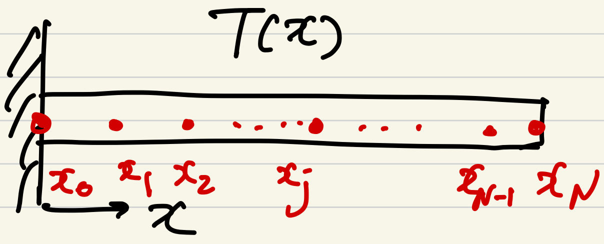
· Banded matrix

diff'l eq. $\xrightarrow[\text{method}]{\text{numerical}}$ system of algebraic eqs.

① ODE (ordinary diff'l eq.)

$$\boxed{\frac{d^2T}{dx^2} - \alpha^2 T = Q(x)}$$

internal source



Introduce a discrete set of points x_j. $j = 0, 1, 2, \cdots, N$

↳ grid points

Find $T(x_j)$ or T_j.

· Finite difference method (FDM)

Taylor series expansion

$h_j = x_{j+1} - x_j$: grid spacing

$h = h_1 = h_2 \cdots$: uniform grid spacing

$$T(x_{j+1}) = T(x_j) + h\frac{dT}{dx}\Big|_j + \frac{1}{2}h^2\frac{d^2T}{dx^2}\Big|_j + \frac{1}{6}h^3\frac{d^3T}{dx^3}\Big|_j + \frac{1}{24}h^4\frac{d^4T}{dx^4}\Big|_j + \cdots$$

$$-\ \big|\ T(x_{j-1}) = \text{''} \quad - \quad \text{''} \quad + \quad \text{''} \quad - \quad \text{''} \quad + \quad \text{''} \cdots$$

$$\Rightarrow T(x_{j+1}) - T(x_{j-1}) = 2h\frac{dT}{dx}\Big|_j + \frac{1}{3}h^3\frac{d^3T}{dx^3}\Big|_j + \cdots$$

$$\Rightarrow \boxed{\frac{dT}{dx}\Big|_j = \frac{T(x_{j+1}) - T(x_{j-1})}{2h}} - \frac{1}{6}h^2\frac{d^3T}{dx^3}\Big|_j + \cdots$$

second-order FDM

leading error term $O(h^2)$

$$+\ \big|\ \text{''} \quad / \quad \text{''}$$

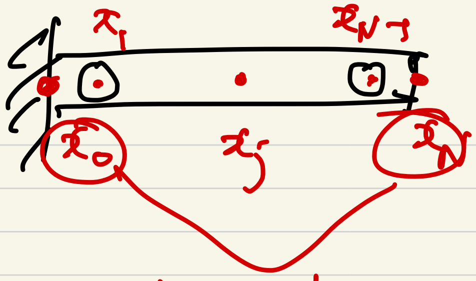
$$T(x_{j+1}) + T(x_{j-1}) = 2T(x_j) + h^2\frac{d^2T}{dx^2}\Big|_j + \frac{1}{12}h^4\frac{d^4T}{dx^4}\Big|_j + \cdots$$

$$\Rightarrow \boxed{\frac{d^2T}{dx^2}\Big|_j = \frac{T(x_{j+1}) - 2T(x_j) + T(x_{j-1})}{h^2}} - \frac{1}{12}h^2\frac{d^4T}{dx^4}\Big|_j + \cdots$$

second-order FDM

leading error term $O(h^2)$

$\rightarrow$ Substitute this into diff'l eq.



Boundary cond.

ⓐ j, $\begin{cases} \dfrac{T_{j+1} - 2T_j + T_{j-1}}{h^2} - \alpha^2 T_j = Q_j, \\ T_0 = 0 \\ T_N = \zeta \end{cases}$

$j = 1, 2, \cdots, N-1$

System of algebraic eqs.

$$\rightarrow \begin{cases} T_{j+1} - (2 + h^2\alpha^2) T_j + T_{j-1} = h^2 Q_j \\ T_0 = 0 \\ T_N = \zeta \end{cases}$$

$j=1$: $T_2 - (2+h^2\alpha^2) T_1 + T_0 = h^2 Q_1$, $\quad T_0 = 0$

$j=2$: $T_3 - (\quad \text{''} \quad) T_2 + T_1 = h^2 Q_2$

$\vdots$

$j = N-1$: $T_N - (\quad \text{''} \quad) T_{N-1} + T_{N-2} = h^2 Q_{N-1}$, $\quad T_N = \zeta$

$$\begin{bmatrix} -(2+h^2\alpha^2) & 1 & 0 & \cdot & \cdots & 0 \\ 1 & -(2+h^2\alpha^2) & 1 & 0 & \cdots & 0 \\ \cdots & 1 & -(2+h^2\alpha^2) & 1 & 0 \cdots & 0 \\ 0 & \cdots & \cdots & 0 & 1 & -(2+h^2\alpha^2) \end{bmatrix} \begin{bmatrix} T_1 \\ T_2 \\ \vdots \\ T_j \\ \vdots \\ T_{N-1} \end{bmatrix} = \begin{bmatrix} h^2 Q_1 \\ h^2 Q_2 \\ \vdots \\ h^2 Q_j \\ \vdots \\ h^2 Q_{N-1} - S \end{bmatrix}$$

tri-diagonal matrix $\quad Ax = b$

banded matrix : non-zero elements only around the main diagonal.

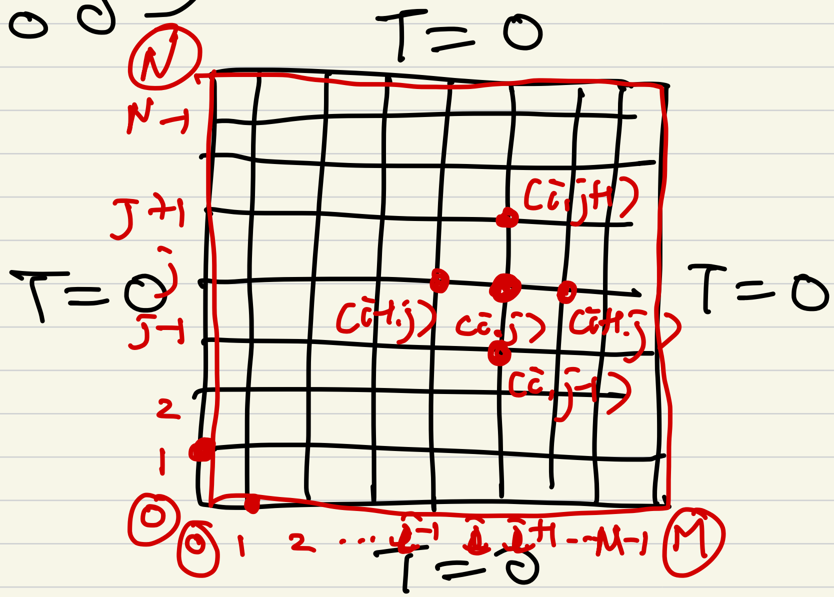
↳ arises from FDM of diff'l eqs.

tri-diagonal matrix : $B[a_i, b_i, c_i]$

$$a_i = 1, \quad b_i = -(2+h^2\alpha^2), \quad c_i = 1$$

→ Sparse matrix

$$\begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix} \qquad \begin{bmatrix} 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 \end{bmatrix} \qquad \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$



② PDE (partial diff'l eq.)

$$\boxed{\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = Q(x,y)}$$

$Q(x,y)$: internal source

$$\left.\begin{aligned} \underbrace{x_{i+1} - x_i}_{=\Delta x_i} &= h \\ \underbrace{y_{j+1} - y_j}_{\Delta y_j} &= h \end{aligned}\right\} \text{uniform grid spacings}$$

2nd-order FDM

$$\rightarrow \boxed{\frac{T_{i+1,j} - 2T_{i,j} + T_{i-1,j}}{h^2} + \frac{T_{i,j+1} - 2T_{i,j} + T_{i,j-1}}{h^2} = Q_{i,j}}$$

$i = 1, 2, \cdots, M-1$

$j = 1, 2, \cdots, N-1$

$(M-1) \times (N-1)$

$$\rightarrow \quad T_{i+1,j} - 4T_{i,j} + T_{i-1,j} + T_{i,j+1} + T_{i,j-1} = h^2 Q_{i,j}$$

$M = 4$

$N = 4$

$$i=1,\ j=1: \quad T_{2,1} - 4T_{1,1} + \overset{=0}{T_{0,1}} + T_{1,2} + \overset{=0}{T_{1,0}} = h^2 Q_{1,1}$$

$$i=2,\ j=1: \quad T_{3,1} - 4T_{2,1} + T_{1,1} + T_{2,2} + \overset{=0}{T_{2,0}} = h^2 Q_{2,1}$$

$$\vdots$$

$$i=3,\ j=3: \quad \overset{=0}{T_{4,3}} - 4T_{3,3} + T_{2,3} + \overset{=0}{T_{3,4}} + T_{3,2} = h^2 Q_{3,3}$$

9 unknowns

$T_{1,1} \cdots T_{3,3}$

$$\Rightarrow \left[\begin{array}{ccc|ccc|ccc}
-4 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
1 & -4 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 1 & -4 & 0 & 0 & 1 & 0 & 0 & 0 \\
\hline
1 & 0 & 0 & -4 & 1 & 0 & 1 & 0 & 0 \\
0 & 1 & 0 & 1 & -4 & 1 & 0 & 1 & 0 \\
0 & 0 & 1 & 0 & 1 & -4 & 0 & 0 & 1 \\
\hline
0 & 0 & 0 & 1 & 0 & 0 & -4 & 1 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 1 & -4 & 1 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & -4
\end{array}\right]
\begin{bmatrix} T_{1,1} \\ T_{2,1} \\ T_{3,1} \\ T_{1,2} \\ T_{2,2} \\ T_{2,3} \\ T_{1,3} \\ T_{2,3} \\ T_{3,3} \end{bmatrix}
=
\begin{bmatrix} h^2 Q_{1,1} \\ h^2 Q_{2,1} \\ \vdots \\ h^2 Q_{3,3} \end{bmatrix}$$

Block-tridiagonal matrix

$$\begin{pmatrix} B_1 & C_1 & 0 & \cdots & 0 \\ A_2 & B_2 & C_2 & 0 \cdots & 0 \\ & \ddots & \ddots & \ddots & 0 \\ 0 & & \ddots & \ddots & \ddots \end{pmatrix}$$

⊙ Solution technique

- diagonal matrix

$$\begin{pmatrix} 7 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$$

- upper triangular matrix

$$\begin{pmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 6 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 7 \\ 8 \\ 9 \end{pmatrix} \uparrow$$

- lower triangular matrix

$$\begin{pmatrix} 1 & 0 & 0 \\ 2 & 3 & 0 \\ 4 & 5 & 6 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 7 \\ 8 \\ 9 \end{pmatrix} \downarrow$$

* Gauss elimination (GE) : to make upper triangular matrix

$$\begin{cases} 4x_2 - x_3 = 5 \\ x_1 + x_2 + x_3 = 6 \\ 2x_1 - 2x_2 + x_3 = 1 \end{cases} \rightarrow \begin{pmatrix} 0 & 4 & -1 \\ 1 & 1 & 1 \\ 2 & -2 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 5 \\ 6 \\ 1 \end{pmatrix}$$

Interchange the first two rows.

$$\begin{pmatrix} 1 & 1 & 1 \\ 0 & 4 & -1 \\ 2 & -2 & 1 \end{pmatrix} \begin{pmatrix} \; \end{pmatrix} = \begin{pmatrix} 6 \\ 5 \\ 1 \end{pmatrix}$$

subtract twice the first row from last row.

$$\begin{pmatrix} 1 & 1 & 1 \\ 0 & 4 & -1 \\ 0 & -4 & -1 \end{pmatrix} \begin{pmatrix} \; \end{pmatrix} = \begin{pmatrix} 6 \\ 5 \\ -11 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 1 & 1 \\ 0 & 4 & -1 \\ 0 & 0 & -2 \end{pmatrix} \begin{pmatrix} \; \end{pmatrix} = \begin{pmatrix} 6 \\ 5 \\ -6 \end{pmatrix}$$

Stop of GE

or, augmented matrix

$$\left(\begin{array}{ccc|c} 0 & 4 & -1 & 5 \\ 1 & 1 & 1 & 6 \\ 2 & -2 & 1 & 1 \end{array}\right) \xrightarrow{GE} \left(\begin{array}{cccc} 1 & 1 & 1 & 6 \\ 0 & 4 & -1 & 5 \\ 0 & 0 & -2 & -6 \end{array}\right)$$

$$\to \begin{pmatrix} 1 & 1 & 1 & 6 \\ 0 & 4 & 0 & 8 \\ 0 & 0 & -2 & -6 \end{pmatrix} \to \begin{pmatrix} 1 & 1 & 1 & 6 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \end{pmatrix}$$

$\to \quad x_1 = 1, \ x_2 = 2, \ x_3 = 3$

"Gauss-Jordan elimination"

- General matrix system

$Ax = b$

$n \times n$

GE

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix}$$

0

To eliminate a_{21}, $l_2 = a_{21}/a_{11}$

multiply the first row by l_2, and subtract from 2nd row.

Continue until all elts. below a_{11} are eliminated.

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ 0 & a_{22}' & \cdots & a_{2n}' \\ 0 & & \vdots & \\ \vdots & & & \\ 0 & a_{n2}' & \cdots & a_{nn}' \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2' \\ \vdots \\ b_n' \end{pmatrix}$$

By same way (we drop (') for convenience)

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ 0 & a_{22} & \cdots & a_{2n} \\ & & \ddots & \vdots \\ 0 & 0 & & a_{nn} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix}$$

stop of GE

$\rightarrow$ backward sweep

$$\begin{cases} x_n = b_n / a_{nn} \\ x_{n-1} = (b_{n-1} - a_{n-1\,n} x_n) / a_{n-1\,n-1} \\ x_j = (b_j - \sum_{k=j+1}^{n} a_{jk} x_k) / a_{jj} \end{cases}$$

$(j = n-1, n-2, \cdots;)$

⊙ Operation counts for obtaining solution of $Ax=b$

$Ax=b$ $\qquad$ A : full matrix

$n \times n$

$l_1 = \dfrac{a_{21}}{a_{11}}$

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ \vdots & & & & \\ a_{n1} & a_{n2} & a_{n3} & \cdots & a_{nn} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix}$$

GE :

To eliminate a_{21} : $1D$, nM, nA

To " the first col. : $(n-1)D$, $n(n-1)M$, $n(n-1)A$

To " " second " : $(n-2)D$, $(n-1)(n-2)M$, $(n-1)(n-2)A$

⋮

To " " $(n-1)^{th}$ col. : $1D$, $1\cdot 2M$, $2A$

Total divisions : $\displaystyle\sum_{k=1}^{n-1} k = \frac{1}{2}n(n-1)$

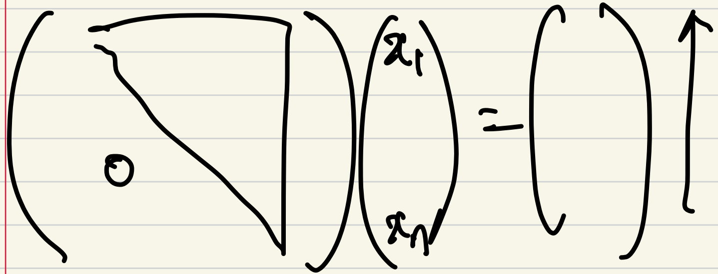
total multiplications: $\sum_{k=1}^{n-1} k(k+1) = \frac{1}{3}(n^3-n)$

$\Theta(n^2/2)$ $\Theta(n^3/3)$ $\left(\sum_{1}^{n} k^2 = \frac{1}{6}n(n+1)(2n+1)\right)$

$\Rightarrow \frac{1}{2}n(n-1)$ D, $\frac{1}{3}(n^3-n)$ M, $\frac{1}{3}(n^3-n)$ A

$\Rightarrow$ $\boxed{\text{Gauss elimination requires } \Theta(n^3/3)}$ ~ $\Theta(n^3)$

Backward sweep



$x_n = b_n/a_{nn}$

$x_j = \left(b_j - \sum_{k=j+1}^{n} a_{jk}x_k\right)/a_{jj}$, $j = n-1, n-2, \dots, 1$

for each j, 1 D, $(n-j)$ M, $(n-j)$ A

total $\sum_{j=1}^{n-1}(n-j) = \frac{1}{2}n(n-1)$ A, $\frac{1}{2}n(n-1)$ M ~ $\Theta(n^2)$

total divisions: n D

negligible as compared to GE requiring $\Theta(n^3)$ for $n \gg 1$.

Gauss-Jordan elimination → exactly same operation counts.

* Tri-diagonal matrix

$$l_1 = \frac{a_2}{b_1} \quad \begin{pmatrix} b_1 & c_1 & 0 & \cdots & 0 \\ a_2 & b_2 & c_2 & 0 \cdots & 0 \\ & a_3 & b_3 & c_3 & 0 \\ \text{\Large 0} & & \ddots & \ddots & \\ & & & a_n & b_n \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} f_1 \\ f_2 \\ \vdots \\ f_n \end{pmatrix}$$

To eliminate a_2,

1D, 2M, 2D

To make U,

$(n-1)D$, $2(n-1)M$, $2(n-1)A$

$$\boxed{GE \to O(n)}$$

after GE,

$$\begin{pmatrix} b_1 & c_1 & & \phi \\ & b_2' & c_2' & \\ \phi & & \ddots & \\ & & & b_n' \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} f_1 \\ f_2' \\ \vdots \\ f_n' \end{pmatrix}$$

backward sweep

$x_n = f_n' / b_n'$

$x_j = (f_j' - c_j' x_{j+1}) / b_j'$

for $j = n-1, \cdots, 1$

→ nD, $(n-1)M$, $(n-1)A$ ⟶ $O(n)$

Total operations : $(2n-1)$ D , $3(n-1)$ M , $3(n-1)$ A

$\Rightarrow$ $\boxed{O(n) \text{ operations for tri-diagonal matrix system.}}$

⑥ Computation of inverse matrix

$AB = I \qquad B = [\underline{b}_1, \underline{b}_2, \cdots, \underline{b}_n]$

$AB = [A\underline{b}_1, A\underline{b}_2, \cdots, A\underline{b}_n] = I$

$\rightarrow A b_1 = e_1, \; A b_2 = e_2, \; \cdots, \; A b_n = e_n,$

$$e_j = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \begin{matrix} \\ \\ \\ \\ \leftarrow j \\ \\ \\ \\ \end{matrix}$$

Augmented matrix $[A \; e_1 \; e_2 \; e_3 \cdots e_n] = [A \; I]$

Perform Gauss-Jordan on this augmented matrix.

ex) $A = \begin{pmatrix} -2 & 1 & 0 \\ 1 & -2 & 1 \\ 0 & 1 & 1 \end{pmatrix}$ $\quad A^{-1} = ?$

tri-diagonal matrix

$Ax = b$

$x = A^{-1} b$

$$\left(\begin{array}{ccc|ccc} -2 & 1 & 0 & 1 & 0 & 0 \\ 1 & -2 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{array}\right) \to \left(\begin{array}{ccc|ccc} -2 & 1 & 0 & 1 & 0 & 0 \\ 0 & -\frac{3}{2} & 1 & \frac{1}{2} & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{array}\right) \to \left(\begin{array}{ccc|ccc} -2 & 1 & 0 & 1 & 0 & 0 \\ 0 & -\frac{3}{2} & 1 & \frac{1}{2} & 1 & 0 \\ 0 & 0 & \frac{5}{3} & \frac{1}{3} & \frac{2}{3} & 1 \end{array}\right)$$

$$\to \cdots \to \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -\frac{3}{5} & -\frac{1}{5} & \frac{1}{5} \\ 0 & 1 & 0 & -\frac{1}{5} & -\frac{2}{5} & \frac{2}{5} \\ 0 & 0 & 1 & -\frac{1}{5} & \frac{2}{5} & \frac{3}{5} \end{array}\right) = A^{-1} : \text{full matrix}$$

I

* inverse of a banded matrix is a full matrix.

Operation counts for A^{-1} : n^3 of each M and A (full matrix A) ~ similar to that of AB

If A^{-1} is not needed, it should not be computed.

⊙ LU decomposition

$$Ax = b \xrightarrow[\text{GE}]{} Ux = c$$

$$Ax = b \to x = A^{-1}b$$

$$\downarrow$$

$$L\underbrace{Ux}_{z} = b \to \begin{array}{ll} Lz = b & O(n^2) \\ Ux = z & O(n^2) \end{array}$$

operation of multiplication of one row by a constant and subtract from another row can be performed by a matrix multiplication.

e.g. multiply 1st row by e_{21} $\left(= - a_{21}/a_{11}\right)$ and add to 2nd row.

$$E_{21} = \begin{pmatrix} 1 & & & 0 \\ e_{21} & 1 & & \\ 0 & & \ddots & \\ \vdots & & & \\ 0 & & & 1 \end{pmatrix} \qquad a_1 = \begin{pmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{n1} \end{pmatrix} \rightarrow E_{21} a_1 = \begin{pmatrix} a_{11} \\ 0 \\ a_{31} \\ \vdots \\ a_{n1} \end{pmatrix}$$

$$E_{31} = \begin{pmatrix} 1 & & & & 0 \\ 0 & 1 & & & \\ e_{31} & & \ddots & & \\ 0 & & & \ddots & \\ \vdots & & & & \\ 0 & & & & 1 \end{pmatrix} \qquad e_{31} = -a_{31}/a_{11} \rightarrow E_{31} a_1 = \begin{pmatrix} a_{11} \\ a_{21} \\ 0 \\ \vdots \\ a_{n1} \end{pmatrix}$$

$$E_{31} E_{21} = \begin{pmatrix} 1 & & & & 0 \\ e_{21} & 1 & & & \\ e_{31} & & 1 & & \\ 0 & & & \ddots & \\ \vdots & 0 & & & \\ 0 & & & & 1 \end{pmatrix} \rightarrow E_{31} E_{21} a_1 = \begin{pmatrix} a_{11} \\ 0 \\ 0 \\ a_{41} \\ \vdots \\ a_{n1} \end{pmatrix}$$

To eliminate the 1st col.,

$$E_1 = \begin{pmatrix} 1 & & & 0 \\ e_{21} & 1 & & \\ e_{31} & & \ddots & \\ \vdots & 0 & & \\ e_{n1} & & & 1 \end{pmatrix} \quad e_{j1} = -a_{j1}/a_{11} \quad E_1 A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ 0 & a'_{22} & \cdots & a'_{2n} \\ \vdots & \vdots & & \\ 0 & a'_{n2} & \cdots & a'_{nn} \end{pmatrix}$$

$\to 0$

To eliminate everything below a'_{22}

$$E_2 = \begin{pmatrix} 1 & & & & 0 \\ 0 & 1 & & & \\ \vdots & e_{32} & 1 & & \\ \vdots & \vdots & 0 & \ddots & \\ 0 & e_{n2} & & & 1 \end{pmatrix} \quad e_{j2} = -a'_{j2}/a'_{22} \quad E_2 E_1 A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ 0 & a'_{22} & \cdots & a'_{2n} \\ \vdots & 0 & a''_{33} & a''_{3n} \\ 0 & 0 & \cdots & a''_{nn} \end{pmatrix}$$

upper triangular matrix

$$\to \underbrace{E_{n-1} E_{n-2} \cdots E_2 E_1}_{= E \text{ : lower triangular matrix}} A = U$$

$EA = U \longrightarrow A = E^{-1} U$?

$$E^* E_{21} A = A$$
$$E^* = E_{21}^{-1}$$

Let's undo E_{21} process.

$$E_{21} = \begin{pmatrix} 1 & & & 0 \\ e_{21} & 1 & & \\ 0 & & \ddots & \\ \vdots & 0 & & \\ 0 & & & 1 \end{pmatrix} \rightarrow E_{21}^{-1} = \begin{pmatrix} 1 & & 0 \\ -e_{21} & 1 & \\ 0 & & \ddots & \\ & & & 1 \end{pmatrix} \qquad \underbrace{E_{21}^{-1} E_{21}}_{I} A = A$$

multiply 1st row by $-e_{21}$ and add to 2nd row.

$$\rightarrow E_1^{-1} = \begin{pmatrix} 1 & & & & 0 \\ -e_{21} & 1 & & & \\ -e_{31} & & \ddots & & \\ \vdots & 0 & & \ddots & \\ -e_{n1} & & & & 1 \end{pmatrix}, \quad E_2^{-1} = \cdots, \quad E_3^{-1} = \cdots.$$

$O(n^3)$

$$EA = U \longrightarrow E_{n-1} E_{n-2} \cdots E_1 A = U$$

$$A = \underbrace{E_1^{-1} E_2^{-1} \cdots E_{n-2}^{-1} E_{n-1}^{-1}}_{} U = LU$$

lower triangular matrix w/ 1's on the main diagonal

$$Ax = b \longrightarrow L \underbrace{Ux}_{z} = b$$

$Lz = b \rightarrow \begin{pmatrix} l_{11} & & & 0 \\ l_{21} & l_{22} & & \\ \vdots & & \ddots & \\ l_{n1} & l_{n2} & \cdots & l_{nn} \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \\ \vdots \\ z_n \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix}$

$\Theta(n^2)$

forward sweep : $z_1 = b_1 / l_{11}$

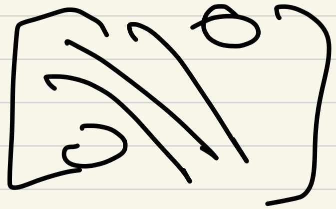
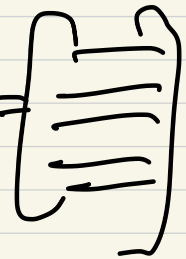
$$z_j = \left(b_j - \sum_{k=1}^{j-1} l_{jk} z_k\right) / l_{jj} \quad , \quad j = 2, 3, \cdots, n$$

$Ux = z$: backward sweep $\rightarrow \Theta(n^2)$ to get x

LU decomposition $\rightarrow \Theta(n^3)$: expensive !

$Ax = b \rightarrow x = A^{-1} b$ } direct solution

$\hookrightarrow LUx = b$

$A =$  $\rightarrow A^{-1} =$ 

$\hookrightarrow$ LU much easier

what if $a_{22} = 0$?

$$\begin{pmatrix} 0 & 4 & -1 \\ 1 & 1 & 1 \\ 2 & -2 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \rightarrow \text{row exchange}$$

$\downarrow$

permutation matrix

$$P = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \rightarrow PA = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 4 & -1 \\ 2 & -2 & 1 \end{pmatrix}$$

$\rightarrow Ax = b$

$\rightarrow PAx = Pb \rightarrow LUx = Pb = b'$