Rock Mechanics & Experiment
암석역학 및 실험

Lecture 2. Stress
Lecture 2. 응력

Lecture 3. Strain/Hooke's Law/Equilibrum Equation
Lecture 3. 변형율, 훅의 법칙, 평형방정식

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Last Lecture

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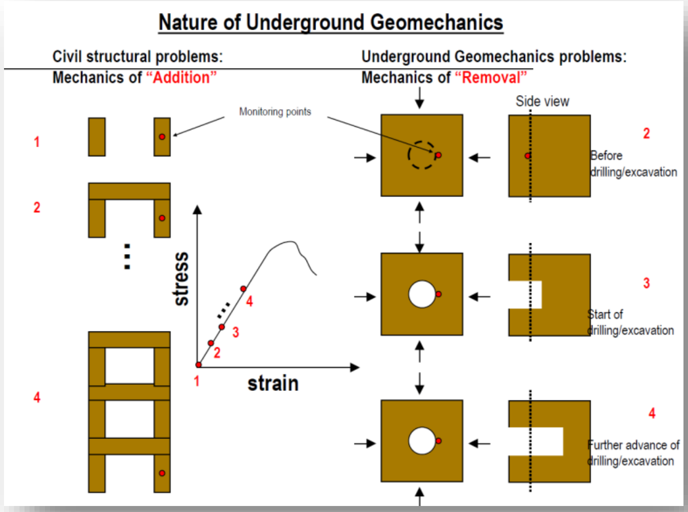
- Introduction to Rock Mechanics/Geomechanics
  - Terminology
  - Area of Applications
  - Nature of Rock Mechanics/Geomechanics
- Applications of Rock Mechanics/Geomechanics
- Methodology to solve Rock Mechanics/Geomechanics problems



Outline

- Stress
- Plane Stress
- Principal Stresses and Maximum Shear Stresses
- Mohr's Circle
- Strain
- Hooke's Law
- Equilibrium Equation

Stress
Normal stress & Shear stress

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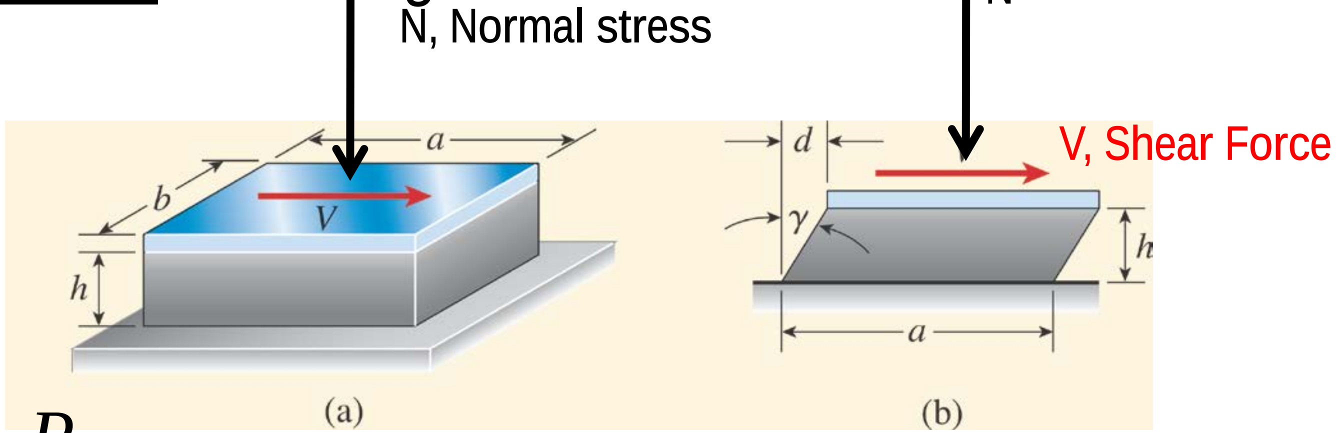
- <u>Stress</u>: *average force per unit area*
- <u>Normal stress</u>: *act in perpendicular to cut surface*
- <u>Shear stress</u>: acts *tangential* to the surface of the material



$$\sigma = \frac{P}{A}$$

$$\tau = \frac{V}{A}$$

P: Axial Force (N)
V: Shear Force
A: cross sectional area (a x b)

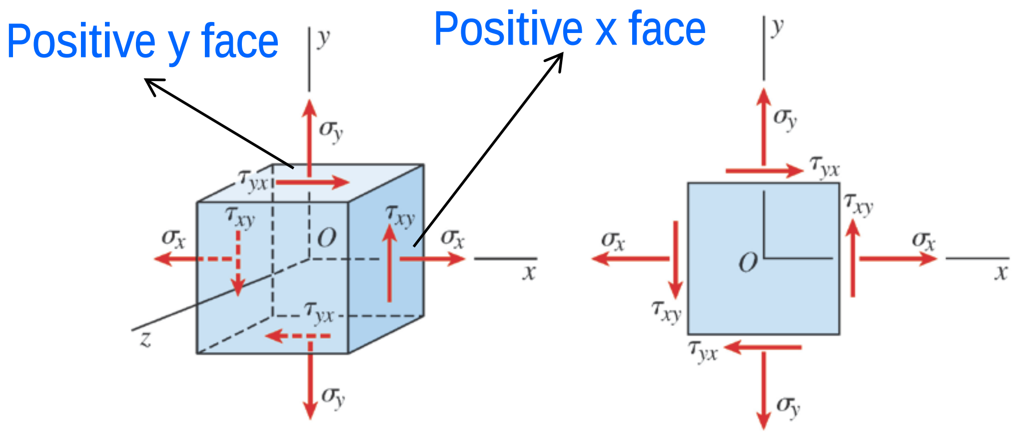
– Unit: N/m^2 = MPa

Stress
Introduction

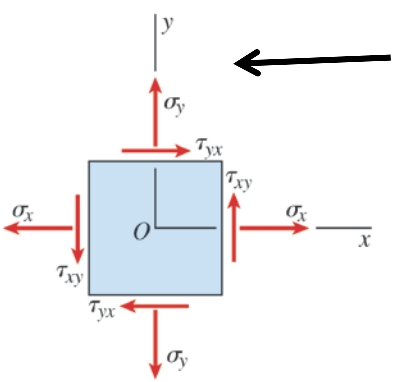
- ONE intrinsic state of stress can be expressed in many many different ways depending on the reference axis (or orientation of element).
  - Similarity to force: One intrinsic state of force (vector) can be expressed similarly depending on the reference axis.
  - Difference from force: we use different transformation equations from those of vectors
  - Stress is NOT a vector BUT a (2nd order) tensor → they do not combine according to the parallelogram law of addition

Stress
Definition

- Normal stress, σ : subscript identify the face on which the stress act. Ex) σ_x
- Shear stress, τ : 1st subscript denotes the face on which the stress acts, and the 2nd gives the direction on that face. Ex) τ_{xy}

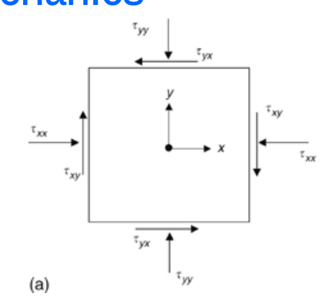


- Sign convention
  - Normal stress:
    typical mechanics: tension (+), compression (-)
    rock/geomechanics: tension (-), compression (+)
  - Shear stress:
    - acts on a positive face of an element in the positive direction of an axis (+) : plus-plus or minus-minus
    - acts on a positive face of an element in the negative direction of an axis (-): plus-minus or minus-plus



← Positive normal & shear stresses in other mechanics

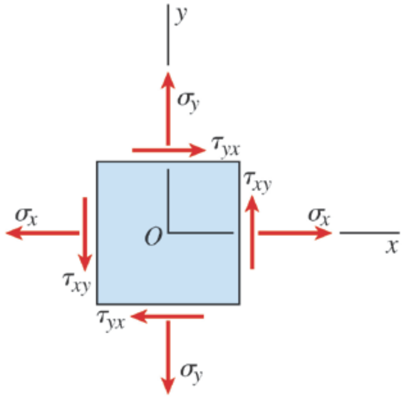
in rock mechanics →



Stress
Definition

- Shear stresses in perpendicular planes are equal in magnitude and directions shown in the below.
  - Derived from the moment equilibrium

$$\tau_{xy} = \tau_{yx}$$



- In 2D (plane stress), we need three (independent) components to describe a complete state of stress

σ_x σ_y τ_{xy}

$$\begin{pmatrix} \sigma_x & \tau_{xy} \\ \tau_{yx} & \sigma_y \end{pmatrix}$$

Stress
Definition

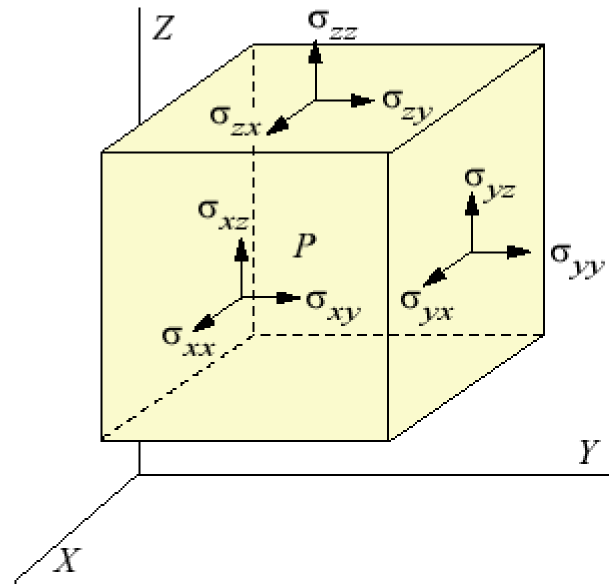
- Stress in 3D

$$\begin{pmatrix} \sigma_x & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \sigma_y & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \sigma_z \end{pmatrix}$$

Tensor form

$$\begin{pmatrix} \sigma_x \\ \sigma_y \\ \sigma_z \\ \tau_{yz} \\ \tau_{xz} \\ \tau_{xy} \end{pmatrix}$$

matrix form



$$\tau_{xy} = \tau_{yx}$$

$$\tau_{xz} = \tau_{zx}$$

$$\tau_{yz} = \tau_{zy}$$

→ 6 independent components

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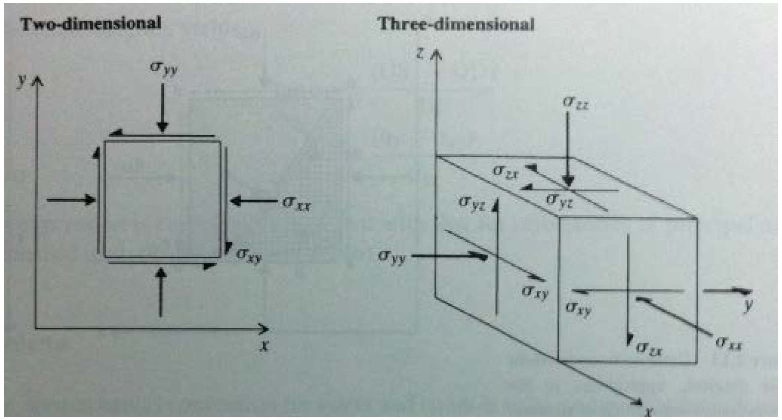
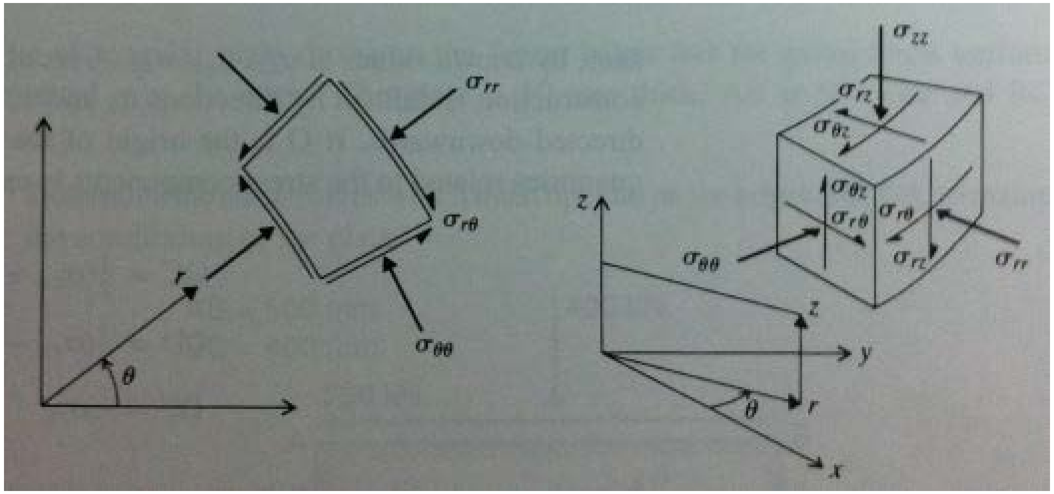
- 2D & 3D Cartesian Coordinates

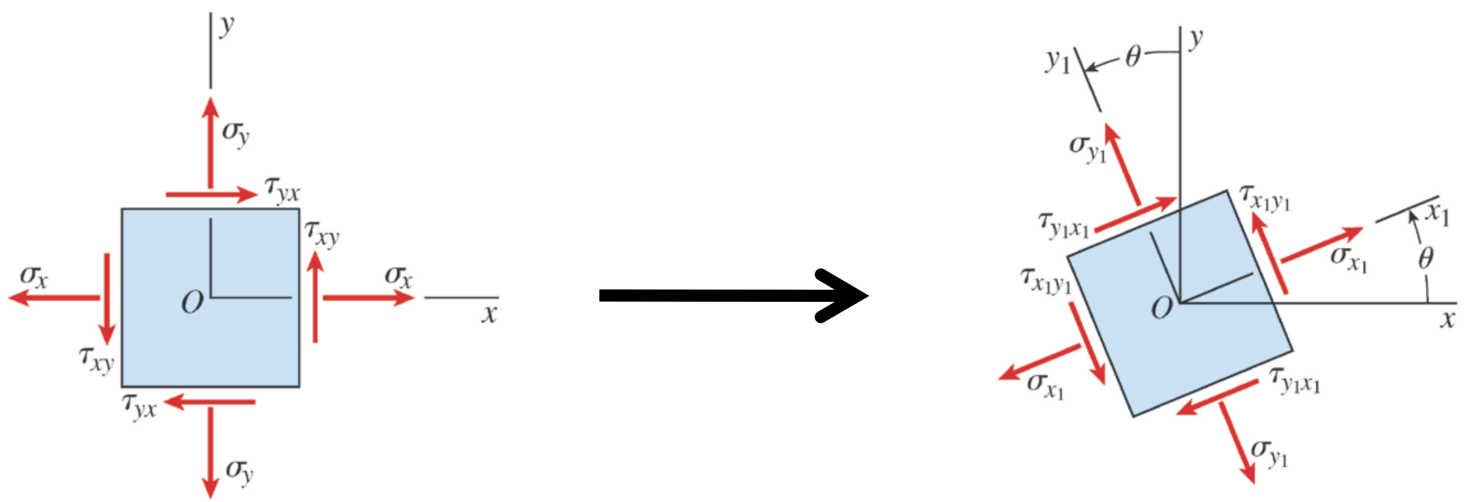


- Polar & Cylindrical coordinates

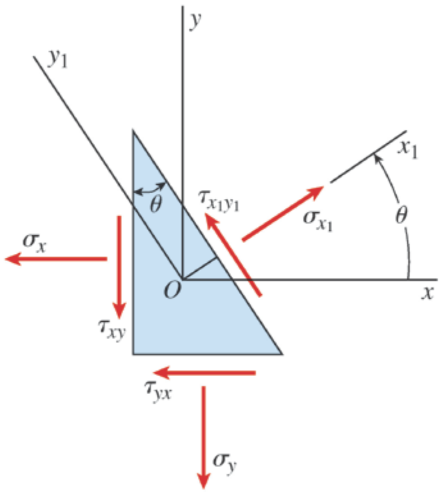


Stress
Stresses on inclined sections

- Stresses acting on inclined sections assuming that σ_x, σ_y, τ_{xy} are known.
  - x_1y_1 axes are rotated counterclockwise through an angle θ
  - Strategy??? →
  - wedge shaped stress element

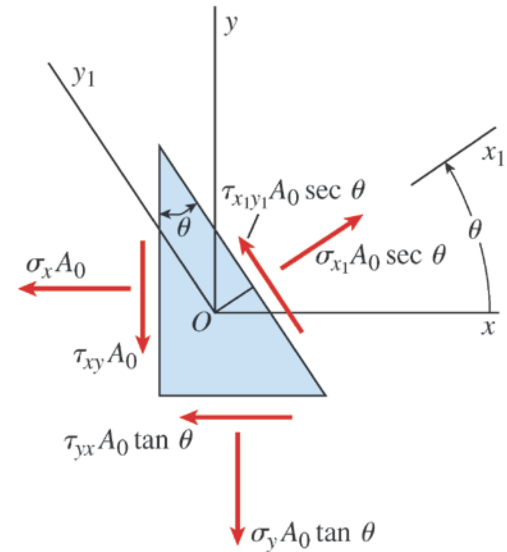


Stress
Stresses on inclined sections



Free Body Diagram →

express in terms of "Force"



- Force Equilibrium Equations in x_1 and y_1 directions

$$\sum F_{x_1} = \sigma_{x_1} A_0 \sec\theta - \sigma_x A_0 \cos\theta - \tau_{xy} A_0 \sin\theta$$
$$- \sigma_y A_0 \tan\theta \sin\theta - \tau_{yx} A_0 \tan\theta \cos\theta = 0$$

$$\sum F_{y_1} = \tau_{x_1 y_1} A_0 \sec\theta + \sigma_x A_0 \sin\theta - \tau_{xy} A_0 \cos\theta$$
$$- \sigma_y A_0 \tan\theta \cos\theta + \tau_{yx} A_0 \tan\theta \sin\theta = 0$$

- Using $\tau_{xy} = \tau_{yx}$ and simplifying

$$\sigma_{x_1} = \sigma_x \cos^2\theta + \sigma_y \sin^2\theta + 2\tau_{xy}\sin\theta\cos\theta$$

$$\tau_{x_1y_1} = -\left(\sigma_x - \sigma_y\right)\sin\theta\cos\theta + \tau_{xy}\left(\cos^2\theta - \sin^2\theta\right)$$

$$\sigma_{x_1} = \sigma_x \qquad \tau_{x_1y_1} = \tau_{xy}$$

  - When θ = 90,

$$\sigma_{x_1} = \sigma_y \qquad \tau_{x_1y_1} = -\tau_{xy}$$

Stress
Transformation Equations

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- From half angle and double angle formulas

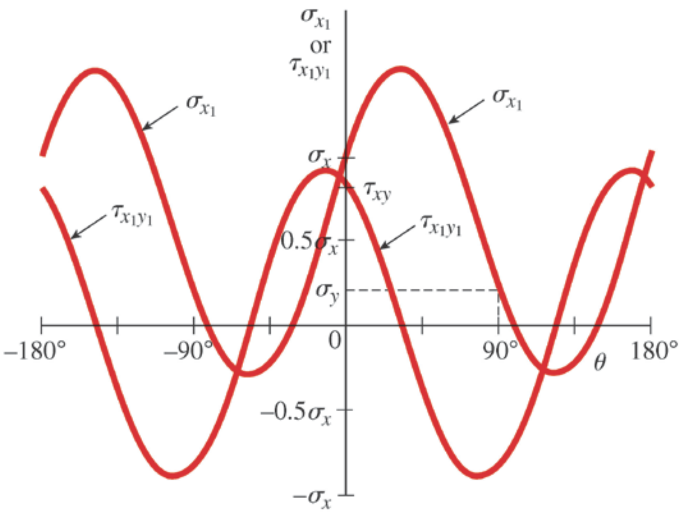
$$\cos^2\theta = \frac{1}{2}(1+\cos 2\theta) \qquad \sin^2\theta = \frac{1}{2}(1-\cos 2\theta) \qquad \sin\theta\cos\theta = \frac{1}{2}\sin 2\theta$$

- Transformation equations for plane stress

$$\sigma_{x_1} = \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2}\cos 2\theta + \tau_{xy}\sin 2\theta \qquad \tau_{x_1 y_1} = -\frac{\sigma_x - \sigma_y}{2}\sin 2\theta + \tau_{xy}\cos 2\theta$$

  – Intrinsic state of stress is the same but the reference axis are different
  – Derived solely from equilibrium→ applicable to stresses in any kind of materials (linear or nonlinear or elastic or inelastic)

Stress
Transformation Equations



With $\boldsymbol{\sigma}_y = 0.2\boldsymbol{\sigma}_x$ & $\boldsymbol{\tau}_{xy} = 0.8\,\boldsymbol{\sigma}_x$

- For σ_{y1}, $\theta \rightarrow \theta + 90$,
  - Making summations

$$\sigma_{y_1} = \frac{\sigma_x + \sigma_y}{2} - \frac{\sigma_x - \sigma_y}{2}\cos 2\theta - \tau_{xy}\sin 2\theta$$

$$\sigma_{x_1} + \sigma_{y_1} = \sigma_x + \sigma_y$$

  - Sum of the normal stresses acting on perpendicular faces of plane stress elements is constant and independent of θ

Stress
Special Cases of Stress State

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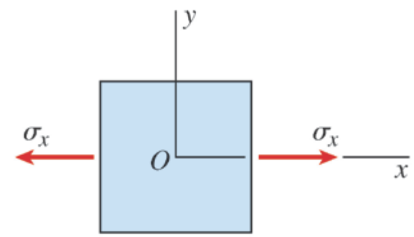
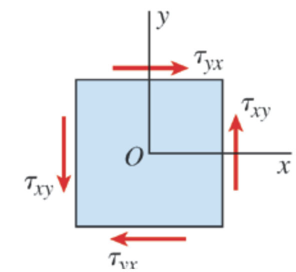
- Uniaxial stress

$$\sigma_{x_1} = \frac{\sigma_x}{2}(1+\cos 2\theta) \qquad \tau_{x_1 y_1} = -\frac{\sigma_x}{2}\sin 2\theta$$



- Pure Shear

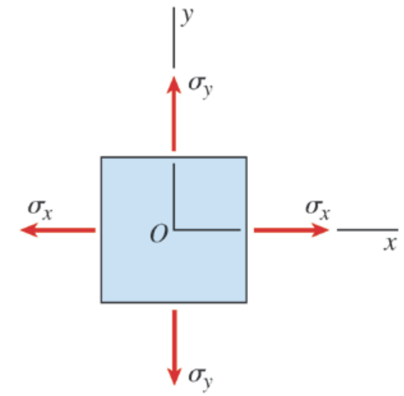
$$\sigma_{x_1} = \tau_{xy}\sin 2\theta \qquad \tau_{x_1 y_1} = \tau_{xy}\cos 2\theta$$



- Biaxial Stress

$$\sigma_{x_1} = \frac{\sigma_x+\sigma_y}{2} + \frac{\sigma_x-\sigma_y}{2}\cos 2\theta$$

$$\tau_{x_1 y_1} = -\frac{\sigma_x-\sigma_y}{2}\sin 2\theta$$

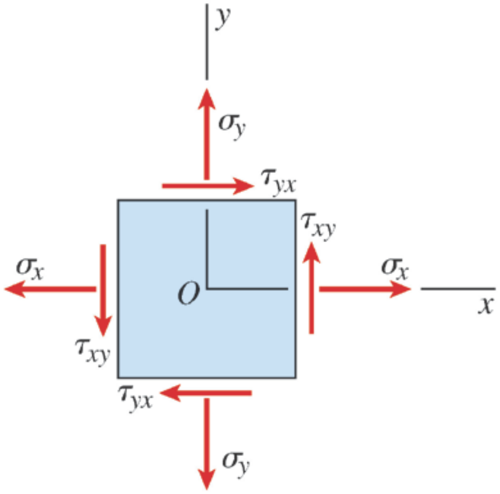


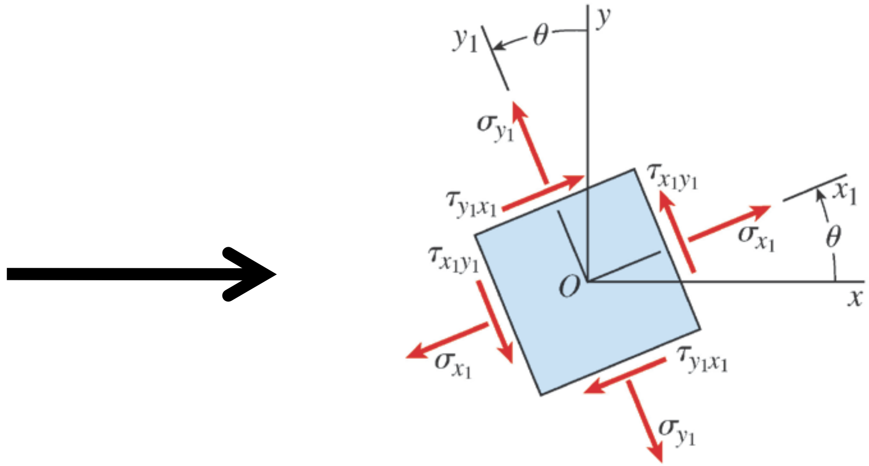
Stress
Stresses on inclined sections

- Stresses acting on inclined sections assuming that σ_x, σ_y, τ_{xy} are known.
  - x_1y_1 axes are rotated counterclockwise through an angle θ

$$\sigma_{x_1} = \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2}\cos 2\theta + \tau_{xy}\sin 2\theta$$

$$\tau_{x_1y_1} = -\frac{\sigma_x - \sigma_y}{2}\sin 2\theta + \tau_{xy}\cos 2\theta$$





- A different way of obtaining transformed stresses
  - For vector

$$\begin{pmatrix} F_{x1} \\ F_{y1} \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} F_x \\ F_y \end{pmatrix}$$

  - For tensor (stress)

$$\begin{pmatrix} \sigma_{x1} & \tau_{x1y1} \\ \tau_{x1y1} & \sigma_{y1} \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} \sigma_x & \tau_{xy} \\ \tau_{xy} & \sigma_y \end{pmatrix} \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}^T$$

$$= \quad \sigma_{x_1} = \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2}\cos 2\theta + \tau_{xy}\sin 2\theta \qquad \tau_{x_1y_1} = -\frac{\sigma_x - \sigma_y}{2}\sin 2\theta + \tau_{xy}\cos 2\theta$$

Mohr's Circle for Plane Stress

- Mohr's Circle
  - Graphical representation of the transformation equation for stress
  - Extremely useful to visualize the relationship between σ_x and τ_{xy}
  - Also used for calculating principal stresses, maximum shear stresses, and stresses on inclined sections
  - Also used for other quantities of similar nature such as strain.

Mohr's Circle for Plane Stress
Equations of Mohr's Circle

- The transformation Equations for plane stress

$$\sigma_{x_1} = \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2}\cos 2\theta + \tau_{xy}\sin 2\theta \qquad \tau_{x_1 y_1} = -\frac{\sigma_x - \sigma_y}{2}\sin 2\theta + \tau_{xy}\cos 2\theta$$

– Rearranging the above equations

$$\sigma_{x_1} - \frac{\sigma_x + \sigma_y}{2} = \frac{\sigma_x - \sigma_y}{2}\cos 2\theta + \tau_{xy}\sin 2\theta$$

$$\tau_{x_1 y_1} = -\frac{\sigma_x - \sigma_y}{2}\sin 2\theta + \tau_{xy}\cos 2\theta$$

– Square both sides of each equation and sum the two equations

$$(\sigma_{x_1} - \frac{\sigma_x + \sigma_y}{2})^2 + \tau_{x_1 y_1}^2 = (\frac{\sigma_x - \sigma_y}{2})^2 + \tau_{xy}^2$$

– Equation of a circle in standard algebraic form

$$(x - x_0)^2 + y^2 = R^2$$

Equations of Mohr's Circle

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$$(\sigma_{x_1} - \frac{\sigma_x + \sigma_y}{2})^2 + \tau_{x_1 y_1}^2 = (\frac{\sigma_x - \sigma_y}{2})^2 + \tau_{xy}^2$$

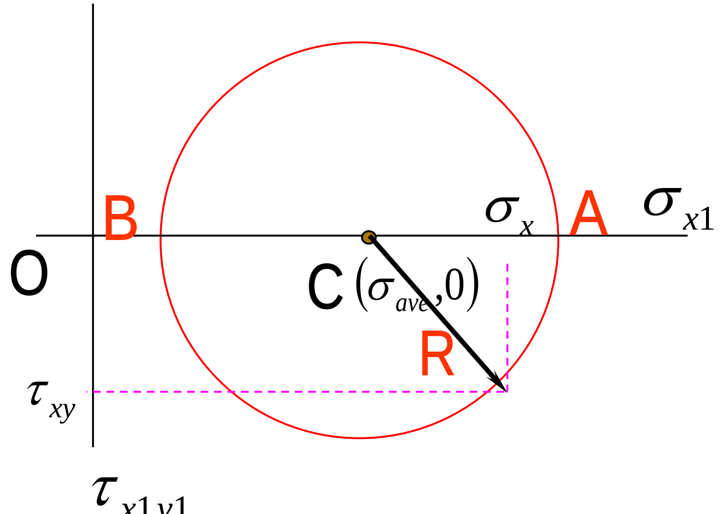
Centre ($\boldsymbol{\sigma}_{ave}$, 0)

$$\sigma_{ave} = \frac{\sigma_x + \sigma_y}{2}$$

(Radius)^2 of a circle

$$R = \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}$$

$$(\sigma_{x_1} - \sigma_{ave})^2 + \tau_{x_1 y_1}^2 = R^2$$



Recognized by Mohr in 1882

Mohr's Circle for Plane Stress
Alternative way of understanding

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- The transformation Equations for plane stress

$$\sigma_{x_1} = \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2}\cos 2\theta + \tau_{xy}\sin 2\theta \qquad \tau_{x_1 y_1} = -\frac{\sigma_x - \sigma_y}{2}\sin 2\theta + \tau_{xy}\cos 2\theta$$

  – In terms of principal stresses (shear stress becomes zero)

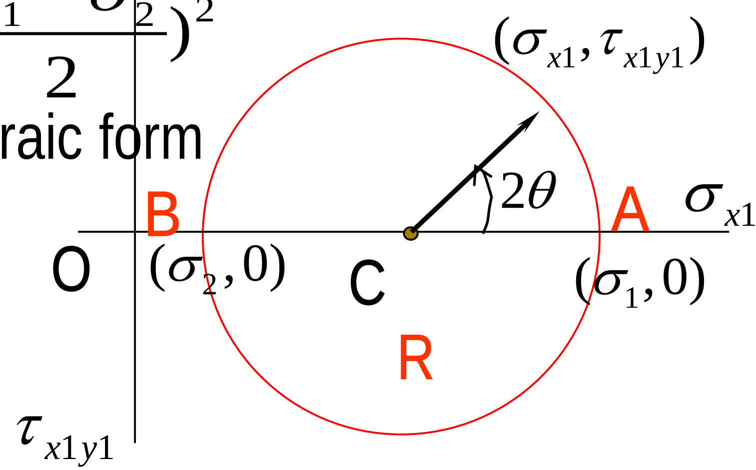
$$\sigma_{x_1} - \frac{\sigma_1 + \sigma_2}{2} = \frac{\sigma_1 - \sigma_2}{2}\cos 2\theta \qquad \tau_{x_1 y_1} = -\frac{\sigma_1 - \sigma_2}{2}\sin 2\theta$$

  – Square both sides of each equation and sum the two equations

$$(\sigma_{x_1} - \frac{\sigma_x + \sigma_y}{2})^2 + \tau_{x_1 y_1}^2 = (\frac{\sigma_1 - \sigma_2}{2})^2$$

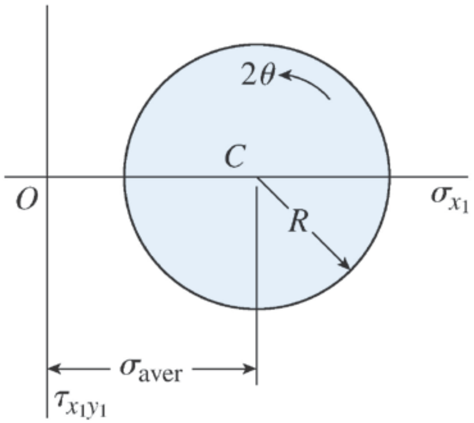
  – Equation of a circle in standard algebraic form

$$(x - x_0)^2 + y^2 = R^2$$

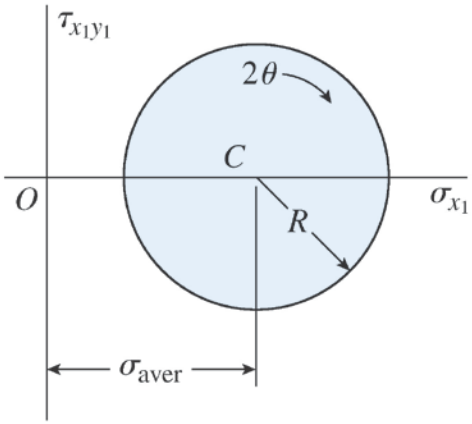


Mohr's Circle for Plane Stress

Two forms of Mohr's Circle



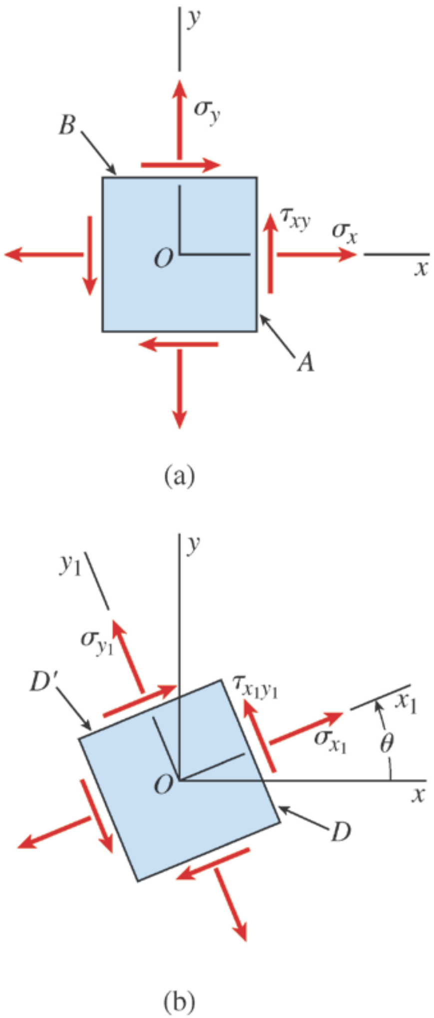
- Shear stress (+) ↓ θ (+) counterclockwise
  - Chosen for this course!



- Shear stress (+) ↑ θ (+) clockwise

Mohr’s Circle for Plane Stress

Construction of Mohr’s Circle



Calculation of R from geometry

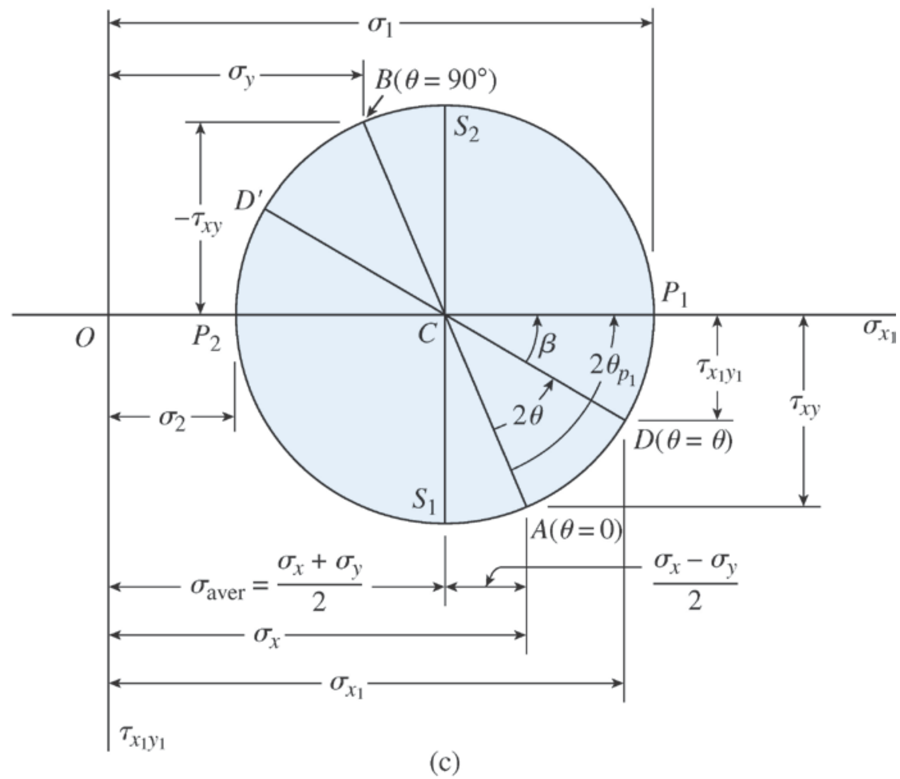
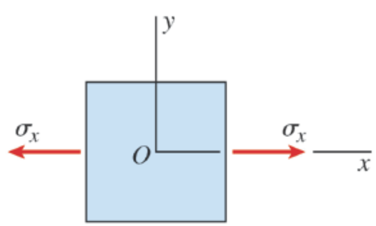


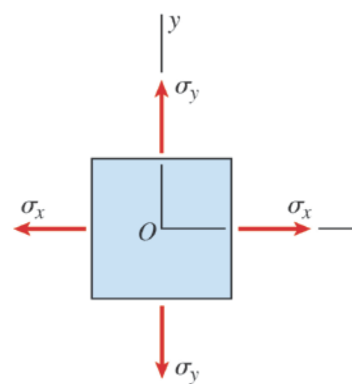
FIG. 7-16 Construction of Mohr’s circle for plane stress

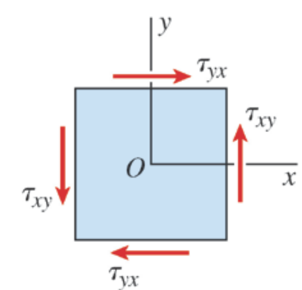
Mohr's Circle for Plane Stress
General Comments

- We can find the stresses acting on any inclined plane, as well as principal stresses and maximum shear stresses from Mohr's Circle.

- Special cases of
  - Uniaxial stresses
  - Biaxial stresses
  - Pure shear







Principal stresses

- Principal Stresses (주응력)
  - Maximum normal stress & Minimum normal stress
  - Strategy?
  - Taking derivatives of normal stress with respect to θ

$$\frac{d\sigma_{x1}}{d\theta} = -\left(\sigma_x - \sigma_y\right)\sin 2\theta + 2\tau_{xy}\cos 2\theta = 0$$

$$\longrightarrow \quad \tan 2\theta_p = \frac{2\tau_{xy}}{\sigma_x - \sigma_y}$$

  - θ_p:orientation of the principal planes (planes on which the principal stresses act)

- Principal stresses can be obtained by substituting θ_p

Principal stresses

- Two values of angle $2\theta_p$: 0 °~ 360 °
  - One : 0 °~ 180 °
  - The other (differ by 180°) : 180 °~ 360 °
- Two values of angle θ_p: 0 °~ 180 °→ Principal angles
  - One : 0 °~ 90 °
  - The other (differ by 90°) : 90 °~ 180 °

→ <u>principal stresses occur on mutually perpendicular planes</u>

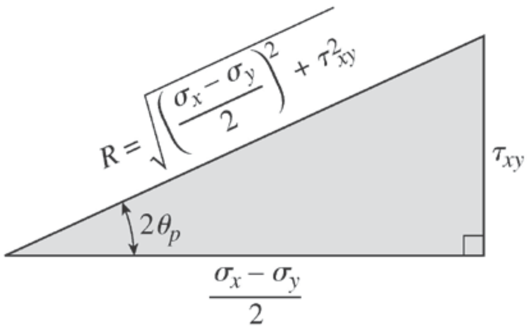
Principal Stresses and Maximum Shear Stresses
Principal stresses

- Calculation of principal stresses

$$\sigma_{x_1} = \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2}\cos 2\theta + \tau_{xy}\sin 2\theta$$

? ?

$$\tan 2\theta_p = \frac{2\tau_{xy}}{\sigma_x - \sigma_y}$$



$$\cos 2\theta_p = \frac{\sigma_x - \sigma_y}{2R}$$

$$\sin 2\theta_p = \frac{\tau_{xy}}{R}$$

$$R = \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^{\,2}}$$

- By substituting,

$$\sigma_{x_1} = \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2}\left(\frac{\sigma_x - \sigma_y}{2R}\right) + \tau_{xy}\left(\frac{\tau_{xy}}{R}\right)$$

$$\sigma_1 = \frac{\sigma_x + \sigma_y}{2} + \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^{\,2}}$$

Larger of two principal stresses
= Maximum Principal Stress

Principal stresses

- The smaller of the principal stresses (= minimum principal stress)

$$\sigma_1 + \sigma_2 = \sigma_x + \sigma_y \longrightarrow \sigma_2 = \frac{\sigma_x + \sigma_y}{2} - \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^{\ 2}}$$

- Putting into shear stress transformation equation

$$\tau_{x_1 y_1} = -\frac{\sigma_x - \sigma_y}{2}\sin 2\theta + \tau_{xy}\cos 2\theta$$

$$0 = -\frac{\sigma_x - \sigma_y}{2}\sin 2\theta + \tau_{xy}\cos 2\theta$$

Same equation for principal angles

  - Shear stresses are <u>zero</u> on the principal stresses

- Principal stresses

$$\sigma_{1,2} = \frac{\sigma_x + \sigma_y}{2} \pm \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^{\ 2}}$$

Principal stresses

- Alternative way of finding the smaller of the principal stresses (= minimum principal stress)

$$\cos(2\theta_p + 180) = -\frac{\sigma_x - \sigma_y}{2R} \qquad \sin(2\theta_p + 180) = -\frac{\tau_{xy}}{R}$$

- By substituting into the transformation equations

$$\sigma_2 = \frac{\sigma_x + \sigma_y}{2} - \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}{}^2}$$

Principal Angles

- Principal angles correspond to principal stresses

$$\theta_{p1} \longrightarrow \sigma_1$$

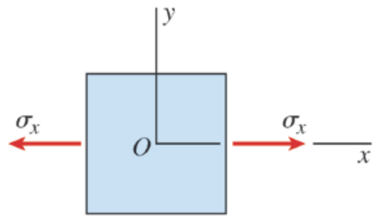
$$\theta_{p2} \longrightarrow \sigma_2$$

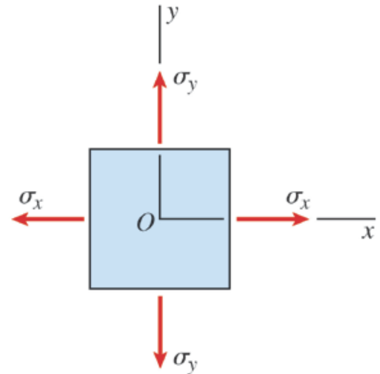
  - Both angles satisfy $\tan 2\theta_p = 0$
  - Procedure to distinguish θ_{p1} from θ_{p2}
    1) Substitute these into transformation equations → tell which is σ_1.
    2) Or find the angle that satisfies

$$\cos 2\theta_p = \frac{\sigma_x - \sigma_y}{2R} \qquad \sin 2\theta_p = \frac{\tau_{xy}}{R}$$

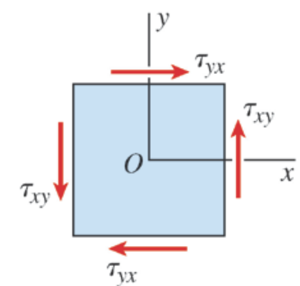
Special cases

- Uniaxial stress & Biaxial stress

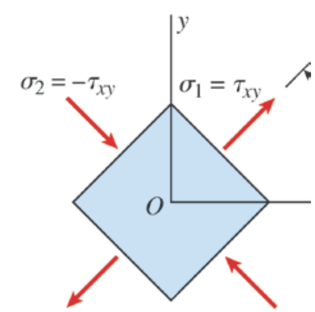




  - Principal planes? $\tan 2\theta_p = \dfrac{2\tau_{xy}}{\sigma_x - \sigma_y}$
  - θ_p = 0 ° and 90° → how do we get this?



- Pure Shear
  - Principal planes?
  - θ_p = 45 ° and 135° → how do we get this?
  - If τ_{xy} is positive, $\sigma_1 = \tau_{xy}$ & $\sigma_2 = -\tau_{xy}$



Maximum Shear Stress

- Maximum Shear Stress?
  - Strategy?
  - Taking derivatives of normal stress with respect to θ

$$\frac{d\tau_{x1y1}}{d\theta} = -\left(\sigma_x - \sigma_y\right)\cos 2\theta - 2\tau_{xy}\sin 2\theta = 0$$

$$\longrightarrow \quad \tan 2\theta_s = -\frac{\sigma_x - \sigma_y}{2\tau_{xy}}$$

  - θ_s:orientation of the planes of the maximum positive and negative shear stresses
    - One : 0 °~ 90 °
    - The other (differ by 90°) : 90 °~ 180 °

-- Maximum positive and maximum negative shear stresses differ only in sign. Why???

Principal Stresses and Maximum Shear Stresses
Maximum Shear Stress

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- Relationship between Principal angles, θ_p and angle of the planes of maximum positive and negative shear stresses, θ_s

$$\tan 2\theta_s = -\frac{1}{\tan 2\theta_p} = -\cot 2\theta_p$$

$$\frac{\sin 2\theta_s}{\cos 2\theta_s} + \frac{\cos 2\theta_p}{\sin 2\theta_p} = 0 \qquad \sin 2\theta_s \sin 2\theta_p + \cos 2\theta_s \cos 2\theta_p = 0$$

$$\cos\left(2\theta_s - 2\theta_p\right) = 0 \qquad 2\theta_s - 2\theta_p = \pm 90°$$

$$\boxed{\theta_s = \theta_p \pm 45°}$$

- The planes of maximum shear stress occur at 45° to the principal planes

Principal Stresses and Maximum Shear Stresses

Maximum Shear Stress

- $\sin 2\theta_s$ & $\cos 2\theta_s$?

$$R = \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}$$

$$\tan 2\theta_s = -\frac{\sigma_x - \sigma_y}{2\tau_{xy}}$$

$$\cos 2\theta_{s1} = \frac{\tau_{xy}}{R} \qquad \sin 2\theta_{s1} = -\frac{\sigma_x - \sigma_y}{2R}$$

$$\tau_{\max} = \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2} \longrightarrow \theta_{s1} = \theta_{p1} - 45°$$

$$\cos 2\theta_{s1} = -\frac{\tau_{xy}}{R} \qquad \sin 2\theta_{s1} = \frac{\sigma_x - \sigma_y}{2R}$$

$$\tau_{\max} = -\sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2} \longrightarrow \theta_{s2} = \theta_{p1} + 45°$$

- Maximum (positive or negative) shear stress, τ_{max}

$$\tau_{\max} = \pm\sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}$$

$$\tau_{\max} = \pm\frac{\sigma_1 - \sigma_2}{2}$$

Maximum positive shear stress is equal to one-half the difference of the principal stress

Principal Stresses and Maximum Shear Stresses
Maximum Shear Stress

- Normal stress at the plane of τ_{max}?

$$\sigma_{x_1} = \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2} \cos 2\theta + \tau_{xy} \sin 2\theta \longleftarrow$$

$$\cos 2\theta_{s1} = \frac{\tau_{xy}}{R}$$

$$\sin 2\theta_{s1} = -\frac{\sigma_x - \sigma_y}{2R}$$

$$\sigma_{x_1} = \frac{\sigma_x + \sigma_y}{2} = \sigma_{aver} = \sigma_{y_1}$$

  – Normal stress acting on the planes of maximum positive shear stresses equal to the average of the normal stresses on the x and y planes.

  – And same normal stress acts on the planes of maximum negative shear stress

- Uniaxial, biaxial or pure shear?

Principal Stresses and Maximum Shear Stresses
In-Plane and Out-of-Plane Shear Stresses

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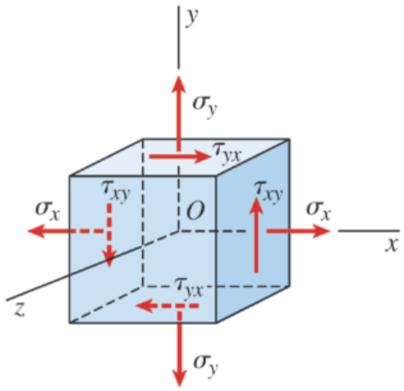
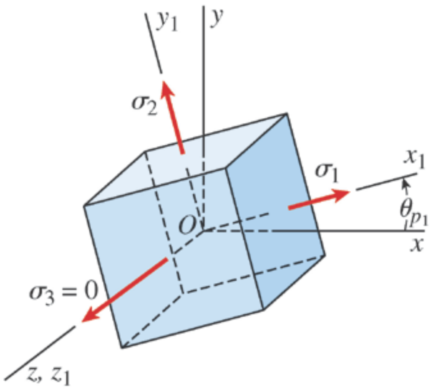
- So far we have dealt only with in-plane shear stress acting in the xy plane.
  - Maximum shear stresses by 45° rotations about the other two principal axes

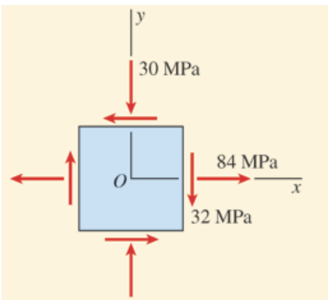
$$(\tau_{\max})_{x1} = \pm\frac{\sigma_2}{2} \qquad (\tau_{\max})_{y1} = \pm\frac{\sigma_1}{2} \qquad (\tau_{\max})_{z1} = \pm\frac{(\sigma_1 - \sigma_2)}{2}$$

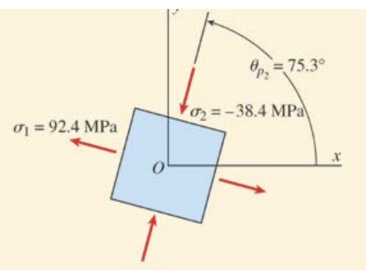
  - The stresses obtained by rotations about the x_1 and y_1 axes are 'out-of-plane shear stresses'

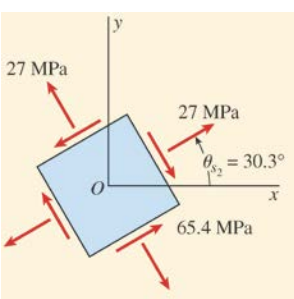


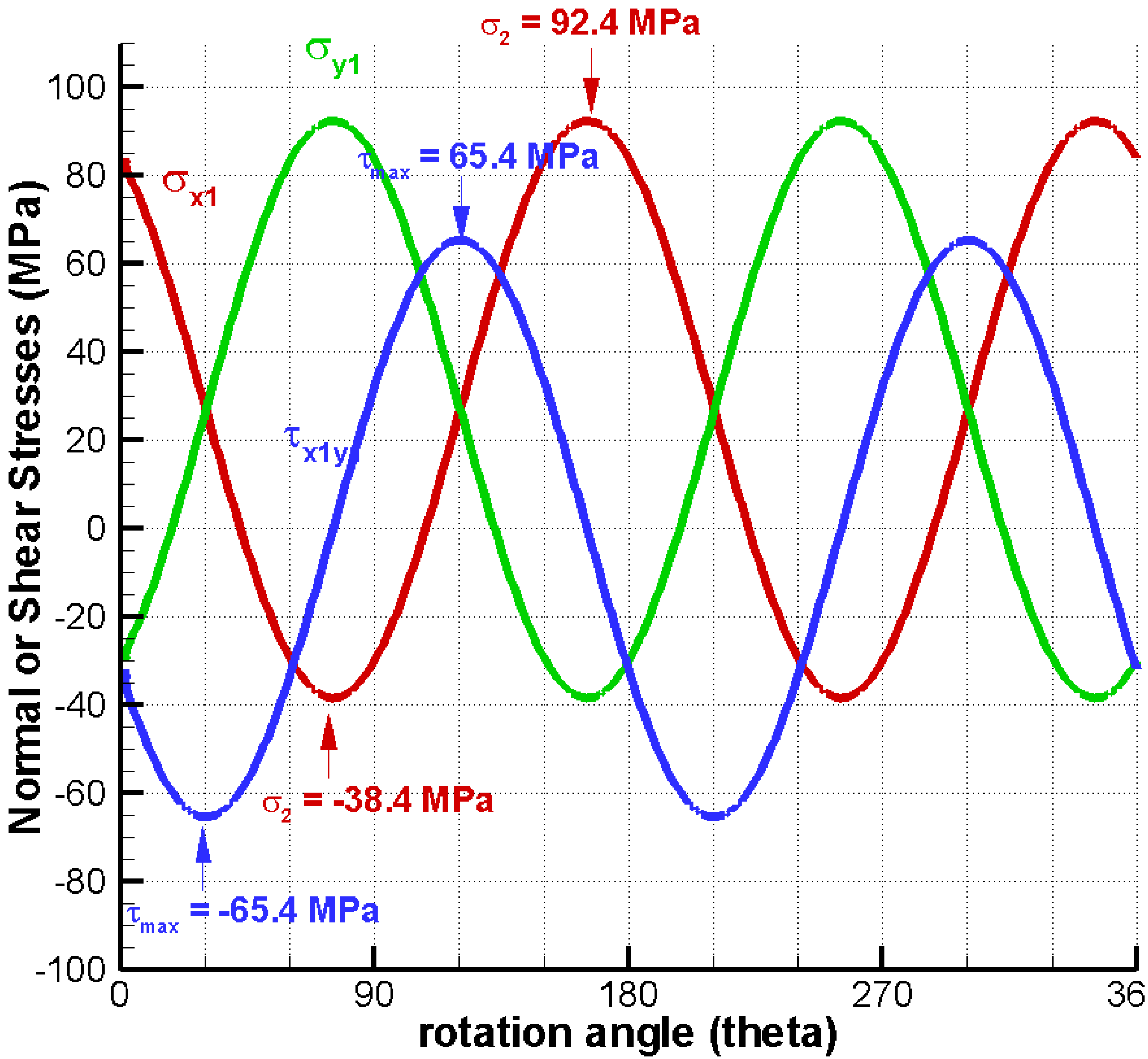


Example

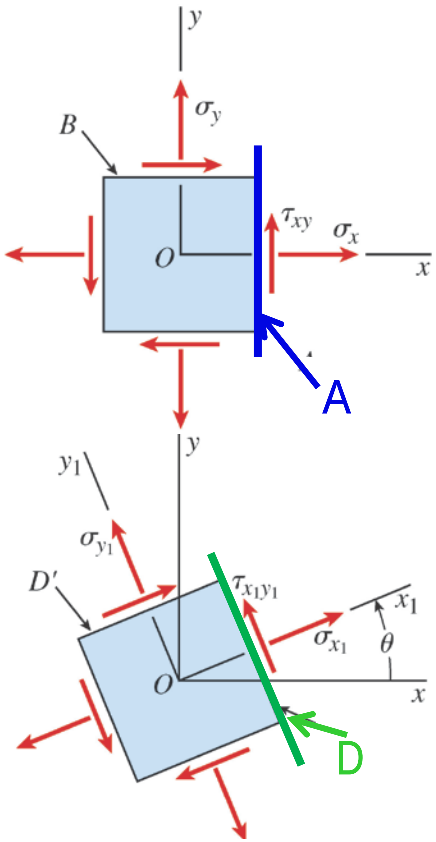

y
30 MPa
84 MPa
O
x
32 MPa

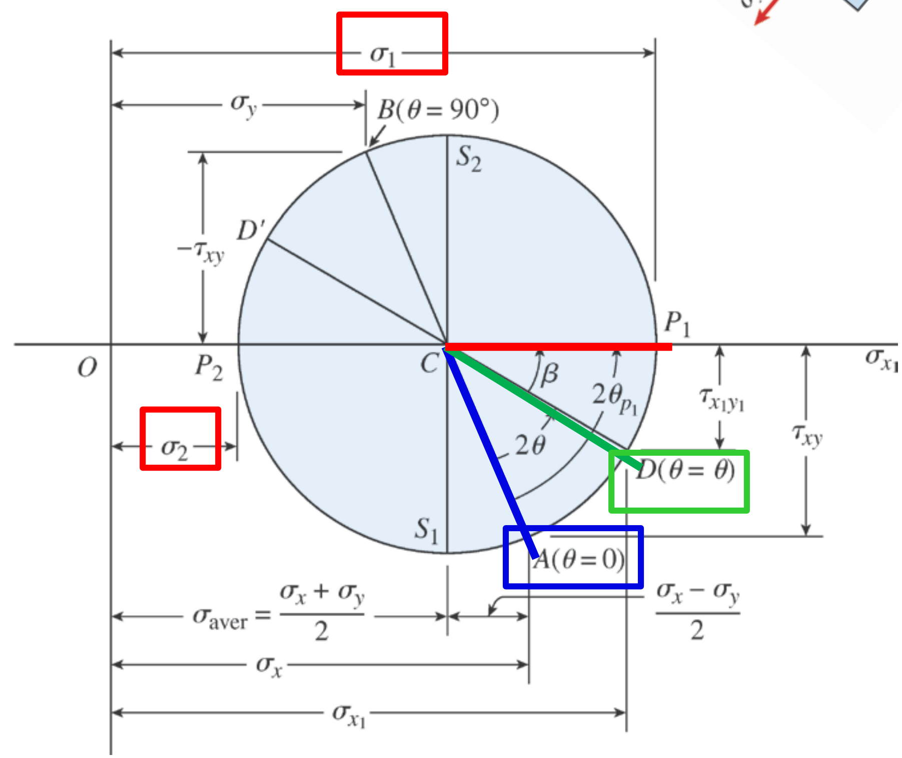

$\theta_{p_2} = 75.3°$
$\sigma_2 = -38.4$ MPa
$\sigma_1 = 92.4$ MPa
O
x

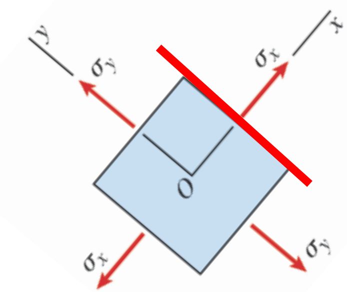

y
27 MPa
27 MPa
$\theta_{s_2} = 30.3°$
O
x
65.4 MPa


σ_2 = 92.4 MPa
σ_{y1}
σ_{x1}
τ_{max} = 65.4 MPa
τ_{x1y1}
σ_2 = -38.4 MPa
τ_{max} = -65.4 MPa
Normal or Shear Stresses (MPa)
100
80
60
40
20
0
-20
-40
-60
-80
-100
0
90
180
270
360
rotation angle (theta)

Mohr's Circle
2D







Mohr’s Circle
3D

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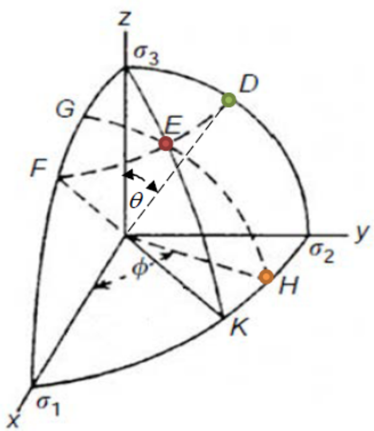
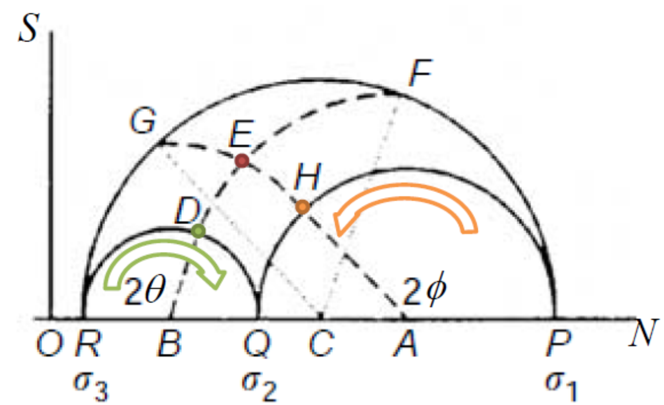
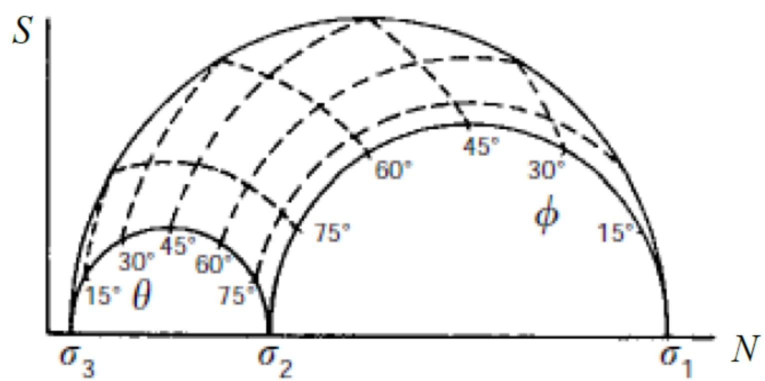
- 3D Mohr’s Circles: Particular stress value exist in the intersections of dotted lines







We can construct a diagram from which the normal and shear tractions acting on any plane can be found by locating the intersections of the circles!

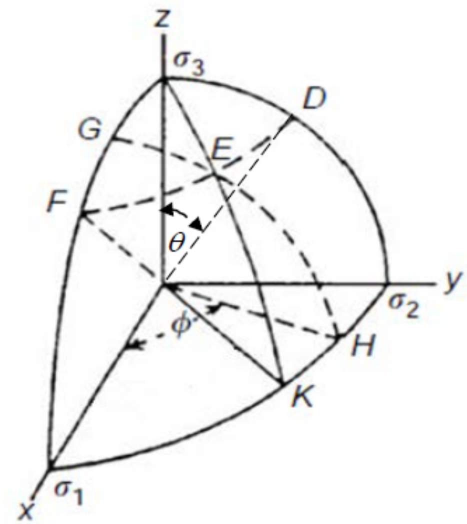
Mohr's Circle
3D - example

For example,
consider the following stress state acting on a point:

$$\sigma_{ij} = \begin{pmatrix} 5 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Question: Calculate the normal and shear stress on the plane with normal vector:

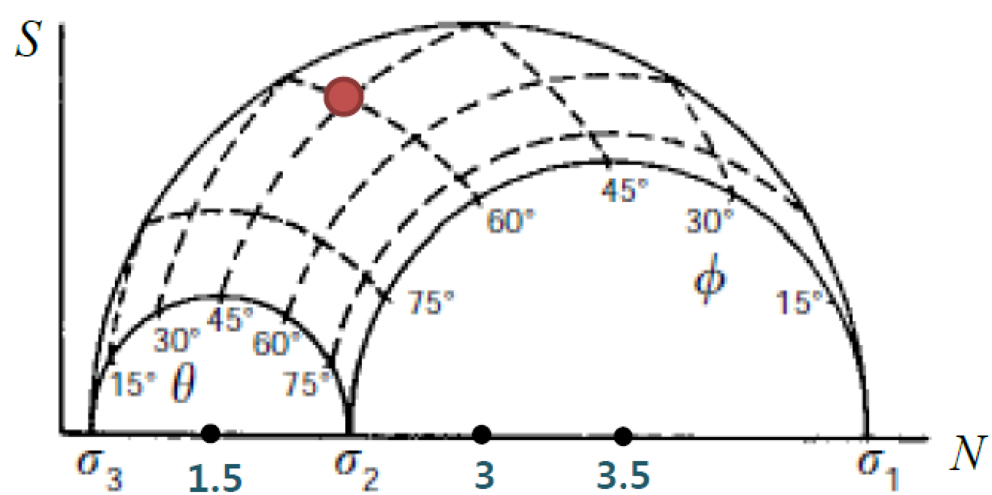
$$n = \left(\frac{1}{2}, \frac{1}{2}, \frac{\sqrt{2}}{2} \right)$$



- **Direction cosines**

$$n_1 = \cos\phi = \frac{1}{2} \qquad \phi = 60°$$

$$n_3 = \cos\theta = \frac{\sqrt{2}}{2} \qquad \theta = 45°$$



Stress
deviatoric stress/stress invariant

- Deviatoric stress

$$s_{ij} = \sigma_{ij} - \frac{\sigma_{kk}}{3}\delta_{ij},$$

$$\begin{bmatrix} s_{11} & s_{12} & s_{13} \\ s_{21} & s_{22} & s_{23} \\ s_{31} & s_{32} & s_{33} \end{bmatrix} = \begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{bmatrix} - \begin{bmatrix} \pi & 0 & 0 \\ 0 & \pi & 0 \\ 0 & 0 & \pi \end{bmatrix}$$

$$= \begin{bmatrix} \sigma_{11} - \pi & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} - \pi & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} - \pi \end{bmatrix}.$$

- Stress invariant

$$I_1 = \sigma_x + \sigma_y + \sigma_z$$
$$I_2 = \sigma_x\sigma_y + \sigma_y\sigma_z + \sigma_z\sigma_x - \tau_{xy}^2 - \tau_{yz}^2 - \tau_{zx}^2$$
$$I_3 = \sigma_x\sigma_y\sigma_z + 2\tau_{xy}\tau_{yz}\tau_{zx} - \sigma_x\tau_{yz}^2 - \sigma_y\tau_{zx}^2 - \sigma_z\tau_{xy}^2$$

$$I_1 = \sigma_1 + \sigma_2 + \sigma_3$$
$$I_2 = \sigma_1\sigma_2 + \sigma_2\sigma_3 + \sigma_3\sigma_1$$
$$I_3 = \sigma_1\sigma_2\sigma_3$$

$$J_1 = s_{kk} = 0,$$
$$J_2 = \tfrac{1}{2}s_{ij}s_{ji} = \tfrac{1}{2}\mathrm{tr}(\boldsymbol{s}^2)$$
$$= \tfrac{1}{2}(s_1^2 + s_2^2 + s_3^2)$$
$$= \tfrac{1}{6}\left[(\sigma_{11} - \sigma_{22})^2 + (\sigma_{22} - \sigma_{33})^2 + (\sigma_{33} - \sigma_{11})^2\right] + \sigma_{12}^2 + \sigma_{23}^2 + \sigma_{31}^2$$
$$= \tfrac{1}{6}\left[(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2\right]$$
$$= \tfrac{1}{3}I_1^2 - I_2 = \frac{1}{2}\left[\mathrm{tr}(\boldsymbol{\sigma}^2) - \frac{1}{3}\mathrm{tr}(\boldsymbol{\sigma})^2\right],$$
$$J_3 = \det(s_{ij})$$
$$= \tfrac{1}{3}s_{ij}s_{jk}s_{ki} = \tfrac{1}{3}\mathrm{tr}(\boldsymbol{s}^3)$$
$$= s_1 s_2 s_3$$
$$= \tfrac{2}{27}I_1^3 - \tfrac{1}{3}I_1 I_2 + I_3 = \tfrac{1}{3}\left[\mathrm{tr}(\boldsymbol{\sigma}^3) - \mathrm{tr}(\boldsymbol{\sigma}^2)\mathrm{tr}(\boldsymbol{\sigma}) + \tfrac{2}{9}\mathrm{tr}(\boldsymbol{\sigma})^3\right].$$

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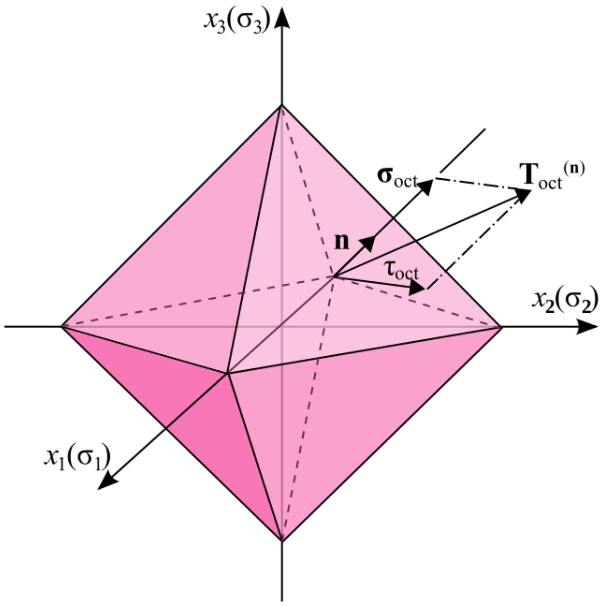
- Octahedral stress



$$\begin{aligned}\sigma_{\text{oct}} &= T_i^{(n)} n_i \\ &= \sigma_{ij} n_i n_j \\ &= \sigma_1 n_1 n_1 + \sigma_2 n_2 n_2 + \sigma_3 n_3 n_3 \\ &= \tfrac{1}{3}(\sigma_1 + \sigma_2 + \sigma_3) = \tfrac{1}{3} I_1\end{aligned}$$

$$\begin{aligned}\tau_{\text{oct}} &= \sqrt{T_i^{(n)} T_i^{(n)} - \sigma_{\text{n}}^2} \\ &= \left[\tfrac{1}{3}(\sigma_1^2 + \sigma_2^2 + \sigma_3^2) - \tfrac{1}{9}(\sigma_1 + \sigma_2 + \sigma_3)^2\right]^{1/2} \\ &= \tfrac{1}{3}\left[(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2\right]^{1/2} = \tfrac{1}{3}\sqrt{2I_1^2 - 6I_2} = \sqrt{\tfrac{2}{3} J_2}\end{aligned}$$

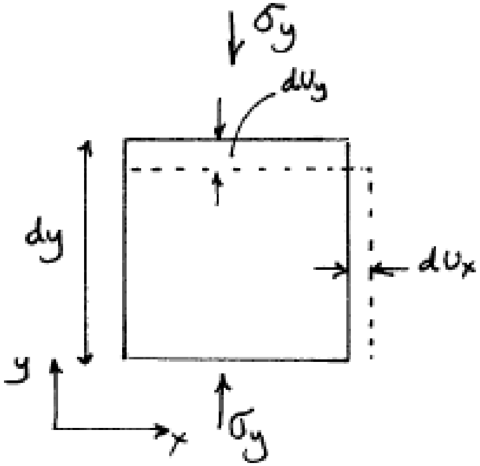
Strain and displacement
Definition

- Definition of strain (dimensionless)
  - Normal strain
  - Shear strain – Engineering strain/ mathematical strain

$$\varepsilon = \frac{\Delta L}{L} = \frac{du}{dx}$$

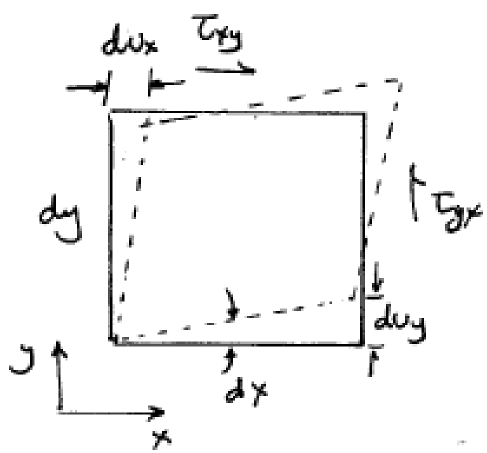
$$\varepsilon_{xx} = \frac{\partial u_x}{\partial x}, \varepsilon_{yy} = \frac{\partial u_y}{\partial y},$$

$$\gamma_{xy} = \frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x}$$



$$\varepsilon_y = \frac{\partial u_y}{\partial y} \simeq \frac{\Delta u_y}{\Delta y}$$

Normal strain



$$\gamma_{xy} = \left(\frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x}\right)$$

Shear strain

Strain and displacement
Definition – 3D

$$\varepsilon_{xx} = \frac{\partial u_x}{\partial x}, \varepsilon_{yy} = \frac{\partial u_y}{\partial y}, \varepsilon_{zz} = \frac{\partial u_z}{\partial z},$$

$$\varepsilon_{xy} = \frac{1}{2}\left(\frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x}\right)$$

$$\varepsilon_{yz} = \frac{1}{2}\left(\frac{\partial u_y}{\partial z} + \frac{\partial u_z}{\partial y}\right)$$

$$\varepsilon_{xz} = \frac{1}{2}\left(\frac{\partial u_x}{\partial z} + \frac{\partial u_z}{\partial x}\right)$$

$$\begin{pmatrix} \varepsilon_{xx} & \varepsilon_{xy} & \varepsilon_{xz} \\ \varepsilon_{yx} & \varepsilon_{yy} & \varepsilon_{yz} \\ \varepsilon_{zx} & \varepsilon_{zy} & \varepsilon_{zz} \end{pmatrix}$$

텐서형식
(Tensor form)

$$\begin{pmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \varepsilon_{zz} \\ 2\varepsilon_{yz} \\ 2\varepsilon_{xz} \\ 2\varepsilon_{xy} \end{pmatrix} = \begin{pmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \varepsilon_{zz} \\ \gamma_{yz} \\ \gamma_{xz} \\ \gamma_{xy} \end{pmatrix}$$

Mathematical or tensorial shear strain → $2\varepsilon_{yz}$

Engineering shear strain → γ_{yz}

행렬형식
(Matrix form)

Strain is also 2nd order tensor, and symmetric

Strain and displacement
Transformation equation for plane strain

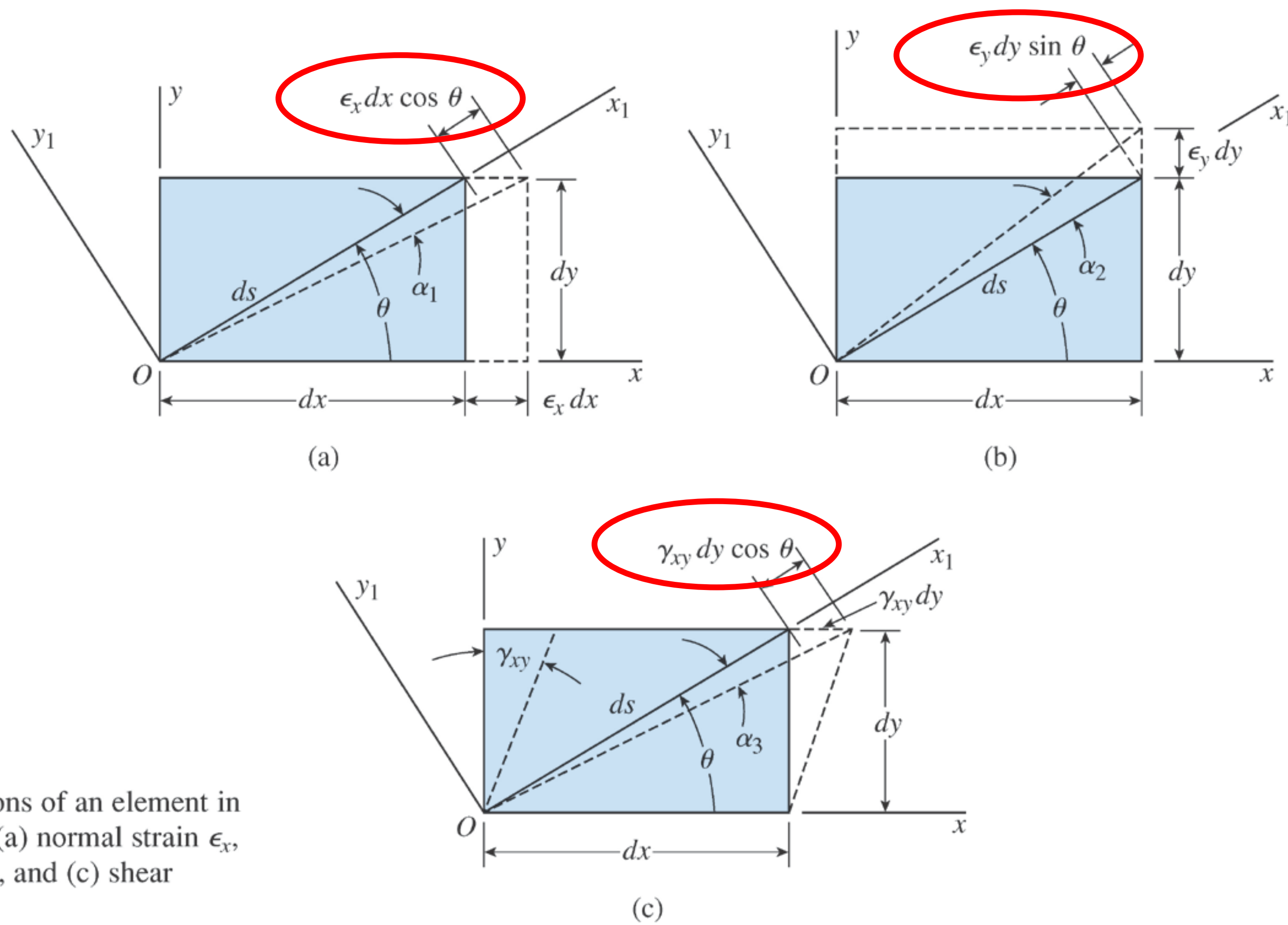


FIG. 7-33 Deformations of an element in plane strain due to (a) normal strain ϵ_x, (b) normal strain ϵ_y, and (c) shear strain γ_{xy}

Strain and displacement
Transformation equation for plane strain

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$$\Delta d = \varepsilon_x dx\cos\theta + \varepsilon_y dy\sin\theta + \gamma_{xy} dy\cos\theta$$

$$\varepsilon_{x1} = \frac{\Delta d}{ds} = \varepsilon_x \frac{dx}{ds}\cos\theta + \varepsilon_y \frac{ds}{ds}\sin\theta + \gamma_{xy} \frac{dy}{ds}\cos\theta$$

$$\varepsilon_{x1} = \varepsilon_x \cos^2\theta + \varepsilon_y \sin^2\theta + \gamma_{xy} \cos\theta\sin\theta$$

Transformation equation for plane strain

- Shear strain γ_{x1y1}:
  - Decrease in angle between lines that were initially along the x1 and y1 axes.

$$\gamma_{x1y1} = \alpha + \beta$$

$$\frac{\gamma_{x_1y_1}}{2} = -(\varepsilon_x - \varepsilon_y)\sin\theta\cos\theta + \frac{\gamma_{xy}}{2}(\cos^2\theta - \sin^2\theta)$$

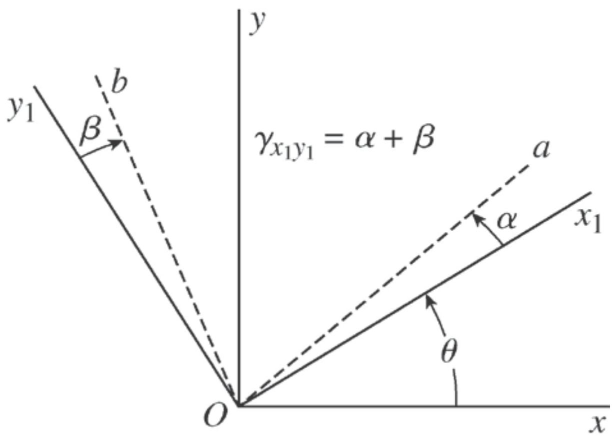


FIG. 7-34 Shear strain $\gamma_{x_1y_1}$ associated with the x_1y_1 axes

Transformation equation for plane strain

- Transformation equations for plane strain

$$\varepsilon_{x_1} = \frac{\varepsilon_x + \varepsilon_y}{2} + \frac{\varepsilon_x - \varepsilon_y}{2}\cos 2\theta + \frac{\gamma_{xy}}{2}\sin 2\theta$$

compare

$$\sigma_{x_1} = \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2}\cos 2\theta + \tau_{xy}\sin 2\theta$$

$$\frac{\gamma_{x_1y_1}}{2} = -\frac{\varepsilon_x - \varepsilon_y}{2}\sin 2\theta + \frac{\gamma_{xy}}{2}\cos 2\theta$$

compare

$$\tau_{x_1y_1} = -\frac{\sigma_x - \sigma_y}{2}\sin 2\theta + \tau_{xy}\cos 2\theta$$

$$\varepsilon_{x_1} + \varepsilon_{y_1} = \varepsilon_x + \varepsilon_y$$

  - Similar to the transformation of plane stress

TABLE 7-1 CORRESPONDING VARIABLES IN THE TRANSFORMATION EQUATIONS FOR PLANE STRESS (EQS. 7-4a AND b) AND PLANE STRAIN (EQS. 7-71a AND b)

Stresses	Strains
σ_x	ϵ_x
σ_y	ϵ_y
τ_{xy}	$\gamma_{xy}/2$
σ_{x_1}	ϵ_{x_1}
$\tau_{x_1y_1}$	$\gamma_{x_1y_1}/2$

Strain and displacement
Transformation equation for plane strain

– Principal Angles

$$\tan 2\theta_p = \frac{\gamma_{xy}}{\varepsilon_x - \varepsilon_y}$$

– Principal Strains

$$\varepsilon_{1,2} = \frac{\varepsilon_x + \varepsilon_y}{2} \pm \sqrt{\left(\frac{\varepsilon_x - \varepsilon_y}{2}\right)^2 + \left(\frac{\gamma_{xy}}{2}\right)^2}$$

– Maximum Shear Strain (and normal strains for the maximum shear)

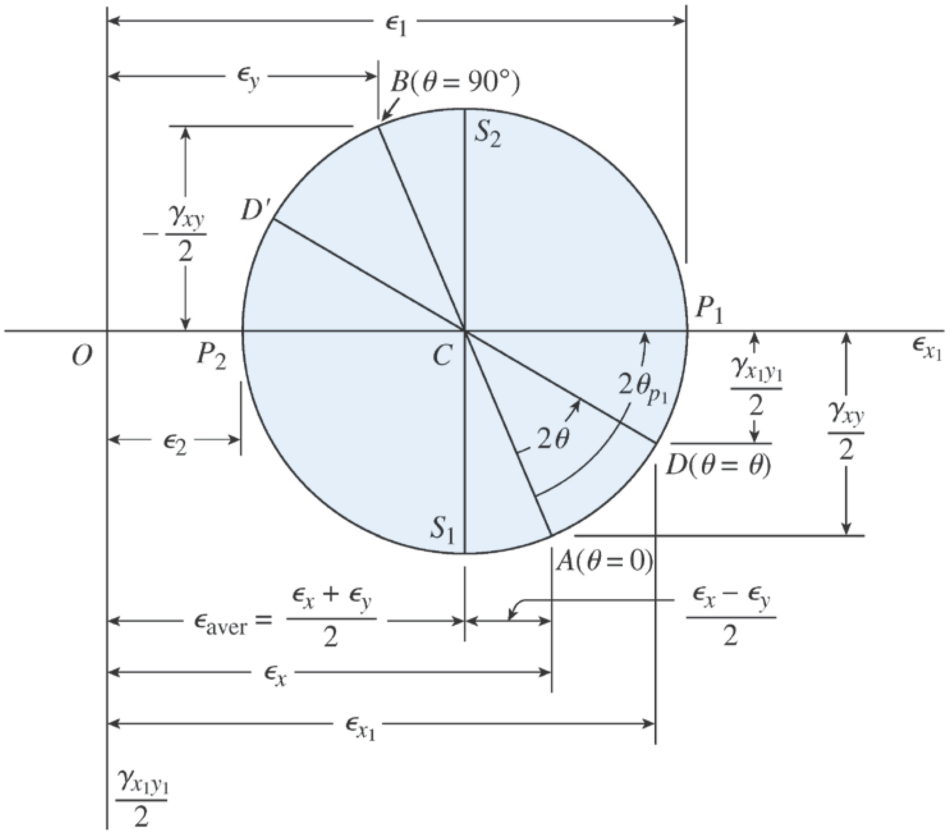
$$\frac{\gamma_{\max}}{2} = \sqrt{\left(\frac{\varepsilon_x - \varepsilon_y}{2}\right)^2 + \left(\frac{\gamma_{xy}}{2}\right)^2}$$

$$\varepsilon_{aver} = \frac{\varepsilon_x + \varepsilon_y}{2}$$

- Mohr's Circle for plane strain ← same as plane stress

TABLE 7-1 CORRESPONDING VARIABLES IN THE TRANSFORMATION EQUATIONS FOR PLANE STRESS (EQS. 7-4a AND b) AND PLANE STRAIN (EQS. 7-71a AND b)

Stresses	Strains
σ_x	ϵ_x
σ_y	ϵ_y
τ_{xy}	$\gamma_{xy}/2$
σ_{x_1}	ϵ_{x_1}
$\tau_{x_1 y_1}$	$\gamma_{x_1 y_1}/2$



Strain and displacement
Strain Measurements

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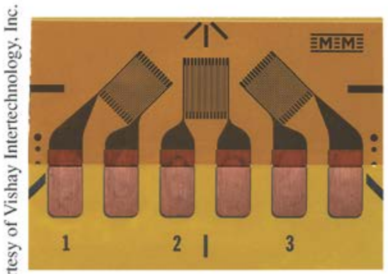
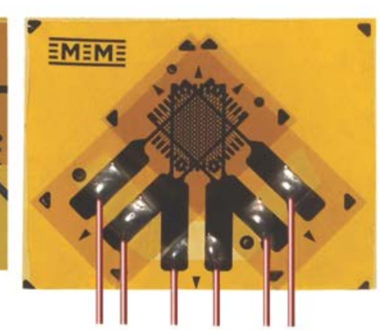
- Strain gages
  - A device for measuring normal strains on the surface of a stressed object (e.g., rock)
  - Electrical resistance of the wire is altered when it stretches or shortens → converted to strain
  - Sensitive: can measure 1×10^{-6}
  - Three measurement → strains in any direction
- Strain rosette
  - A group of three gages arranged in a particular direction

Images courtesy of Vishay Intertechnology, Inc.

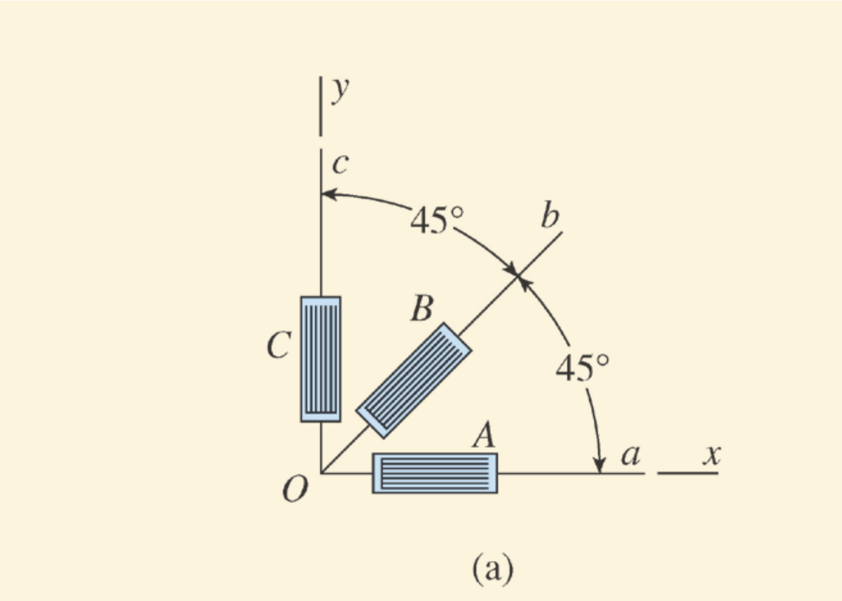


(a) 45 strain gages three-element rosette



(b) Three-element strain-gage rosettes prewired

- A strain rosette is bonded to the surface of rock before it is loaded. With normal strains ε_a, ε_b and ε_c how to obtain ε_{x1}, ε_{y1} and γ_{x1y1}?



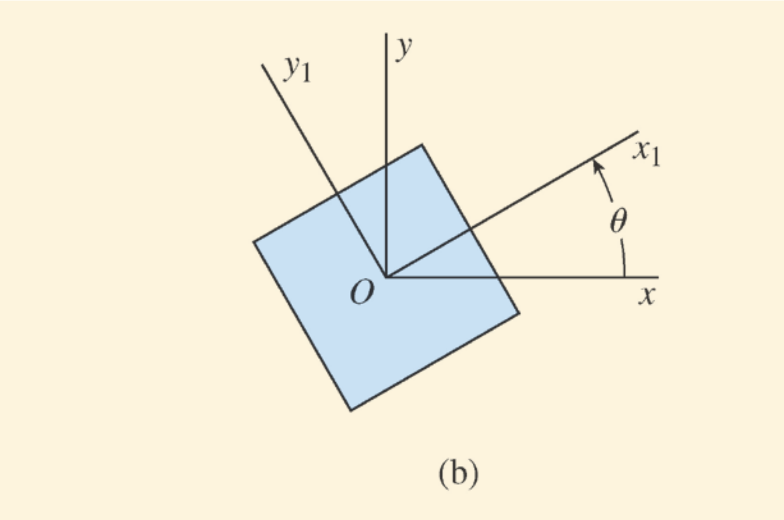


FIG. 7-38 Example 7-8. (a) 45° strain rosette, and (b) element oriented at an angle θ to the xy axes

Strain and displacement

- Strain transformation equation derived solely from the consideration of geometry.
  - No need to know material properties
- Determining Stresses from Strain
  - Apply Hooke's law → need to know material properties

Hooke's Law

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- Hooke's Law in 1D

$$\sigma = E\varepsilon$$

- Shear modulus (전단계수) G

$$\tau_{xy} = G\gamma_{xy}$$

- Generalized Hooke's law (isotropy)
  - Isotropic rock has two independent parameters (E, v)
  - Shear modulus can be related to elastic modulus and Poisson's ratio

$$G = \frac{E}{2(1+\nu)}$$

$$\begin{pmatrix} \varepsilon_x \\ \varepsilon_y \\ \varepsilon_z \\ \gamma_{yz} \\ \gamma_{xz} \\ \gamma_{xy} \end{pmatrix} = \begin{pmatrix} \frac{1}{E} & -\frac{\nu}{E} & -\frac{\nu}{E} & 0 & 0 & 0 \\ -\frac{\nu}{E} & \frac{1}{E} & -\frac{\nu}{E} & 0 & 0 & 0 \\ -\frac{\nu}{E} & -\frac{\nu}{E} & \frac{1}{E} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{G} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{G} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{G} \end{pmatrix} \begin{pmatrix} \sigma_x \\ \sigma_y \\ \sigma_z \\ \tau_{yz} \\ \tau_{xz} \\ \tau_{xy} \end{pmatrix}$$

Hooke's Law
inverse form

$$\begin{bmatrix}\sigma_{xx}\\ \sigma_{yy}\\ \sigma_{zz}\\ \sigma_{xy}\\ \sigma_{yz}\\ \sigma_{zx}\end{bmatrix} = \frac{E(1-\nu)}{(1+\nu)(1-2\nu)}\begin{bmatrix} 1 & \nu/(1-\nu) & \nu/(1-\nu) & 0 & 0 & 0\\ \nu/(1-\nu) & 1 & \nu/(1-\nu) & 0 & 0 & 0\\ \nu/(1-\nu) & \nu/(1-\nu) & 1 & 0 & 0 & 0\\ 0 & 0 & 0 & \frac{(1-2\nu)}{2(1-\nu)} & 0 & 0\\ 0 & 0 & 0 & 0 & \frac{(1-2\nu)}{2(1-\nu)} & 0\\ 0 & 0 & 0 & 0 & 0 & \frac{(1-2\nu)}{2(1-\nu)}\end{bmatrix}\begin{bmatrix}\varepsilon_{xx}\\ \varepsilon_{yy}\\ \varepsilon_{zz}\\ \gamma_{xy}\\ \gamma_{yz}\\ \gamma_{zx}\end{bmatrix} \quad (2.41)$$

[D]

The inverse forms of equations 2.40, usually called Lamé's equations, are obtained from equation 2.41, i.e.

$$\sigma_{xx} = \lambda\Delta + 2G\varepsilon_{xx}, \text{ etc.}$$
$$\sigma_{xy} = G\gamma_{xy}, \text{ etc.}$$

where λ is Lamé's constant, defined by

$$\lambda = \frac{2\nu G}{(1-2\nu)} = \frac{\nu E}{(1+\nu)(1-2\nu)}$$

and Δ is the volumetric strain.

Hooke's Law

- Normal strains under plane stress

Normal strain, ε_x = $\frac{1}{E}\sigma_x$ + $-\frac{\nu}{E}\sigma_y$

$$\varepsilon_x = \frac{1}{E}\left(\sigma_x - \nu\sigma_y\right)$$

E: Elastic Modulus or Young's Modulus

v: Poisson's ratio

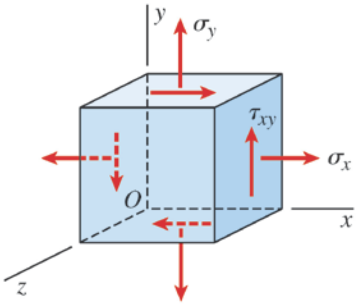
  - Similarly

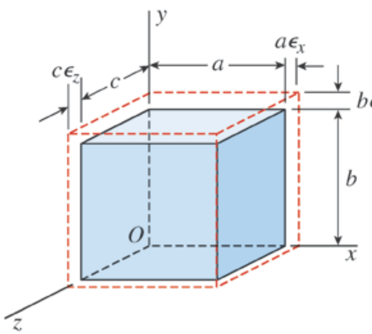
$$\varepsilon_y = \frac{1}{E}\left(\sigma_y - \nu\sigma_x\right) \qquad \varepsilon_z = -\frac{\nu}{E}\left(\sigma_x + \sigma_y\right)$$

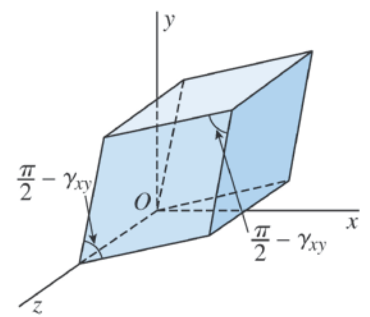
- Shear strains under plane stress
  - Shear strain is the decrease of angle
  - σ_x and σ_y has no effect

$$\gamma_{xy} = \frac{\tau_{xy}}{G}$$

G: Shear Modulus







Hooke's law

- Hooke's Law for Plane Stress
  - Strains in terms of stresses (plane stress)

Normal strain in z-direction can be non-zero

$$\varepsilon_x = \frac{1}{E}\left(\sigma_x - \nu\sigma_y\right) \qquad \varepsilon_y = \frac{1}{E}\left(\sigma_y - \nu\sigma_x\right) \qquad \varepsilon_z = -\frac{\nu}{E}\left(\sigma_x + \sigma_y\right) \qquad \gamma_{xy} = \frac{\tau_{xy}}{G}$$

  - Stresses in terms of strains (plane stress)

Normal stress in z-direction is non-zero

$$\sigma_x = \frac{E}{1-\nu^2}\left(\varepsilon_x + \nu\varepsilon_y\right) \qquad \sigma_y = \frac{E}{1-\nu^2}\left(\varepsilon_y + \nu\varepsilon_x\right) \qquad \sigma_z = 0 \qquad \tau_{xy} = G\gamma_{xy}$$

  - They contain three material properties, but only two are independent.

$$G = \frac{E}{2(1+\nu)}$$

Triaxial Stress

- Strains in terms of Triaxial Stress

$$\varepsilon_x = \frac{\sigma_x}{E} - \frac{\nu}{E}(\sigma_y + \sigma_z)$$

$$\varepsilon_y = \frac{\sigma_y}{E} - \frac{\nu}{E}(\sigma_z + \sigma_x)$$

$$\varepsilon_z = \frac{\sigma_z}{E} - \frac{\nu}{E}(\sigma_x + \sigma_y)$$

- Stresses in terms of strains

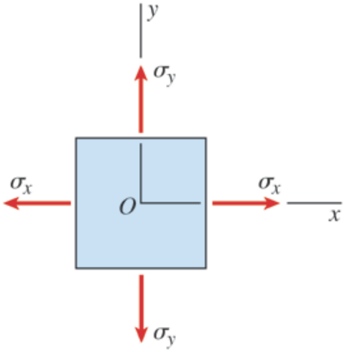
$$\sigma_x = \frac{E}{(1+\nu)(1-2\nu)}\left[(1-\nu)\varepsilon_x + \nu(\varepsilon_y + \varepsilon_z)\right]$$

$$\sigma_y = \frac{E}{(1+\nu)(1-2\nu)}\left[(1-\nu)\varepsilon_y + \nu(\varepsilon_z + \varepsilon_x)\right]$$

$$\sigma_z = \frac{E}{(1+\nu)(1-2\nu)}\left[(1-\nu)\varepsilon_z + \nu(\varepsilon_x + \varepsilon_y)\right]$$

Hooke's Law
Triaxial Stress

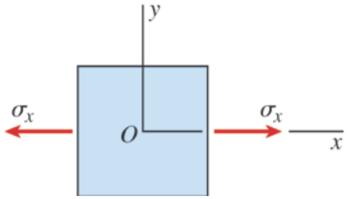
– Biaxial Stress $\sigma_x \neq 0,\ \sigma_y \neq 0,\ \tau_{xy} = 0$



$$\varepsilon_x = \frac{1}{E}\left(\sigma_x - \nu\sigma_y\right) \qquad \varepsilon_y = \frac{1}{E}\left(\sigma_y - \nu\sigma_x\right) \qquad \varepsilon_z = -\frac{\nu}{E}\left(\sigma_x + \sigma_y\right) \qquad \gamma_{xy} = 0$$

$$\sigma_x = \frac{E}{1-\nu^2}\left(\varepsilon_x + \nu\varepsilon_y\right) \qquad \sigma_y = \frac{E}{1-\nu^2}\left(\varepsilon_y + \nu\varepsilon_x\right) \qquad \sigma_z = 0 \qquad \tau_{xy} = 0$$

– Uniaxial Stress $\sigma_x \neq 0,\ \sigma_y = 0,\ \tau_{xy} = 0$



$$\varepsilon_x = \frac{1}{E}\sigma_x \qquad \varepsilon_y = \varepsilon_z = -\nu\frac{\sigma_x}{E} \qquad \gamma_{xy} = 0$$

$$\sigma_x = E\varepsilon_x \qquad \sigma_y = \sigma_z = \tau_{xy} = 0$$

– Pure Shear $\sigma_x = 0,\ \sigma_y = 0,\ \tau_{xy} \neq 0$

$$\varepsilon_x = \varepsilon_y = \varepsilon_z = 0 \qquad \gamma_{xy} = \frac{\tau_{xy}}{G}$$

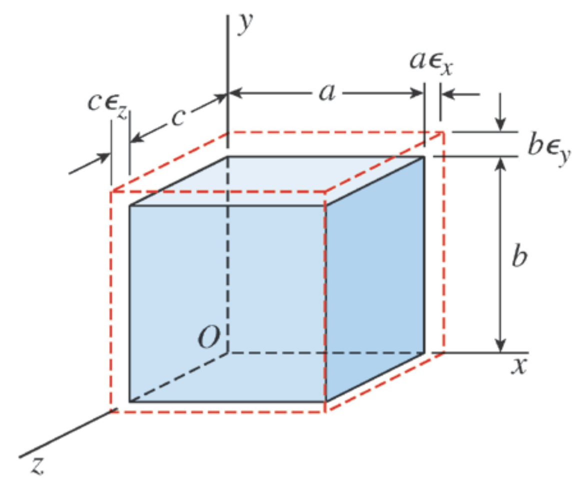
$$\sigma_x = \sigma_y = \sigma_z = 0 \qquad \tau_{xy} = G\gamma_{xy}$$

Hooke's Law
Volume Change

- When a solid undergoes strains, its volume will change
  - The original volume $V_0 = abc$
  - Final volume after deformation $V_1 = (a + a\varepsilon_x)(b + b\varepsilon_y)(c + c\varepsilon_z) = abc(1+\varepsilon_x)(1+\varepsilon_y)(1+\varepsilon_z)$
    $= V_0(1+\varepsilon_x)(1+\varepsilon_y)(1+\varepsilon_z)$
  - Upon expanding the terms in the right hand side

$$V_1 = V_0(1 + \varepsilon_x + \varepsilon_y + \varepsilon_z + \varepsilon_x\varepsilon_y + \varepsilon_x\varepsilon_z + \varepsilon_y\varepsilon_z + \varepsilon_x\varepsilon_y\varepsilon_z)$$

  - With small strains $V_1 = V_0(1 + \varepsilon_x + \varepsilon_y + \varepsilon_z)$
  - Volume change $\Delta V = V_1 - = V_0(\varepsilon_x + \varepsilon_y + \varepsilon_z)$
    - Shear strain produce no change in volume



- The unit volume change (= dilatation).

$$e = \frac{\Delta V}{V_0} = \varepsilon_x + \varepsilon_y + \varepsilon_z$$

$$\varepsilon_x = \frac{1}{E}\left(\sigma_x - \nu\sigma_y\right)$$

$$\varepsilon_y = \frac{1}{E}\left(\sigma_y - \nu\sigma_x\right)$$

$$\varepsilon_z = -\frac{\nu}{E}\left(\sigma_x + \sigma_y\right)$$

  - (+) expansion, (-) contraction
  - Unit volume change in terms of stress
    - uniaxial

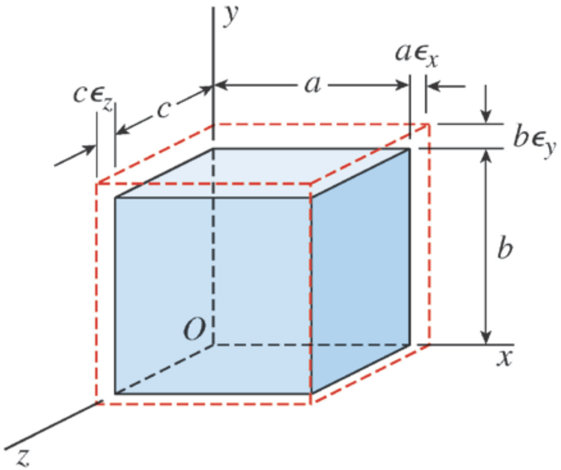
$$e = \frac{\Delta V}{V_0} = \frac{\sigma_x}{E}(1 - 2\nu)$$

    - plane stress or biaxial

$$e = \frac{\Delta V}{V_0} = \frac{1 - 2\nu}{E}(\sigma_x + \sigma_y)$$

    - Triaxial stress

$$e = \frac{1 - 2\nu}{E}(\sigma_x + \sigma_y + \sigma_z)$$



Hooke's Law
Strain-Energy Density in Plane Stress

- Strain Energy Density, u, in Plane Stress
  - Strain energy stored in a unit volume of the material

$$u = \frac{1}{2}(\sigma_x \varepsilon_x + \sigma_y \varepsilon_y + \tau_{xy}\gamma_{xy})$$

  - Strain energy density in terms of stresses alone

$$u = \frac{1}{2E}({\sigma_x}^2 + {\sigma_y}^2 - 2\nu\sigma_x\sigma_y) + \frac{{\tau_{xy}}^2}{2G}$$

  - Strain energy density in terms of strains alone

$$u = \frac{E}{2(1-\nu^2)}({\varepsilon_x}^2 + {\varepsilon_y}^2 + 2\nu\varepsilon_x\varepsilon_y) + \frac{G{\gamma_{xy}}^2}{2}$$

  - Strain energy density in uniaxial stress $\quad u = \frac{{\sigma_x}^2}{2E} \qquad u = \frac{E{\varepsilon_x}^2}{2}$
  - Strain energy density in pure shear $\quad u = \frac{{\tau_{xy}}^2}{2G} \qquad u = \frac{G{\gamma_{xy}}^2}{2}$

Hooke's Law
Strain-Energy Density in Triaxial Stress

- Strain Energy Density, *u*, in Triaxial Stress (no shear stress)

$$u = \frac{1}{2}(\sigma_x \varepsilon_x + \sigma_y \varepsilon_y + \sigma_z \varepsilon_z)$$

  – Strain Energy Density in terms of stresses

$$u = \frac{1}{2E}(\sigma_x{}^2 + \sigma_y{}^2 + \sigma_z{}^2) - \frac{\nu}{E}(\sigma_x \sigma_y + \sigma_x \sigma_z + \sigma_y \sigma_z)$$

  – Strain Energy Density in terms of strains

$$u = \frac{E}{2(1+\nu)(1-2\nu)}\left[(1-\nu)(\varepsilon_x{}^2 + \varepsilon_y{}^2 + \varepsilon_z{}^2) + 2\nu(\varepsilon_x \varepsilon_y + \varepsilon_x \varepsilon_z + \varepsilon_y \varepsilon_z)\right]$$

Hooke's Law
General Perspective - Anisotropy

- The most general case
  - Stress and Strain are linearly related

$$\begin{pmatrix} \varepsilon_x \\ \varepsilon_y \\ \varepsilon_z \\ \gamma_{yz} \\ \gamma_{xz} \\ \gamma_{xy} \end{pmatrix} = \begin{pmatrix} S_{11} & S_{12} & S_{13} & S_{14} & S_{15} & S_{16} \\ S_{21} & S_{22} & S_{23} & S_{24} & S_{25} & S_{26} \\ S_{31} & S_{32} & S_{33} & S_{34} & S_{35} & S_{36} \\ S_{41} & S_{42} & S_{43} & S_{44} & S_{45} & S_{46} \\ S_{51} & S_{52} & S_{53} & S_{54} & S_{55} & S_{56} \\ S_{61} & S_{62} & S_{63} & S_{64} & S_{65} & S_{66} \end{pmatrix} \begin{pmatrix} \sigma_x \\ \sigma_y \\ \sigma_z \\ \tau_{yz} \\ \tau_{xz} \\ \tau_{xy} \end{pmatrix}$$

- Compliance matrix has 21 independent parameters
  (By the symmetry of stress tensor, strain tensor and consideration of strain energy)

Hooke's Law
General Perspective - Anisotropy

Coupling of normal in different directions

Coupling of normal in the same directions

Coupling of normal & Shear

$$
\begin{pmatrix} \varepsilon_x \\ \varepsilon_y \\ \varepsilon_z \\ \gamma_{yz} \\ \gamma_{xz} \\ \gamma_{xy} \end{pmatrix}
=
\begin{pmatrix}
\frac{1}{E_x} & -\frac{\nu_{yx}}{E_y} & -\frac{\nu_{zx}}{E_z} & \frac{\eta_{x,yz}}{G_{yz}} & \frac{\eta_{x,xz}}{G_{xz}} & \frac{\eta_{x,xy}}{G_{xy}} \\
-\frac{\nu_{xy}}{E_x} & \frac{1}{E_y} & -\frac{\nu_{zy}}{E_z} & \frac{\eta_{y,yz}}{G_{yz}} & \frac{\eta_{y,xz}}{G_{xz}} & \frac{\eta_{y,xy}}{G_{xy}} \\
-\frac{\nu_{xz}}{E_x} & -\frac{\nu_{yz}}{E_y} & \frac{1}{E_z} & \frac{\eta_{z,yz}}{G_{yz}} & \frac{\eta_{z,xz}}{G_{xz}} & \frac{\eta_{z,xy}}{G_{xy}} \\
\frac{\eta_{yz,x}}{E_x} & \frac{\eta_{yz,y}}{E_y} & \frac{\eta_{yz,z}}{E_z} & \frac{1}{G_{yz}} & \frac{\mu_{yz,xz}}{G_{xz}} & \frac{\mu_{yz,xy}}{G_{xy}} \\
\frac{\eta_{xz,x}}{E_x} & \frac{\eta_{xz,y}}{E_y} & \frac{\eta_{xz,z}}{E_z} & \frac{\mu_{xz,yz}}{G_{yz}} & \frac{1}{G_{xz}} & \frac{\mu_{xz,xy}}{G_{xy}} \\
\frac{\eta_{xy,x}}{E_x} & \frac{\eta_{xy,y}}{E_y} & \frac{\eta_{xy,z}}{E_z} & \frac{\mu_{xy,yz}}{G_{yz}} & \frac{\mu_{xy,xz}}{G_{xz}} & \frac{1}{G_{xy}}
\end{pmatrix}
\begin{pmatrix} \sigma_x \\ \sigma_y \\ \sigma_z \\ \tau_{yz} \\ \tau_{xz} \\ \tau_{xy} \end{pmatrix}
$$

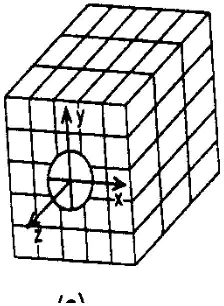
Coupling of shear in different directions

Coupling of shear in the same directions

Lekhnitskii(1963) & Hudson (1997)

Orthotropic
Three orthogonal planes of elastic symmetry

$$\begin{pmatrix} \varepsilon_x \\ \varepsilon_y \\ \varepsilon_z \\ \gamma_{yz} \\ \gamma_{xz} \\ \gamma_{xy} \end{pmatrix} = \begin{pmatrix} \frac{1}{E_x} & -\frac{\nu_{yx}}{E_y} & -\frac{\nu_{zx}}{E_z} & 0 & 0 & 0 \\ -\frac{\nu_{xy}}{E_x} & \frac{1}{E_y} & -\frac{\nu_{zy}}{E_z} & 0 & 0 & 0 \\ -\frac{\nu_{xz}}{E_x} & -\frac{\nu_{yz}}{E_y} & \frac{1}{E_z} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{G_{yz}} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{G_{xz}} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{G_{xy}} \end{pmatrix} \begin{pmatrix} \sigma_x \\ \sigma_y \\ \sigma_z \\ \tau_{yz} \\ \tau_{xz} \\ \tau_{xy} \end{pmatrix}$$



- Three orthogonal planes elastic symmetry
- 9 independent constants

Transversely Isotropic
One axis of symmetry

- Transversely isotropic – 5 independent parameters

$$\begin{bmatrix} \varepsilon_x \\ \varepsilon_y \\ \varepsilon_z \\ \gamma_{yz} \\ \gamma_{zx} \\ \gamma_{xy} \end{bmatrix} = \begin{bmatrix} \frac{1}{E} & -\frac{\nu'}{E'} & -\frac{\nu}{E} & 0 & 0 & 0 \\ -\frac{\nu'}{E'} & \frac{1}{E'} & -\frac{\nu'}{E'} & 0 & 0 & 0 \\ -\frac{\nu}{E} & -\frac{\nu'}{E'} & \frac{1}{E} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{G'} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{2(1+\nu)}{E} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{G'} \end{bmatrix} \begin{bmatrix} \sigma_x \\ \sigma_y \\ \sigma_z \\ \tau_{yz} \\ \tau_{zx} \\ \tau_{xy} \end{bmatrix}$$

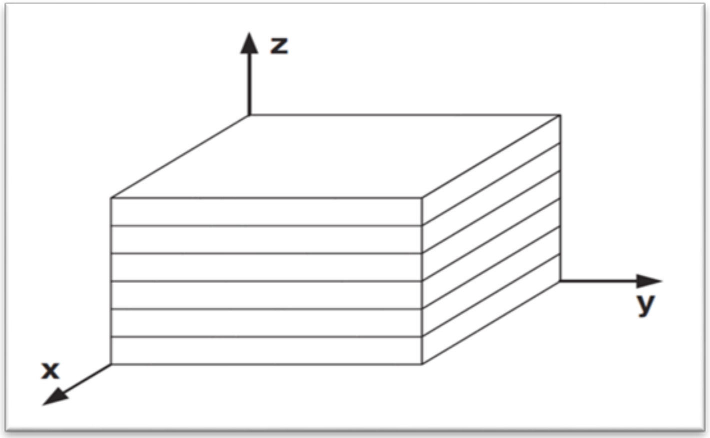
$$E_x = E_z = E$$

$$E_y = E'$$

$$\nu_{xz} = \nu_{zx} = \nu$$

$$\nu_{yx} = \nu_{yz} = \nu'$$

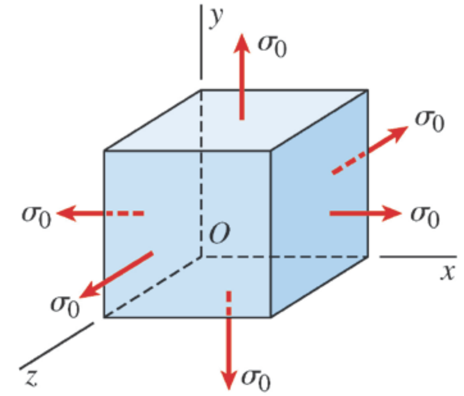
$$G_{xy} = G_{yz} = G'$$



(H. Gercek, 2006)

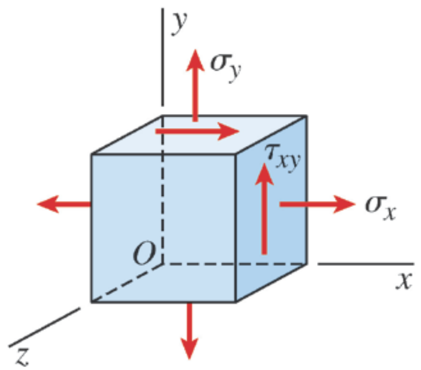
- Hydrostatic Stress :
  - when three normal stresses are equal $\sigma_x = \sigma_y = \sigma_z = \sigma_0$
  - Any plane cut through the element will be subjected to the same normal stress σ_0
  - Normal Strain $\varepsilon_0 = \dfrac{\sigma_0}{E}(1-2\nu)$
  - Unit volume change

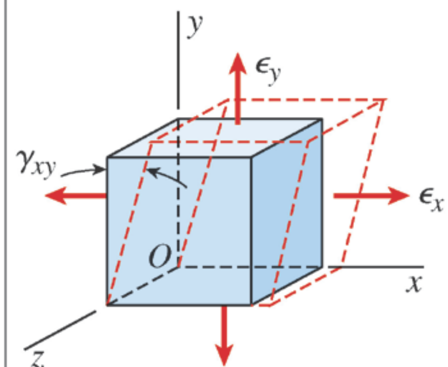
$$e = 3\varepsilon_0 = \frac{3\sigma_0}{E}(1-2\nu) = \frac{\sigma_0}{K}$$



  - Bulk modulus (of elasticity), K $K = \dfrac{E}{3(1-2\nu)} = \dfrac{1}{\beta}$ → Compressibility, **β**
  - Uniform pressure in all directions: <u>Hydrostatic</u>
    - An object submerged in water or deep rock within the earth

Plane Stress & Plane stress condition





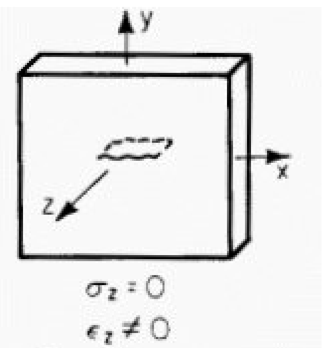
	Plane stress	Plane strain
Stresses	$\sigma_z = 0 \quad \tau_{xz} = 0 \quad \tau_{yz} = 0$ σ_x, σ_y, and τ_{xy} may have nonzero values	$\tau_{xz} = 0 \quad \tau_{yz} = 0$ σ_x, σ_y, σ_z, and τ_{xy} may have nonzero values
Strains	$\gamma_{xz} = 0 \quad \gamma_{yz} = 0$ ϵ_x, ϵ_y, ϵ_z, and γ_{xy} may have nonzero values	$\epsilon_z = 0 \quad \gamma_{xz} = 0 \quad \gamma_{yz} = 0$ ϵ_x, ϵ_y, and γ_{xy} may have nonzero values

Plane Stress & Plane stress condition
Plane stress condition

- Stress and strain in different dimensions are coupled → we need a special consideration – plane strain and plane stress
- Plane stress
  - 3rd dimensional stress goes zero
  - Thin plate stressed in its own plane

$$\begin{pmatrix} \sigma_{xx} & \sigma_{xy} & 0 \\ \sigma_{yx} & \sigma_{yy} & 0 \\ 0 & 0 & 0 \end{pmatrix} \rightarrow \begin{pmatrix} \varepsilon_{xx} & \varepsilon_{xy} & 0 \\ \varepsilon_{yx} & \varepsilon_{yy} & 0 \\ 0 & 0 & \varepsilon_{zz} \end{pmatrix}$$

$$\begin{pmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \gamma_{xy} \end{pmatrix} = \begin{pmatrix} \frac{1}{E} & -\frac{\nu}{E} & 0 \\ -\frac{\nu}{E} & \frac{1}{E} & 0 \\ 0 & 0 & \frac{2(1+\nu)}{E} \end{pmatrix} \begin{pmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{xy} \end{pmatrix}$$



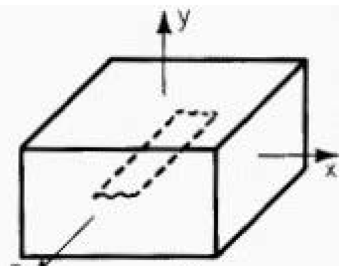
Plane Stress
(Thin Body)

Plane Stress & Plane stress condition
Plane strain condition

- Plane strain
  - 3^{rd} dimensional strain goes zero
  - Stresses around drill hole or 2D tunnel

$$\begin{pmatrix} \varepsilon_{xx} & \varepsilon_{xy} & 0 \\ \varepsilon_{yx} & \varepsilon_{yy} & 0 \\ 0 & 0 & 0 \end{pmatrix} \rightarrow \begin{pmatrix} \sigma_{xx} & \sigma_{xy} & 0 \\ \sigma_{yx} & \sigma_{yy} & 0 \\ 0 & 0 & \sigma_{zz} \end{pmatrix}$$

$$\begin{pmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \gamma_{xy} \end{pmatrix} = \begin{pmatrix} \frac{(1-\nu^2)}{E} & -\frac{\nu(1+\nu)}{E} & 0 \\ -\frac{\nu(1+\nu)}{E} & \frac{(1-\nu^2)}{E} & 0 \\ 0 & 0 & \frac{2(1+\nu)}{E} \end{pmatrix} \begin{pmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{xy} \end{pmatrix}$$



$\epsilon_z = 0$
$\sigma_z = \nu(\sigma_x + \sigma_y)$
Plane Strain
(Thick Body)

Stress equilibrium Equation

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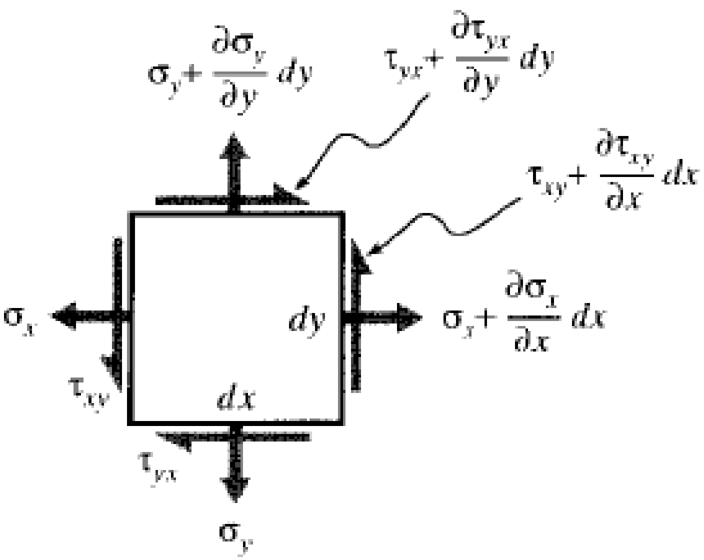
– Sum of traction, body forces (and moment) are zero (static case)



$$\sum F_i = m\frac{\partial^2 u_i}{\partial t} \xrightarrow{\text{Very slow loading}} \sum F_i = 0$$

$$\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z} + \rho b_x = 0$$

$$\frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} + \frac{\partial \tau_{zy}}{\partial z} + \rho b_y = 0$$

$$\frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \sigma_{zz}}{\partial z} + \rho b_z = 0$$

$$\sum M_i = 0 \longrightarrow \tau_{xy} = \tau_{yx}$$

– b_x, b_y, b_z are components of acceleration due to gravity.

Governing Equation
1D

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- Strain-displacement relationship $$\varepsilon = \frac{du}{dx}$$
- Stress-strain relationship $$\varepsilon = \frac{1}{E}\sigma$$
- Equillibrium Equation $$\frac{\partial \sigma_{xx}}{\partial x} + \rho b_x = 0$$
- Final equation for elasticity

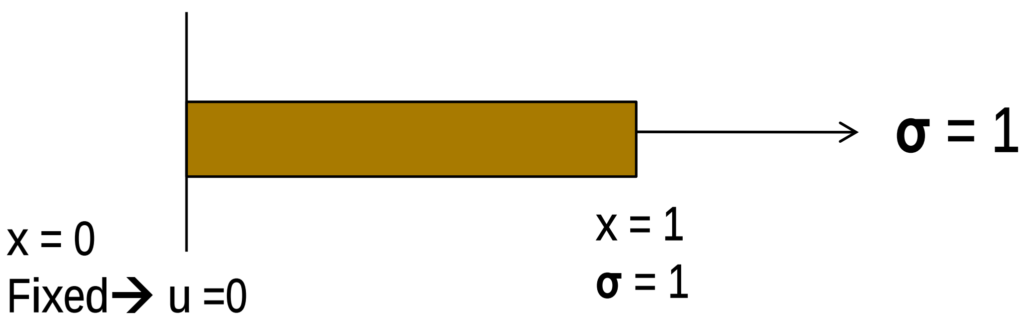
$$E\frac{\partial^2 u_x}{\partial x^2} + \rho b_x = 0$$

Governing Equation

1D - example

– With no body force, and static case

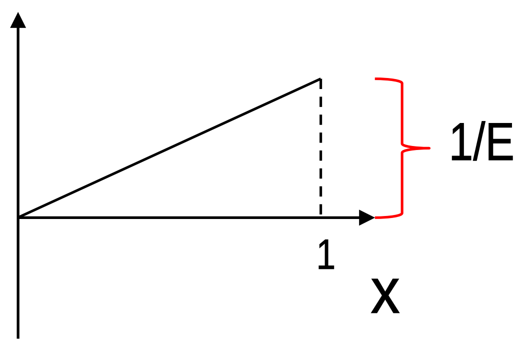
$$E\frac{\partial^2 u_x}{\partial x^2} + \rho b_x = \rho\frac{\partial^2 u_x}{\partial t^2} \qquad E\frac{\partial^2 u_x}{\partial x^2} = 0$$



$$E\frac{\partial u_x}{\partial x} = C_1$$ → From **σ** = 1, C_1=1

$$Eu_x = x + C_2$$ → From u = 0 at x=0, C_2=0

$$u_x = \frac{x}{E}$$



Governing Equation (Navier's Equation)

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- 3D - Navier's equation

$$G\left(\frac{\partial^2 u_x}{\partial x^2}+\frac{\partial^2 u_x}{\partial y^2}+\frac{\partial^2 u_x}{\partial z^2}\right)+(\lambda+G)\left(\frac{\partial^2 u_x}{\partial x^2}+\frac{\partial^2 u_y}{\partial x\partial y}+\frac{\partial^2 u_z}{\partial x\partial z}\right)+\rho b_x=0$$

$$G\left(\frac{\partial^2 u_y}{\partial x^2}+\frac{\partial^2 u_y}{\partial y^2}+\frac{\partial^2 u_y}{\partial z^2}\right)+(\lambda+G)\left(\frac{\partial^2 u_x}{\partial x\partial y}+\frac{\partial^2 u_y}{\partial y^2}+\frac{\partial^2 u_z}{\partial y\partial z}\right)+\rho b_y=0$$

$$G\left(\frac{\partial^2 u_x}{\partial x^2}+\frac{\partial^2 u_x}{\partial y^2}+\frac{\partial^2 u_x}{\partial z^2}\right)+(\lambda+G)\left(\frac{\partial^2 u_x}{\partial x\partial z}+\frac{\partial^2 u_y}{\partial y\partial z}+\frac{\partial^2 u_z}{\partial z^2}\right)+\rho b_z=0$$

$$\lambda=\frac{E\nu}{(1+\nu)(1-2\nu)}$$

References

- Gere JM, Goodno BJ, 2009, Mechanics of Materials, SI Edition, 7th Ed, Cengage Learning, 1002p