Separable DE

- Separable equation

$$g\left(y\right)y' = f\left(x\right)$$

$$\Rightarrow \quad \boxed{g\left(y\right)dy = f\left(x\right)dx}$$

  - Integrating both sides with respect to *x*

$$\int g\left(y\right)\frac{dy}{dx}dx = f\left(x\right)dx + c$$

$$\int g\left(y\right)dy = f\left(x\right)dx + c$$

  - If *f* and *g* are continuous, the integrals will exist.

Example of Separable DE (1)

- Example 1

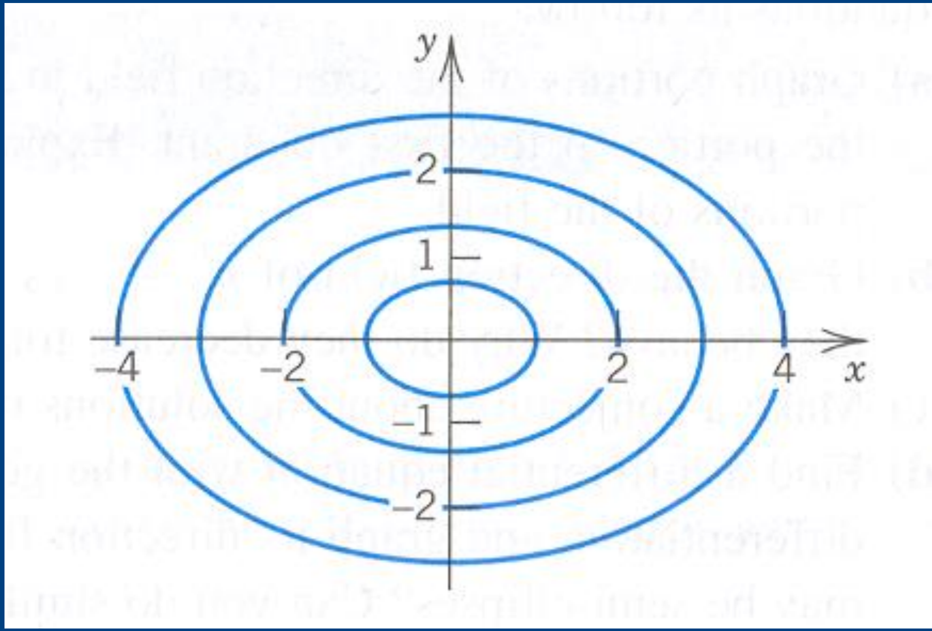
$$9yy' + 4x = 0$$

$$\Rightarrow \quad 9y\,dy = -4x\,dx$$

  - Integrating both sides

$$\frac{9}{2}y^2 = -2x^2 + c^*,$$

$$\frac{x^2}{9} + \frac{y^2}{4} = c, \left(c = \frac{c^*}{18} \right)$$



Example of Separable DE (2)

- Example 2

$$y' = 1 + y^2$$

$$\Rightarrow \quad \frac{dy}{1+y^2} = dx$$

$$\arctan y = x + c,$$
$$y = \tan(x + c)$$

  - Must introduce *c* immediately when the integration is performed.

$$\Rightarrow \quad y = \tan x + c$$

Example of Separable DE (3)

- Exponential Growth or Decay

$$y' = ky$$

Positive or negative

$$\Rightarrow \quad \frac{dy}{y} = k\,dx \qquad \ln|y| = kx + \tilde{c}$$

  - where

$$\left(\ln|y|\right)' = y'/y$$

$$\begin{cases} \left(\ln|y|\right)' = \left(\ln y\right)' = y'/y, \left(y > 0\right) \\ \left(\ln|y|\right)' = \left\{\ln\left(-y\right)\right\}' = \left(-y\right)'/\left(-y\right) = y'/y, \left(y < 0\right) \end{cases}$$

Example of Separable DE (4)

- Exponential Growth or Decay
  - Taking exponentials

$$|y| = e^{kx+\tilde{c}} = e^{kx}e^{\tilde{c}},$$

$$\Rightarrow \quad y = ce^{kx}$$

  - where

$$c = +e^{\tilde{c}} \left(y > 0 \right),$$

$$c = -e^{\tilde{c}} \left(y < 0 \right),$$

$$c = 0 \quad \Rightarrow \quad y \equiv 0$$

Example of Separable DE (5)

- Initial Value Problem

$$y' = -\frac{y}{x}, y(1) = 1$$

$$\Rightarrow \quad \frac{dy}{y} = -\frac{dx}{x},$$

$$\ln|y| = -\ln|x| + \tilde{c} = \ln\frac{1}{|x|} + \tilde{c}$$

  - Taking exponentials

$$\Rightarrow \quad y = cx$$

  - From IC

$$\Rightarrow \quad c = 1, y = x$$

Example of Separable DE (6)

- Heat Conduction

$$y' = -2xy, y(0) = 1$$

$$\Rightarrow \frac{dy}{y} = -2x dx, \ln|y| = -x^2 + \tilde{c},$$

$$|y| = e^{-x^2 + \tilde{c}}$$

  – Setting

$$c = +e^{\tilde{c}} \left(y > 0 \right),$$

$$c = -e^{\tilde{c}} \left(y < 0 \right),$$

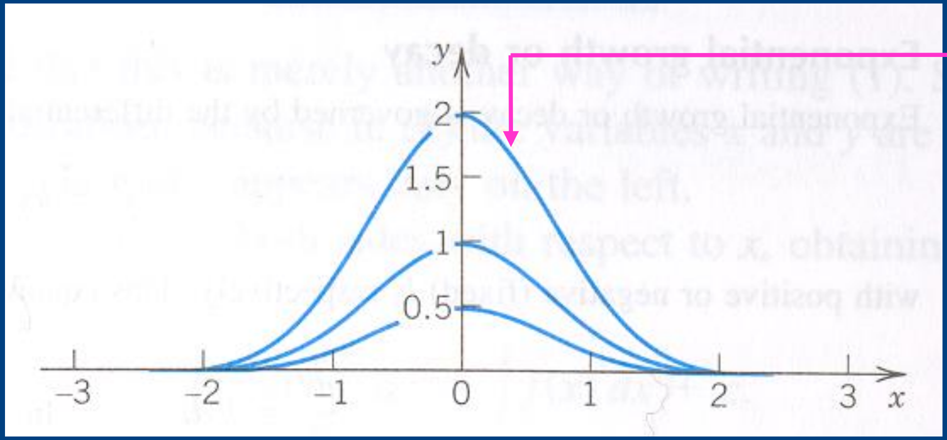
$$c = 0 \quad \Rightarrow \quad y \equiv 0$$

$$\Rightarrow y = ce^{-x^2}$$

Example of Separable DE (7)

- Bell-shaped Curve (Heat Conduction)
  - From IC

$$\Rightarrow \quad y = e^{-x^2}$$



c>0

Reduction to Separable Form

- Not separable,but can be made separable
  - Introducing a new unknown function

$$y' = g\left(\frac{y}{x}\right)$$

Any differentiable function of y/x: $(y/x)^3$, cos(y/x), …

  - Setting

$$y/x = u$$

$$\Rightarrow \quad y = ux,\ y' = u'x + u$$

  - Substituting

$$u'x + u = g(u),\ u'x = g(u) - u$$

  - Separation

$$\frac{du}{g(u) - u} = \frac{dx}{x}$$

Reduction to Separable Form

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  - Substituting

$$u'x + u = g(u), u'x = g(u) - u$$

  - Separation

$$\frac{du}{g(u) - u} = \frac{dx}{x}$$

Example of Reduction (1)

$$2xyy' = y^2 - x^2$$

$$\Rightarrow \quad y' = \frac{y^2}{2xy} - \frac{x^2}{2xy} = \frac{1}{2}\left(\frac{y}{x} - \frac{x}{y}\right)$$

– Applying $y' = u'x + u$

$$\Rightarrow \quad u'x + u = \frac{1}{2}\left(u - \frac{1}{u}\right)$$

– Simplification

$$u'x = -\frac{1}{2}\left(u + \frac{1}{u}\right) = -\frac{u^2+1}{2u}$$

– Separation

$$\frac{2udu}{1+u^2} = -\frac{dx}{x}$$

Example of Reduction (2)

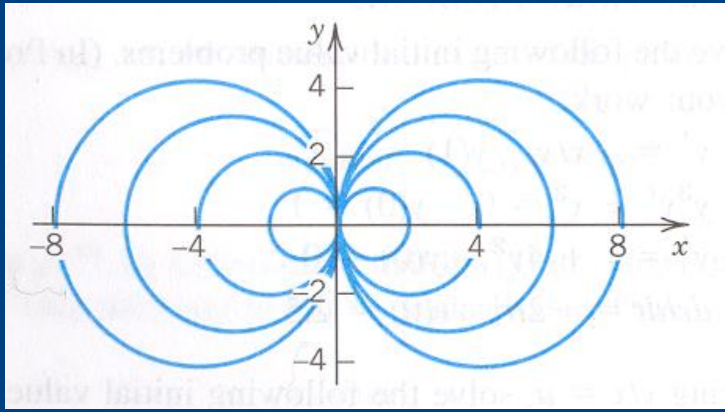
– Integrating

$$\ln\left(1+u^2\right) = -\ln|x| + c^*, 1+u^2 = \frac{c}{x}$$

– Replacing *u*

$$1+\left(\frac{y}{x}\right)^2 = \frac{c}{x}, x^2+y^2 = cx,$$

$$\Rightarrow \left(x-\frac{c}{2}\right)^2 + y^2 = \frac{c^2}{4}$$



Transformation Technique (1)

- Using $v = ay + bx + k$
  - Example

$$\left(2x - 4y + 5\right) y' + x - 2y + 3 = 0$$

  - Trying $v = x - 2y$

$$\Rightarrow \quad 2y = x - v,\ y' = \frac{1}{2}\left(1 - v'\right)$$

  - Substituting

$$\left(2v + 5\right) v' = 4v + 11$$

  - Separation

$$\frac{4v+10}{4v+11} dv = \frac{4v+11-1}{4v+11} dv = \left(1 - \frac{1}{4v+11}\right) dv = 2dx$$

Transformation Technique (2)

– Integrating

$$v - \frac{1}{4}\ln\left|4v + 11\right| = 2x + c^*$$

– Replacing

$$4x + 8y + \ln\left|4x - 8y + 11\right| = c$$

Implicit general solution

Modeling: Separable DE (1)

- Radiocarbon dating
  - Content of radioactive carbon ${}_6C^{14}$: decaying when an organism dies
  - Half life of ${}_6C^{14}$: 5730 yrs
  - Math model and sol.

$$y' = ky \quad \Rightarrow \quad y(t) = y_o e^{kt}$$

  - Determination of *k*

$$y_o e^{k \cdot 5730} = \frac{1}{2} y_o \quad \Rightarrow \quad e^{k \cdot 5730} = \frac{1}{2}, k = \frac{\ln(1/2)}{5730} = -0.000121$$

  - Time after which 25% of the original amount is still present

$$\mathrm{y_o e^{-0.000121t}} = \frac{1}{4} \mathrm{y_o}, \mathrm{t} = \frac{\ln(1/4)}{-0.000121} = 11,460(\mathrm{yrs})$$

Modeling: Separable DE (2)

- Mixing Problem
  - Tank contains 200 gal water, 40 lb of salt dissolved
  - 5 gal of brine (2 lb of salt per each gal) flows in per minute
  - Same amount flows out
  - Math model: amount of the salt at time $y(t)$

y' = salt inflow rate – salt outflow rate

  - Salt inflow rate = 10 (lb/min)
  - Salt outflow rate: 1 gal contains salt $y(t)/200 \times 5$ outflowing gal

$$\Rightarrow \quad 5y(t)/200 = y(t)/40 = 0.025y(t)$$

  - Final DE and IC

$$y' = 10 - 0.025y, \; y(0) = 40$$

Modeling: Separable DE (3)

– Separation

$$y' = -0.025(y-400), \quad \frac{dy}{y-400} = -0.025dt$$

– Integrating

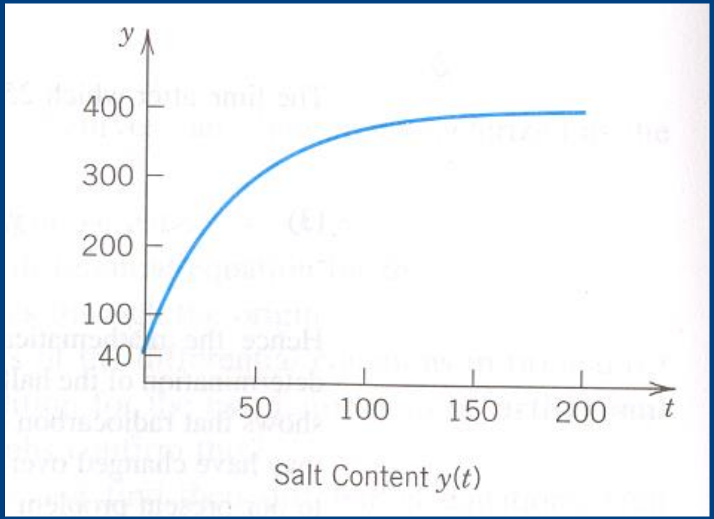
$$\ln|y-400| = -0.025t + \tilde{c}$$

– Taking exponentials

$$y - 400 = ce^{-0.025t}$$

– From IC

$$y(t) = 400 - 360e^{-0.025t} \; [\text{lb}]$$



Modeling: Separable DE (4)

- Heating Problem
  - Time rate of change of the temperature is proportional to the difference between its temperature and that of surroundings

$$\frac{dT}{dt} = k\left(T - T_A\right) = k\left(T - 32\right)$$

Newton's law of cooling

  - Separation

$$\Rightarrow \frac{dT}{T-32} = k\,dt,\ \ln\left|T-32\right| = kt + \tilde{c},$$

$$T\left(t\right) = 32 + ce^{kt}\ \left(c = e^{\tilde{c}}\right)$$

  - From IC

$$T\left(0\right) = 66 \Rightarrow c = 34,\ T\left(t\right) = 32 + 34e^{kt}$$

Modeling: Separable DE (5)

- Heating Problem
  - Determination of *k*

$$T(2) = 32 + 34e^{k\cdot 2} = 63,$$

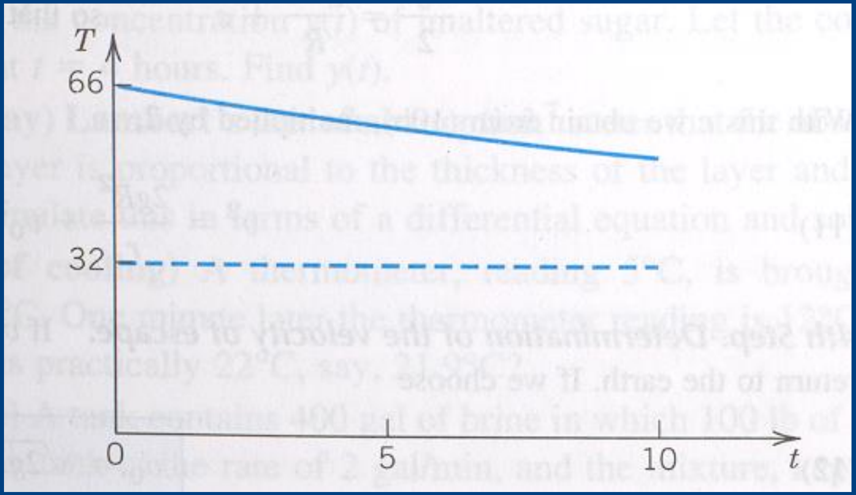
$$T(2) = 63 \Rightarrow e^{2k} = \frac{63-32}{34} = 0.911765,$$

$$k = \frac{1}{2}\ln 0.911765 = -0.046187$$

  - Temperature after 10 hrs of shutoff

$$\Rightarrow T(10) = 32 + 34e^{-0.046187\cdot 10}$$

$$= 53.4\left[{}^\circ\mathrm{F}\right]$$



Modeling: Separable DE (6)

- Velocity of escape
  - Newton's law of gravitation

$$F = G\frac{mM}{r^2} \qquad \left(G = 66.73\times10^{-12}\,\mathrm{m}^3/\mathrm{kg}\cdot\mathrm{s}^2\right)$$

  - At the earth's surface

$$W = mg = m\frac{GM}{R^2},\ g = \frac{GM}{R^2}$$ (*R*: earth's radius)

  - Rewriting

$$F = GM\frac{m}{r^2} = gR^2\frac{m}{r^2} = m\frac{gR^2}{r^2}$$

Acceleration due to gravitational force

Modeling: Separable DE (7)

- Velocity of escape
  - Acceleration

$$a(r) = -\frac{gR^2}{r^2}$$

Acts in the negative r-direction

  - Chain rule

$$a = \frac{dv}{dt} = \frac{dv}{dr}\frac{dr}{dt} = \frac{dv}{dr}v$$

  - Substituting

$$\frac{dv}{dr}v = -\frac{gR^2}{r^2}$$

Modeling: Separable DE (8)

- Velocity of escape
  - Separation

$$vdv = -gR^2 \frac{dr}{r^2}, \frac{v^2}{2} = \frac{gR^2}{r} + c$$

  - From initial velocity

$$r = R, v = v_o \quad \Rightarrow \quad \frac{v_o^2}{2} = \frac{gR^2}{R} + c, c = \frac{v_o^2}{2} - gR$$

  - Replacing *c*

$$\Rightarrow \quad v^2 = \frac{2gR^2}{r} + v_o^2 - 2gR$$

Modeling: Separable DE (9)

- Velocity of escape
  - Determination of velocity of escape

If $v_o = \sqrt{2gR}$ then $\dfrac{2gR^2}{r} > 0 \Rightarrow v^2 > 0$

  - Numerical value

$$v_o = \sqrt{2gR} = 11.2\,\mathrm{km/s} = 6.96\,\mathrm{mi/s}$$

Exact DE (1)

- Exact DE

$$M(x,y)\,dx + N(x,y)\,dy = 0$$

$$du = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy$$

- The given DE

$$\Rightarrow \quad du = 0$$

- General sol.

$$u(x,y) = c$$

Exact DE (2)

– To be an exact DE

$$\frac{\partial u}{\partial x} = M, \frac{\partial u}{\partial y} = N$$

– Assuming *M* and *N* are defined and have continuous 1st partial derivatives in xy-plane

$$\Rightarrow \quad \frac{\partial M}{\partial y} = \frac{\partial^2 u}{\partial y \partial x}, \frac{\partial M}{\partial x} = \frac{\partial^2 u}{\partial x \partial y}$$

– By assumption of continuity

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

← *Necessary, sufficient condition to be an exact DE*

Exact DE (3)

– Way to find *u*(*x*,*y*)

$$u = \int M dx + k(y)$$

Constant of integration

$$u = \int N dy + l(y)$$

- Example of Exact DE

$$\left(x^3 + 3xy^2\right)dx + \left(3x^2y + y^3\right)dy = 0$$

– Exactness

$$M = x^3 + 3xy^2, N = 3x^2y + y^3$$

$$\frac{\partial M}{\partial y} = 6xy, \frac{\partial N}{\partial x} = 6xy$$

Example of Exact DE (1)

– Implicit sol.

$$u = \int M dx + k(y) = \int \left(x^3 + 3xy^2\right) dx + k(y)$$

$$= \frac{1}{4}x^4 + \frac{3}{2}x^2 y^2 + k(y)$$

– To find *k*(*y*)

$$\frac{\partial u}{\partial y} = 3x^2 y + \frac{dk}{dy} = N = 3x^2 y + y^3$$

$$\frac{dk}{dy} = y^3, k = \frac{1}{4}y^4 + \tilde{c}$$

Example of Exact DE (2)

– Final sol.

$$\Rightarrow \quad u = \frac{1}{4}\left(x^4 + 6x^2y^2 + y^4\right) = c$$

– Checking the sol.

$$\frac{1}{4}\left(4x^3 + 12xy^2 + 12x^2yy' + 4y^3y'\right) = 0$$

$$\Rightarrow \quad M + Ny' = 0$$

Example of Exact DE (3)

- Initial Value Problem

$$(\sin x \cosh y)dx - (\cos x \sinh y)dy = 0, y(0) = 3$$

  - Solution procedure

$$u = \int \sin x \cosh y dx + k(y) = -\cos x \cosh y + k(y)$$

  - To find *k*(*y*)

$$\frac{\partial u}{\partial y} = -\cos x \sinh y + \frac{dk}{dy}$$

$$\frac{dk}{dy} = 0, k = const.$$

Example of Exact DE (4)

- Initial Value Problem
  - General solution

$$u = const., \cos x \cosh y = const.$$

  - Particular solution

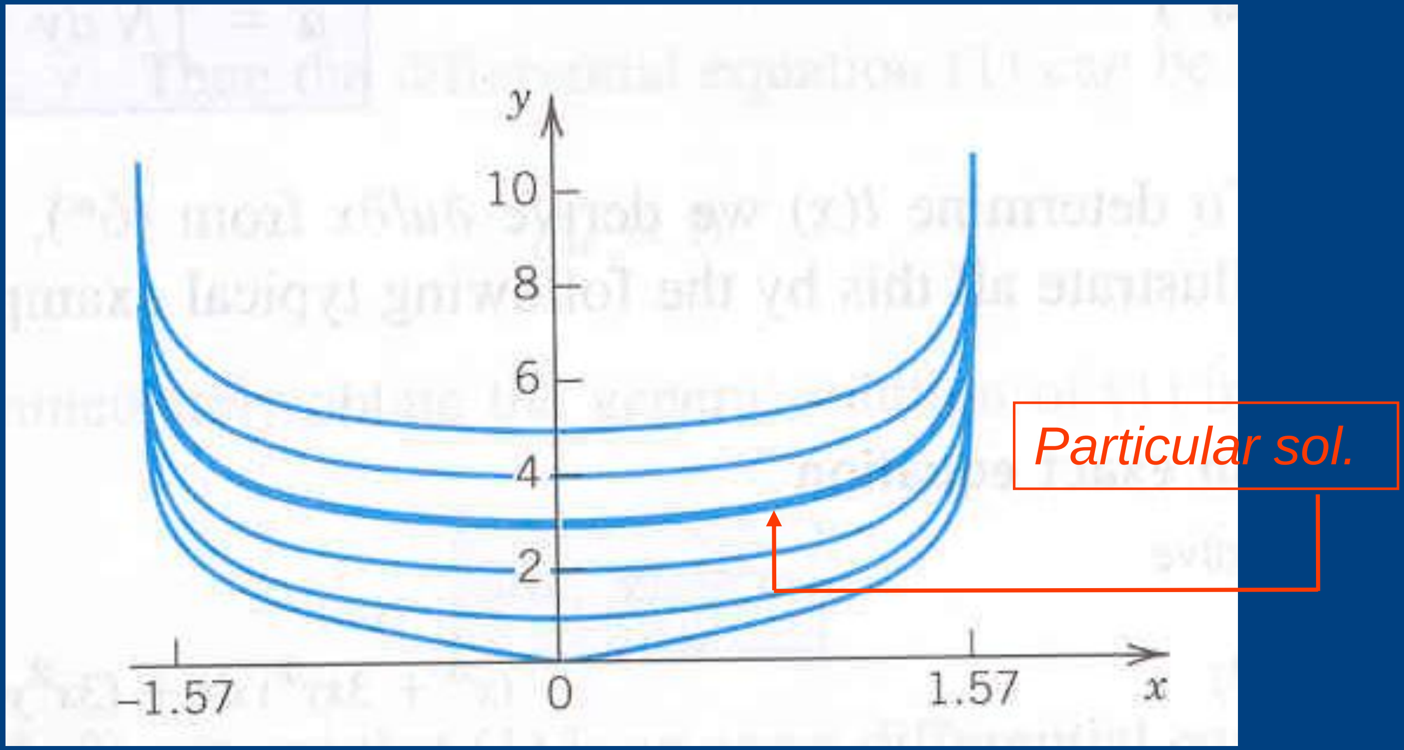
$$\cos 0 \cosh 3 = 10.07 = const.,$$

$$\Rightarrow \quad \cos x \cosh y = 10.07$$

  - Checking

$$\left(\cos x \cosh y\right)' = -\sin x \cosh y + \cos x \left(\sinh y\right) y' = 0$$

Example of Exact DE (5)



Example of Non-Exact DE

- Non-Exact DE

$$-ydx + xdy = 0$$

$$M = -y, N = x$$

$$\frac{\partial M}{\partial y} = -1, \frac{\partial N}{\partial x} = 1$$

  - Solution procedure

$$u = \int M dx + k(y) = -xy + k(y)$$

$$\frac{\partial u}{\partial y} = -x + k'(y)$$

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Integrating Factor (1)

- Non-exact DE in the previous example

$$-ydx + xdy = 0$$

  - Multiply by a certain factor

$$\frac{-ydx + xdy}{x^2} = -\frac{y}{x^2}dx + \frac{1}{x}dy = d\left(\frac{y}{x}\right) = 0$$

Exact DE

  - General solution

$$\Rightarrow \quad \frac{y}{x} = c = const.$$

Integrating Factor (2)

- Non-exact DE in the previous example
  - Checking of exactness

$$M = -\frac{y}{x^2}, \frac{\partial M}{\partial y} = -\frac{1}{x^2},$$

$$N = \frac{1}{x}, \frac{\partial N}{\partial x} = -\frac{1}{x^2}$$

Integrating Factor (3)

- Reduction to an exact DE
  - Nonexact DE

$$P(x,y)\,dx + Q(x,y)\,dy = 0$$

  - Multiply by a function *F*

$$FPdx + FQdy = 0$$

Exact DE

Integrating factor

Example of Integrating Factor (1)

- Previous non-exact DE
  - Non-exact DE

$$-ydx + xdy = 0$$

  - Integrating factor found

$$F = \frac{1}{x^2}$$

  - Reduced exact DE

$$FPdx + FQdy = \frac{-ydx + xdy}{x^2} = d\left(\frac{y}{x}\right) = 0$$

$$\Rightarrow \quad \frac{y}{x} = const.$$

Example of Integrating Factor (2)

- Previous non-exact DE
  - Other integrating factors

$$\frac{1}{y^2} \Rightarrow \frac{-ydx + xdy}{y^2} = -d\left(\frac{x}{y}\right)$$

$$\frac{1}{xy} \Rightarrow \frac{-ydx + xdy}{xy} = -d\left(\ln\frac{x}{y}\right)$$

$$\frac{1}{x^2 + y^2} \Rightarrow \frac{-ydx + xdy}{x^2 + y^2} = d\left(\arctan\frac{y}{x}\right)$$

Finding Integrating Factors (1)

- By inspection or some trials
- General case
  - Reduced exact DE

$$FPdx + FQdy = 0$$

  - To be exact

$$\frac{\partial}{\partial y}(FP) = \frac{\partial}{\partial x}(FQ)$$

  - By product rule

$$F_y P + FP_y = F_x Q + FQ_x$$

Finding Integrating Factors (2)

- General case
  - Assume $F=F(x)$, then

$$F_y = 0, F_x = F' = \frac{dF}{dx}$$

  - Criterion becomes

$$\frac{1}{F}\frac{dF}{dx} = \frac{1}{Q}\left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\right) = R(x)$$

  - Integrating factor F

$$F(x) = \exp \int R(x) dx$$

Finding Integrating Factors (3)

- General case
  - In case $F=F(y)$

$$\frac{1}{F}\frac{dF}{dy}=\frac{1}{P}\left(\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}\right)=\tilde{R}\left(y\right)$$

  - Integrating factor F

$$F\left(y\right)=\exp\int\tilde{R}\left(y\right)dy$$

Example of finding Integrating Factor (1)

- Initial Value Problem

$$2\sin\left(y^2\right)dx + xy\cos\left(y^2\right)dy = 0,\ y(2) = \sqrt{\frac{\pi}{2}}$$

  - Exactness

$$P = 2\sin\left(y^2\right),\ Q = xy\cos\left(y^2\right)$$

$$P_y = 4y\cos\left(y^2\right) \neq Q_x = y\cos\left(y^2\right)$$

Not exact

Example of finding Integrating Factor (2)

– Apply $F = F(x)$

$$R = \frac{1}{Q}\left(P_y - Q_x\right)$$

$$= \frac{1}{xy\cos\left(y^2\right)}\left[4y\cos\left(y^2\right) - y\cos\left(y^2\right)\right] = \frac{3y}{xy} = \frac{3}{x}$$

– Integrating factor

$$F\left(x\right) = \exp\int R\left(x\right)dx = \exp\int \frac{3}{x}dx = x^3$$

– Reduced exact DE

$$2x^3\sin\left(y^2\right)dx + x^4 y\cos\left(y^2\right)dy = 0$$

Example of finding Integrating Factor (3)

– Check for exactness again

$$\frac{\partial}{\partial y}\left[2x^3\sin\left(y^2\right)\right]=4x^3y\cos\left(y^2\right)=\frac{\partial}{\partial x}\left[x^4y\cos\left(y^2\right)\right]$$

– Solution procedure for exact DE

$$u=\int 2x^3\sin\left(y^2\right)dx=\frac{1}{2}x^4\sin\left(y^2\right)+k\left(y\right)$$

– To find *k*(*y*)

$$u_y=x^4y\cos\left(y^2\right)+k'\left(y\right)=x^4y\cos\left(y^2\right)$$

$$\frac{dk}{dy}=0, k=const.$$

Example of finding Integrating Factor (4)

– General sol.

$$u(x,y)=\frac{1}{2}x^4\sin\left(y^2\right)=c=const.$$

– Particular sol.

$$\frac{1}{2}2^4\sin\frac{\pi}{2}=8=c$$

$$\frac{1}{2}x^4\sin\left(y^2\right)=8$$

$$\Rightarrow \quad x^4\sin\left(y^2\right)=16$$

Example of finding Integrating Factor (5)