Linear DE (1)

- Linear DE

$$\boxed{y' + p(x)y = r(x)}$$

--- the equation is linear in unknown function *y* and its derivative *y'* ; *p*(*x*) and *r*(*x*) can be any given function of x

  - Homogeneous DE

$$r(x) \equiv 0$$

  - Otherwise, non-homogeneous

Linear DE (2)

- General sol. of homogeneous linear DE

$$y' + p(x)y = 0$$

- Separation

$$\frac{dy}{y} = -p(x)dx$$

$$\ln|y| = -\int p(x)dx + c^*$$

- Taking exponentials

$$\Rightarrow \quad y(x) = ce^{-\int p(x)dx} \left(c = \pm e^{c^*}\right)$$

- Trivial sol.

$$c = 0, y(x) = 0$$

Linear DE (3)

- General sol. of Non-homogeneous linear DE

$$\left(py - r\right)dx + dy = 0$$

- Integrating factor *F*(*x*)

$$P = py - r, Q = 1$$

$$\frac{1}{F}\frac{dF}{dx} = p\left(x\right)$$

$$F\left(x\right) = e^{\int pdx}$$

- Exact DE

$$e^{\int pdx}\left(y' + py\right) = \left(e^{\int pdx}y\right)' = e^{\int pdx}r$$

Linear DE (4)

- Integrating the second and the third components

$$e^{\int pdx}y = \int e^{\int pdx} rdx + c$$

- Obtaining *y*(*x*)

$$y(x) = e^{-h}\left[\int e^{h} rdx + c\right], h = \int p(x)dx$$

--- numerical methods may be needed for the integration or the DE itself

Example of linear DE (1)

- Linear non-homogeneous DE

$$y' - y = e^{2x}$$

$$p = -1, r = e^{2x}, h = \int p dx = -x$$

  - Solution formula

$$y(x) = e^{x}\left[\int e^{-x} e^{2x} dx + c\right] = e^{x}\left[e^{x} + c\right] = ce^{x} + e^{2x}$$

  - More direct way, multiply by *e^h*

$$(y' - y)e^{-x} = \left(ye^{-x}\right)' = e^{2x}e^{-x} = e^{x}$$

$$ye^{-x} = e^{x} + c$$

$$\Rightarrow \quad y = e^{2x} + ce^{x}$$

Example of linear DE (2)

- Mixing Problem
  - Tank contains 1000 gal water, 200 lb of salt dissolved
  - 50 gal of brine (1+cos*t* lb of salt per each) flows in per minute
  - Same amount flows out
  - Math model: amount of the salt at time *y*(*t*)

y' = salt inflow rate – salt outflow rate

  - Salt inflow rate = 50 (1+cos*t)* (lb/min)
  - Salt outflow rate: 1 gal contains salt *y*(*t*)/1000 × 50 outflowing gal

$$\Rightarrow \quad 50y(t)/1000 = 0.05y(t)$$

  - Final DE and IC

$$\boxed{y' = 50(1+\cos t) - 0.05y}$$

Example of linear DE (3)

- Linear non-homogeneous DE

$$y' + 0.05y = 50\left(1 + \cos t\right)$$

$$p = 0.05, h = 0.05t$$

  - Solution formula

$$y = e^{-0.05t}\left[\int e^{0.05t}50\left(1 + \cos t\right)dt + c\right]$$

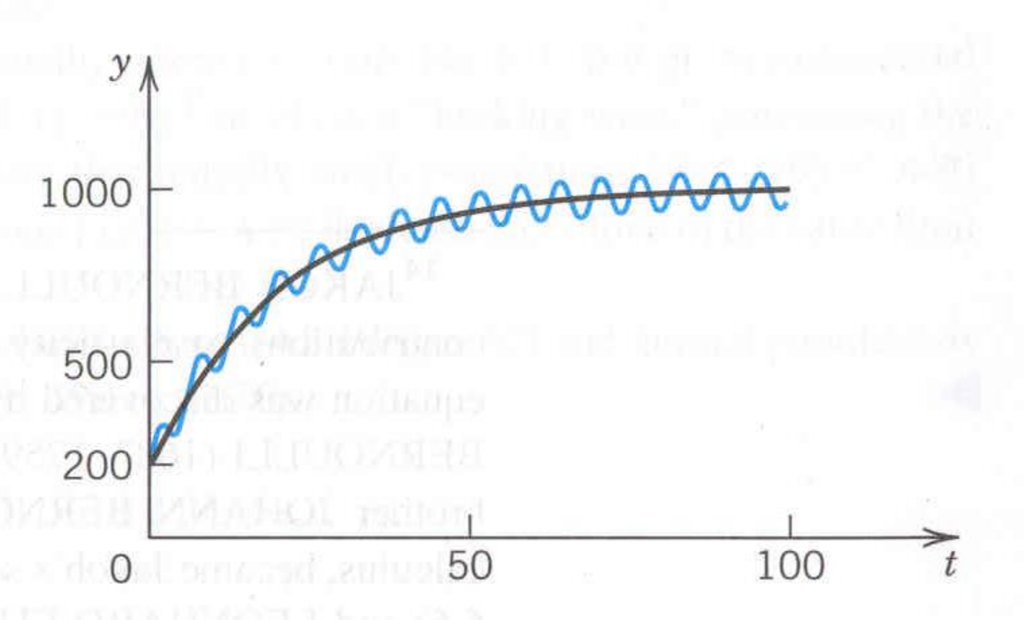
$$= e^{-0.05t}\left[e^{0.05t}\left(1000 + a\cos t + b\sin t\right) + c\right]$$

$$= 1000 + 2.494\cos t + 49.88\sin t + ce^{-0.05t}$$

  - From IC

$$\Rightarrow \quad y\left(t\right) = 1000 + 2.494\cos t + 49.88\sin t - 802.5e^{-0.05t}$$

Example of linear DE (4)



- Mean value

$$y' + 0.05y = 50$$

➡ $$y(t) = 1000 - 800e^{-0.05t}$$

Example of linear DE (5)

$$\boxed{y' + 2y = e^x\left(3\sin 2x + 2\cos 2x\right)}$$

$$p = 2, h = 2x$$

- Solution formula

$$y = e^{-2x}\left[\int e^{2x}e^x\left(3\sin 2x + 2\cos 2x\right)dx + c\right]$$

$$= e^{-2x}\left[e^{3x}\sin 2x + c\right]$$

$$\Rightarrow \qquad = ce^{-2x} + e^x\sin 2x$$

Example of linear DE (6)

- Initial Value Problem

$$\boxed{y' + y\tan x = \sin 2x, y(0) = 1}$$

$$p = \tan x, r = \sin 2x, = 2\sin x \cos x,$$

$$h = \int p dx = \int \tan x dx = \ln\left|\sec x\right|$$

  - Solution formula

$$e^{h} = \sec x, e^{-h} = \cos x, e^{h} r = (\sec x)(2\sin x \cos x) = 2\sin x$$

$$y(x) = \cos x\left[2\int \sin x dx + c\right] = c\cos x - 2\cos^2 x$$

  - From IC

➡ $$y = 3\cos x - 2\cos^2 x$$

Nonlinear DE

- Reduction to Linear DE --- Bernoulli Eq.

$$\boxed{y' + p(x)\,y = g(x)\,y^{a}}$$

(*a* any real number)

--- if *a*=0 or *a*=1, linear; otherwise, nonlinear

- Nonlinear case, set

$$u(x) = \left[y(x)\right]^{1-a}$$

- Differentiate and substitute with the given DE

$$u' = (1-a)\,y^{-a}y' = (1-a)\,y^{-a}\left(gy^{a} - py\right)$$

- Simplification

$$u' = (1-a)\left(g - py^{1-a}\right)$$

$$\Rightarrow \quad u' + (1-a)\,pu = (1-a)\,g$$

Example of Nonlinear DE (1)

- Verhulst Eq.

$$\boxed{y' - Ay = -By^2}$$

(*A*,*B* positive const.)

$$a = 2, u = y^{-1}$$

  - Solution procedure of nonlinear DE

$$u' = -y^{-2}y' = -y^{-2}\left(-By^2 + Ay\right) = B - Ay^{-1}$$

$$u' + Au = B$$

  - Solution procedure of non-homogeneous linear DE

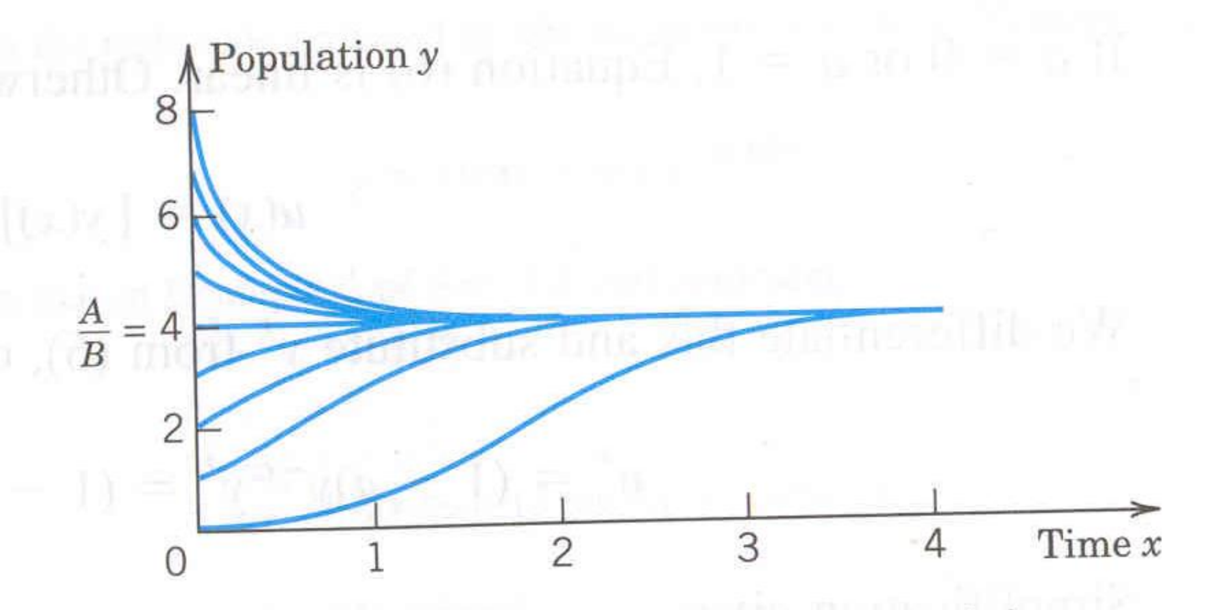
$$p = A, h = Ax.r = B$$

$$u = e^{-Ax}\left[\int Be^{Ax}dx + c\right] = e^{-Ax}\left[\frac{B}{A}e^{Ax} + c\right] = ce^{-Ax} + \frac{B}{A}$$

Example of Nonlinear DE (2)

$$\Rightarrow \quad y=\frac{1}{u}=\frac{1}{\left(B/A\right)+ce^{-Ax}}$$

Logistic law of population growth



Physical Meaning of DE

- Linear DE

$$y' + p(x)\,y = r(x)$$

--- *x*: time

r(*x*): force ← *Input*

y(*x*): displacement, current ← *Output*

– Solution of non-homogeneous DE

$$y(x) = e^{-h}\int e^{h} r\,dx + ce^{-h}$$

Total Output = Response to the Input + Response to the Initial Data

Modeling: Electric Circuit

- Elements of a Circuit
  - Electromotive force: source of electric energy (generator, battery)
  - Resistor: light-bulb
  - Inductor: opposes change in current, an inertia effect
  - Capacitor: stores energy

- Governing Relation
  - Ohm's Law: voltage drop across a resistor is proportional to the instantaneous current

$$E_R = RI$$

resistance

Electric Circuit (1)

- Governing Relation
  - Voltage drop across an inductor: proportional to the instantaneous time rate of change of the current

$$E_L = L\frac{dI}{dt}$$

Inductance (Henry)

  - Voltage drop across a capacitor: proportional to the instantaneous electric charge on the capacitor

$$E_C = \frac{1}{C}Q$$

charge (Coulomb)

capacitance (Farad)

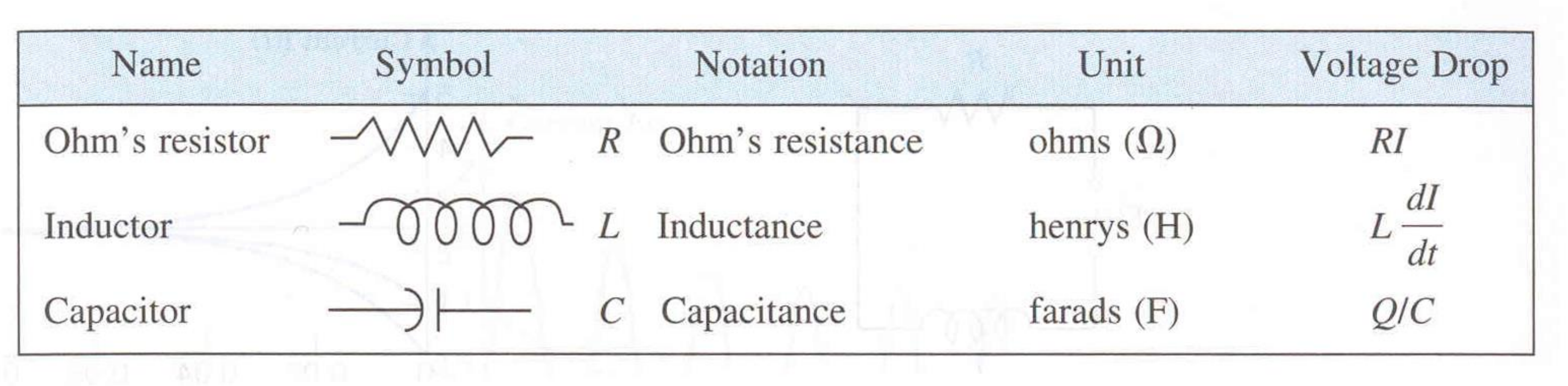
Electric Circuit (2)

- Between the charge and current

$$I(t) = \frac{dQ}{dt}$$

- Voltage drop across a capacitor

$$E_C = \frac{1}{C}\int_{t_0}^{t} I(t^*)dt^*$$



Name	Symbol	Notation	Unit	Voltage Drop
Ohm's resistor		R Ohm's resistance	ohms (Ω)	RI
Inductor		L Inductance	henrys (H)	$L\frac{dI}{dt}$
Capacitor		C Capacitance	farads (F)	Q/C

Kirchhoff’s Voltage Law

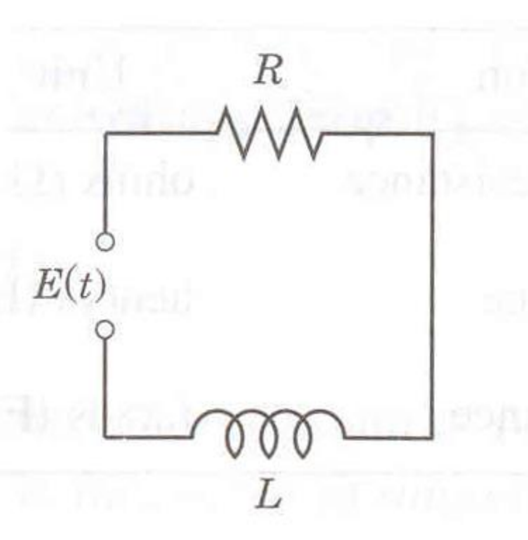
- The algebraic sum of all the instantaneous voltage drops around any closed loop is zero.
- The voltage drop impressed on a closed loop is equal to the sum of the voltage drops in the rest of the loop

- RL Circuit
  - Constant electromotive force
  - Periodic electromotive force

$$L\frac{dI}{dt}+RI=E\left(t\right)$$

  - Numerical values assigned

$$\frac{dI}{dt}+50I=120$$



Example: RL Circuit (1)

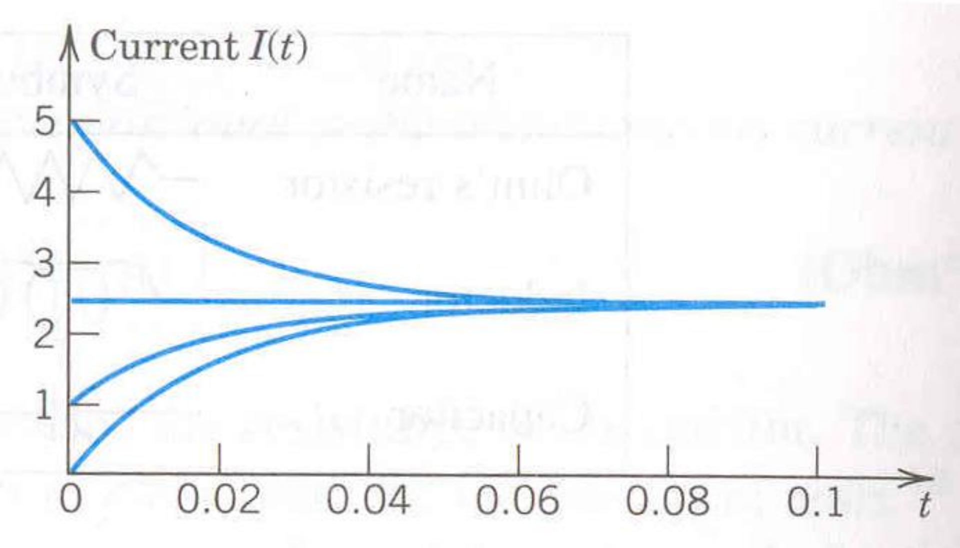
- Using integrating factor e^{50t}

$$\frac{d}{dt}\left(e^{50t}I\right)=120e^{50t},\ e^{50t}I=\frac{120}{50}e^{50t}+c,$$

$$I=2.4+ce^{-50t}$$

- Or solution procedure of non-homogeneous DE

$$I(t)$$
$$=e^{-50t}\left[\int e^{50t}120dt+c\right]$$
$$=\frac{120}{50}+ce^{-50t}$$



Example: RL Circuit (2)

- Without numerical values

$$\frac{dI}{dt}+\frac{R}{L}I=\frac{E}{L}$$

$$I(t)=e^{-\alpha t}\left[\int e^{\alpha t}\frac{E}{L}dt+c\right],\alpha=\frac{R}{L}$$

- Constant electromotive force $E=E_0$

Steady-state sol.

$$I(t)=e^{-\alpha t}\left[\frac{E_0}{L}\cdot\frac{L}{R}e^{\alpha t}+c\right]=\frac{E_0}{R}+ce^{-\alpha t}$$

Inductive time const

- From IC $I(0)=0$

$$I(t)=\frac{E_0}{R}\left(1-e^{-\alpha t}\right)=\frac{E_0}{R}\left(1-e^{-t/\tau_L}\right)$$

Example: RL Circuit (3)

- Periodic electromotive force *E* (*t*) =E_0sin ωt

$$I\left(t\right)=e^{-\alpha t}\left[\frac{E_0}{L}\int e^{\alpha t}\sin\omega t dt+c\right],\alpha=\frac{R}{L}$$

- Integrating by parts

$$I\left(t\right)=ce^{-\left(R/L\right)t}+\frac{E_0}{R^2+\omega^2L^2}\left(R\sin\omega t-\omega L\cos\omega t\right)$$

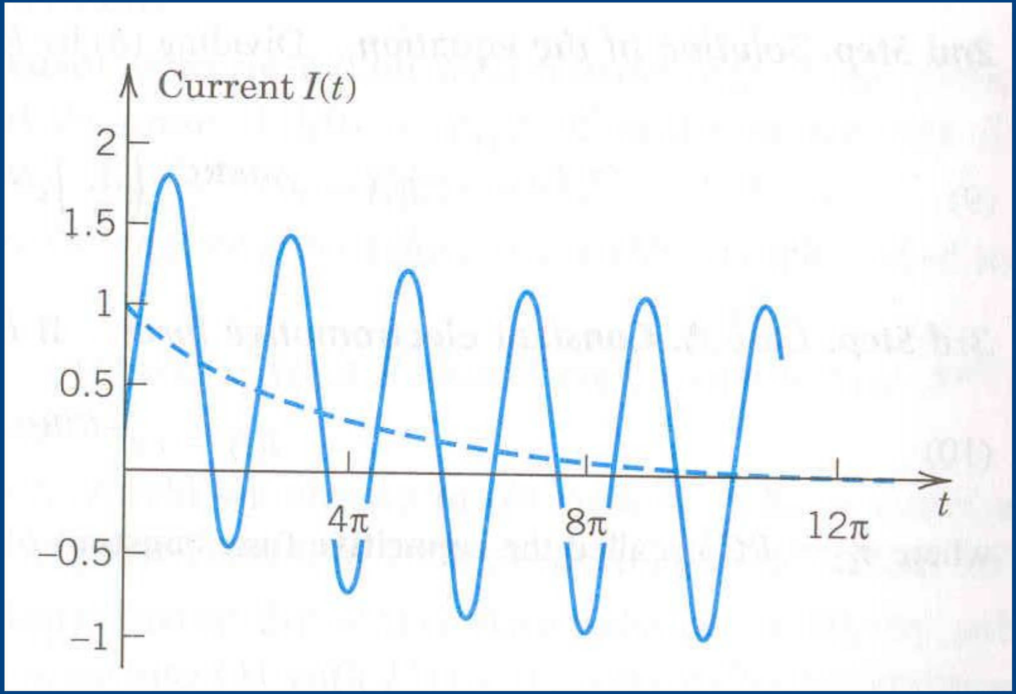
$$I\left(t\right)=ce^{-\left(R/L\right)t}+\frac{E_0}{\sqrt{R^2+\omega^2L^2}}\sin\left(\omega t-\delta\right),$$

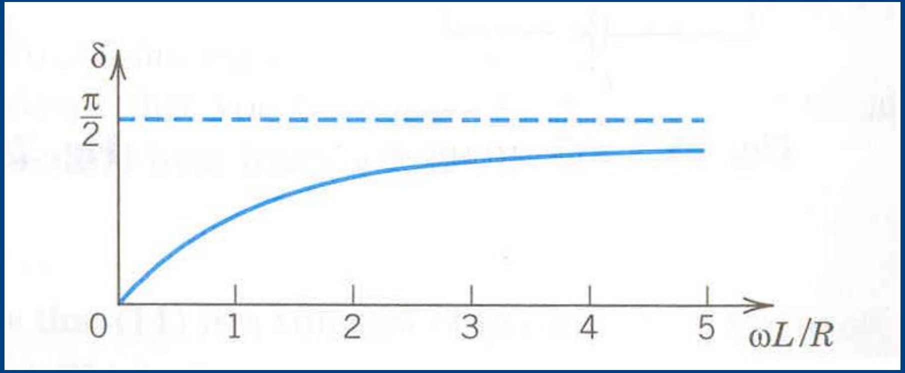
Transient-state sol.

Steady-state sol.

$$\delta=\arctan\frac{\omega L}{R}$$

Example: RL Circuit (4)

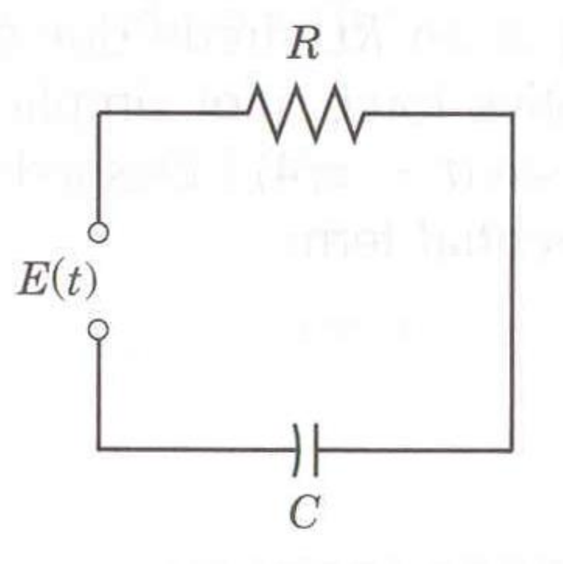




Example: RC Circuit (1)

- RC Circuit
  - Governing Eqn.

$$RI+\frac{1}{C}\int Idt=E\left(t\right)$$



  - Differentiating

$$R\frac{dI}{dt}+\frac{1}{C}I=\frac{dE}{dt}$$

  - Solution procedure of non-homogeneous DE

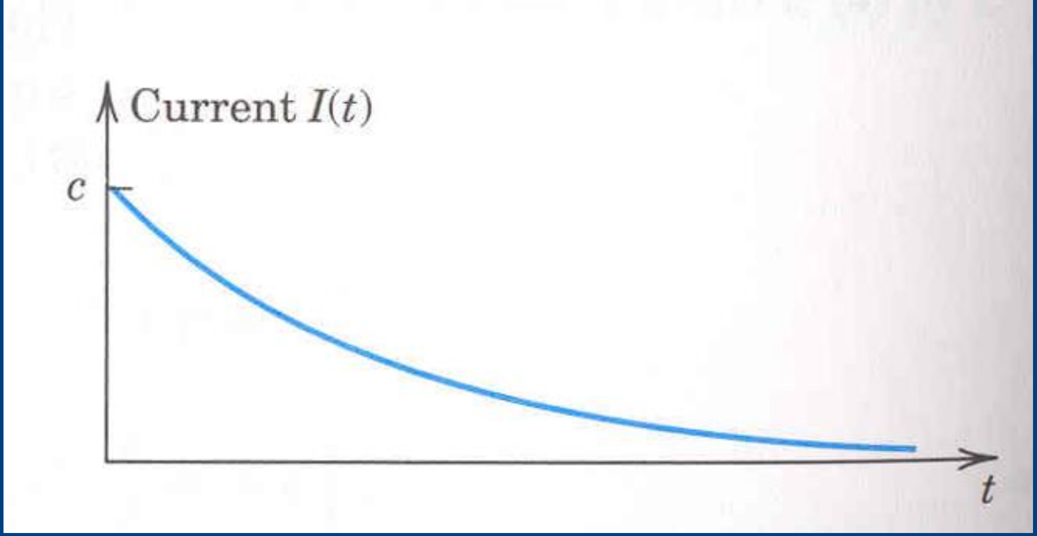
$$I\left(t\right)=e^{-t/(RC)}\left(\frac{1}{R}\int e^{t/(RC)}\frac{dE}{dt}dt+c\right)$$

  - If *E* = const

$$I\left(t\right)=ce^{-t/(RC)}=ce^{-t/\tau_C}$$

Capacitive time const

Example: RC Circuit (2)



– If $E = E_0 \sin \omega t$

$$\frac{dE}{dt} = \omega E_0 \cos \omega t$$

Example: RC Circuit (3)

- Substituting and integrating by parts

$$I\left(t\right)=ce^{-t/\left(RC\right)}+\frac{\omega E_0 C}{1+\left(\omega RC\right)^2}\left(\cos\omega t+\omega RC\sin\omega t\right)$$

$$=ce^{-t/\left(RC\right)}+\frac{\omega E_0 C}{\sqrt{1+\left(\omega RC\right)^2}}\sin\left(\omega t+\delta\right)$$

Transient-state sol.

Steady-state sol.

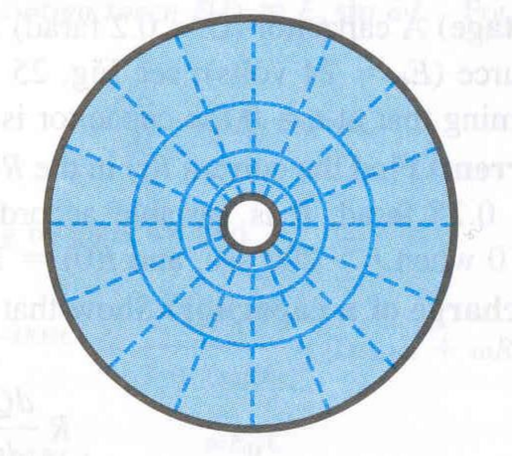
$$\delta=-\frac{1}{\omega RC}$$

- RLC circuit --- 2nd order DE

Orthogonal Trajectories

- Orthogonal Trajectories

  --- that intersect given curves at right angles
  - Meridians and parallels on the globe
  - Steepest descent and the contour lines on a map
  - Electric force curves are orthogonal to the equipotential lines
  - Fluid flow, heat conduction, etc.



How to Find Trajectories

- Using DE: given curve

$$y' = f(x, y)$$

- Orthogonal curve

$$y' = -\frac{1}{f(x, y)}$$

- General sol. --- family of curves

$$F(x, y, c) = 0$$

One-parameter family of curves

Example of Finding Trajectories (1)

$$\boxed{y = cx^2}$$

One-parameter family of curves

- Parameter representation

$$F\left(x, y, c\right) = y - cx^2 = 0$$

- Differentiating

$$y' = 2cx$$

- Solve for c algebraically, differentiate, and multiply by x^2

$$\frac{y}{x^2} = c, \frac{y'}{x^2} - 2\frac{y}{x^3} = 0$$

$$y' = \frac{2y}{x}$$

Example of Finding Trajectories (2)

- Orthogonal curve

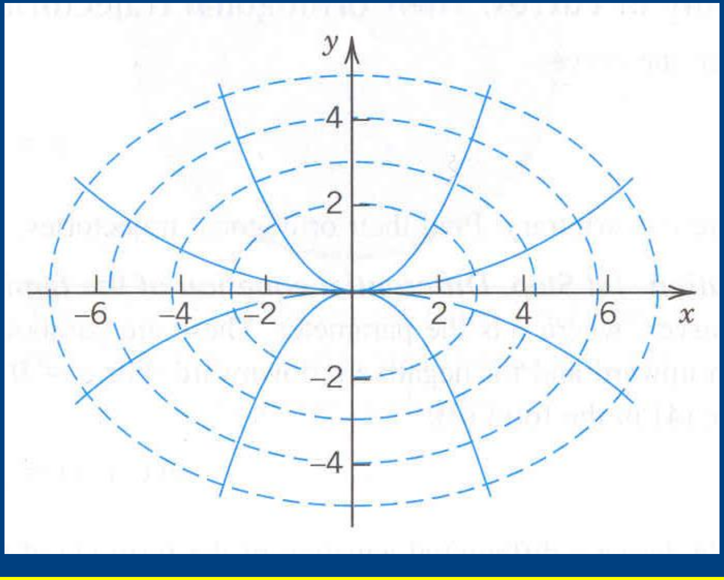
$$y' = -\frac{x}{2y}$$

- Separation

$$2ydy = -xdx, y^2 = -\frac{x^2}{2} + c^*$$

- Usual form

$$\frac{1}{2}x^2 + y^2 = c^*$$



Homogeneous Linear DE of 2nd Order

$$\boxed{y'' + p(x)y' + q(x) = r(x)}$$

- Homogeneous DE

$$y'' + p(x)y' + q(x) = 0$$

- Example of a non-homogeneous linear DE

$$y'' + 4y = e^{-x}\sin x$$

- Example of a homogeneous linear DE

$$\left(1 - x^2\right)y'' - 2xy' + 6y = 0$$

- Example of a non-linear DE

$$x\left(y''y + y'^2\right) + 2y'y = 0$$

Homogeneous Linear DE of 2nd Order

- Example of a non-linear DE

$$y'' = \sqrt{y'^2 + 1}$$

- Superposition or Linearity Principle
  - DE

$$y'' - y = 0$$

  - Solution

$$y = e^x, y = e^{-x},$$

$$y = -3e^x + \frac{2}{5}e^{-x}$$

  - Linear combination of the known solutions y_1, y_2

$$y = c_1 y_1 + c_2 y_2$$

Superposition Principle

--- Any linear combination of two solutions is again a solution on I. Sums and constant multiples of solutions are again solutions.

➡ Does not hold for non-homogeneous and nonlinear DE

– Example of non-homogeneous DE

$$y'' + y = 1$$

– Solution

$$y = 1 + \cos x, \; y = 1 + \sin x$$

– But, not a solution

$$2(1 + \cos x), (1 + \cos x) + (1 + \sin x)$$

Superposition Principle

- Example of non-linear DE

$$y''y - xy' = 0$$

- Solution

$$y = x^2, y = 1$$

- But, not a solution

$$-x^2, x^2 + 1$$

Initial Value Problem (1)

- Solution of 2nd order homogeneous linear DE

$$\boxed{y = c_1 y_1 + c_2 y_2}$$

- Two IC's

$$\boxed{y\left(x_0\right) = K_0, y'\left(x_0\right) = K_1}$$

- Example of IVP

$$y'' - y = 0, y\left(0\right) = 4, y'\left(0\right) = -2$$

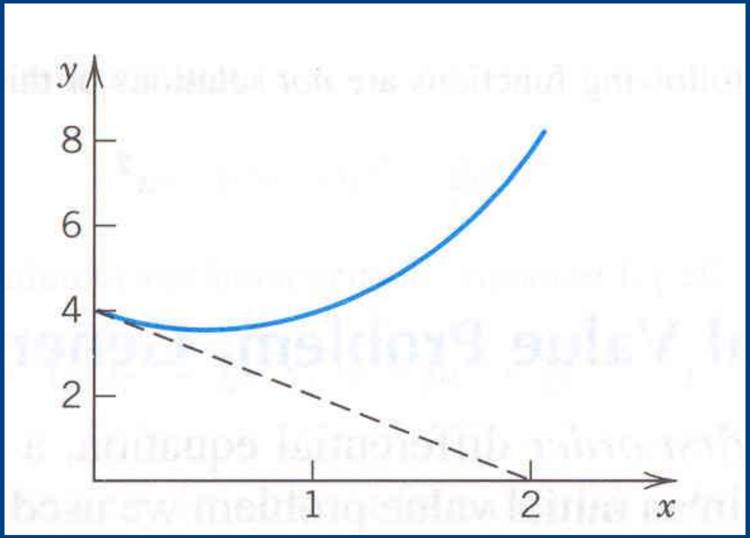
- Two known solutions e^x, e^{-x}

$$y = c_1 e^x + c_2 e^{-x}$$

- Particular solution

$$y = e^x + 3e^{-x}$$

Initial Value Problem (2)



- Wrong selection of known solutions e^x, e^{-x}

$$y = c_1 e^x + c_2 l e^x$$

→ Cannot solve the DE

Basis (Fundamental System)

- General sol.of homogeneous linear DE

$$y = c_1 y_1 + c_2 y_2$$

- Not proportional to the known solutions

$$y_1, y_2 \quad \longleftarrow \quad \boxed{\textit{Basis}}$$

- Particular solution can be obtained by assigning c_1, c_2

$$y = c_1 y_1 + c_2 y_2$$

- Linearly independent y_1, y_2 on I

$$k_1 y_1 (x) + k_2 y_2 (x) = 0 \quad \Rightarrow \quad k_1 = 0, k_2 = 0$$

Basis (Fundamental System)

- Linearly dependent

$$y_1 = -\frac{k_2}{k_1} y_2$$