit{Initial Value Prob.}$$

Initial Value Problem

- Typical form

$$y' = f(x, y), \, y(x_o) = y_o$$

- Geometrical Example
  - Curve passing through (1, 1) in the x-y plane having slope –y/x

$$y' = -\frac{y}{x} \quad \Longrightarrow \quad y = \frac{c}{x} \quad \boxed{\textit{general solution}}$$

$$y = \frac{1}{x} \quad \boxed{\textit{particular solution}}$$

Directional Field (1)

- 1st Order ODE

$$F\left(x, y, y'\right) = 0$$

implicit form

$$y' = f\left(x, y\right)$$

explicit form

- Geometrical Meaning
  - Plotting approximate solution curves without actually solving it
  - Slope ⟹ lineal elements ⟹ *directional field, slope field*

- Plotting by Computer
  - Computer Algebra System (CAS)

Directional Field (2)

- Plotting by Hand
  - 1st Step: Draw Isoclines (curves of equal inclination).

  $$f(x, y) = k = const$$

  - 2nd Step: Draw lineal elements along isoclines.

  → *directional field, slope field*

  - 3rd Step: Sketch approximate solution.

Example of Directional Field (1)

- Example

$$y' = xy$$

- Isoclines: equilateral hyperbolas

$$xy = k$$

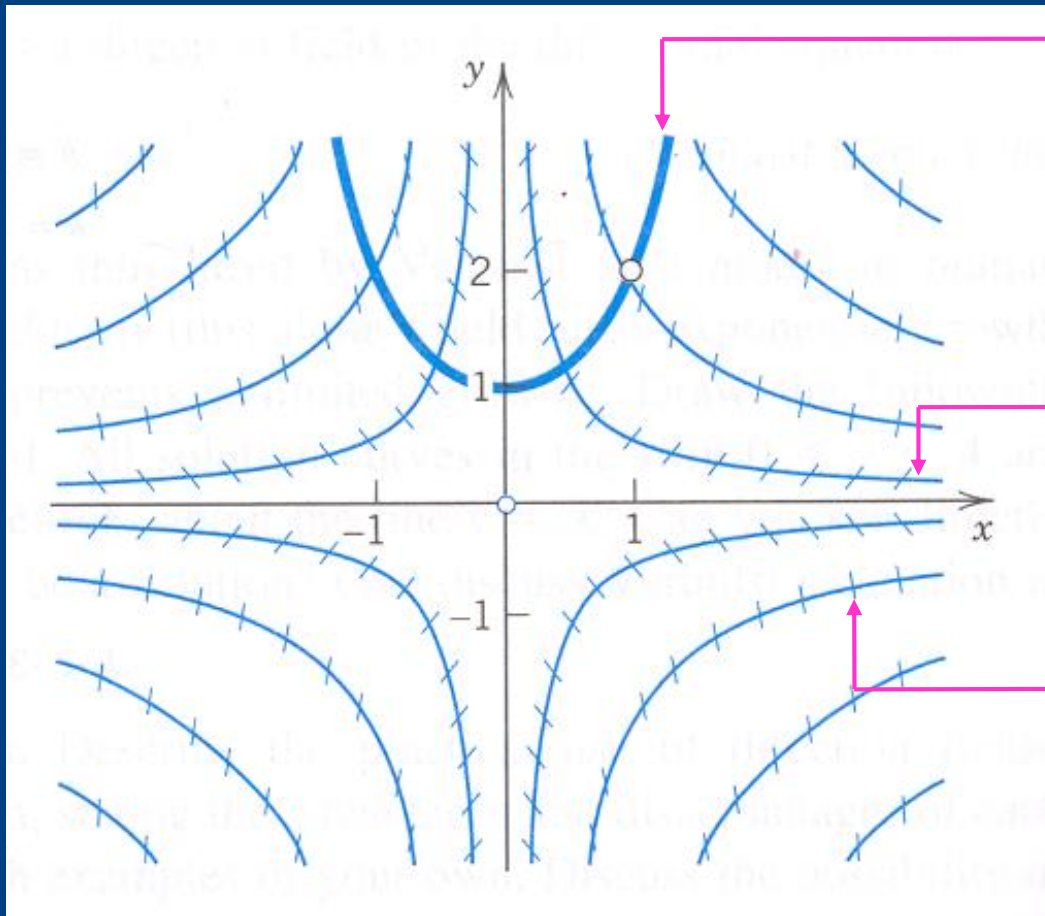- Exact solution (Closed-form solution)

$$y(x) = ce^{x^2/2}$$

- Particular solution passing through (1,2)

$$y(x) = 2e^{(x^2-1)/2}$$

Example of Directional Field (2)

Approximate solution curve

Isocline

Lineal element

Example of Directional Field (3)

- Van der Pol Equations (electronics)

$$y' = 0.1\left(1 - x^2\right) - \frac{x}{y}$$