Homogeneous Linear DE of 2nd Order

$$y'' + p(x)y' + q(x) = r(x)$$

– Homogeneous DE

$$y'' + p(x)y' + q(x) = 0$$

– Example of a non-homogeneous linear DE

$$y'' + 4y = e^{-x}\sin x$$

– Example of a homogeneous linear DE

$$(1-x^2)y'' - 2xy' + 6y = 0$$

– Example of a non-linear DE

$$x(y''y + y'^2) + 2y'y = 0$$

Homogeneous Linear DE of 2nd Order

– Example of a non-linear DE

$$y'' = \sqrt{y'^2 + 1}$$

- Superposition or Linearity Principle
  – DE

$$y'' - y = 0$$

  – Solution

$$y = e^x, y = e^{-x},$$

$$y = -3e^x + \frac{2}{5}e^{-x}$$

  – Linear combination of the known solutions y_1, y_2

$$y = c_1 y_1 + c_2 y_2$$

Superposition Principle

--- Any linear combination of two solutions is again a solution on I. Sums and constant multiples of solutions are again solutions.

⇨ Does not hold for non-homogeneous and nonlinear DE

– Example of non-homogeneous DE

$$y'' + y = 1$$

– Solution

$$y = 1 + \cos x, \; y = 1 + \sin x$$

– But, not a solution

$$2(1 + \cos x), (1 + \cos x) + (1 + \sin x)$$

Superposition Principle

– Example of non-linear DE

$$y''y - xy' = 0$$

– Solution

$$y = x^2, y = 1$$

– But, not a solution

$$-x^2, x^2 + 1$$

Initial Value Problem (1)

– Solution of 2nd order homogeneous linear DE

$$y = c_1 y_1 + c_2 y_2$$

– Two IC's

$$y(x_0) = K_0, y'(x_0) = K_1$$

– Example of IVP

$$y'' - y = 0, y(0) = 4, y'(0) = -2$$

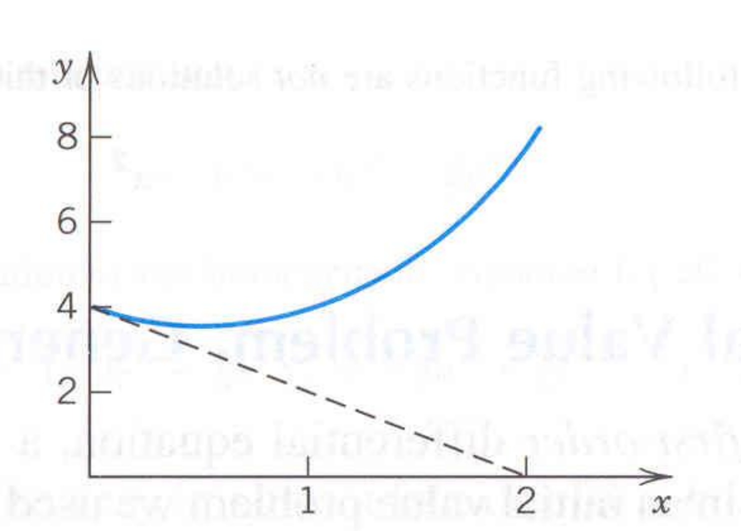
– Two known solutions e^x, e^{-x}

$$y = c_1 e^x + c_2 e^{-x}$$

– Particular solution

$$y = e^x + 3e^{-x}$$

Initial Value Problem (2)



– Wrong selection of known solutions e^x, e^{-x}

$$y = c_1 e^x + c_2 l e^x$$

➡ Cannot solve the DE

Basis (Fundamental System)

– General sol.of homogeneous linear DE

$$y = c_1 y_1 + c_2 y_2$$

– Not proportional to the known solutions

$$y_1, y_2 \longleftarrow \boxed{\textit{Basis}}$$

– Particular solution can be obtained by assigning c_1, c_2

$$y = c_1 y_1 + c_2 y_2$$

– Linearly independent y_1, y_2 on I

$$k_1 y_1(x) + k_2 y_2(x) = 0 \Rightarrow k_1 = 0, k_2 = 0$$

Basis (Fundamental System)

– Linearly dependent

$$y_1 = -\frac{k_2}{k_1} y_2$$

Basis (Fundamental System)

– General sol.of homogeneous linear DE
--- a pair y_1, y_2 on I of linearly independent sol.

– Example

$$y'' - y = 0$$

– General sol.

$$\Rightarrow \quad y = c_1 e^{x} + c_2 e^{-x}$$

– Example

$$y'' + y = 0$$

– General sol.

$$\Rightarrow \quad y = c_1 \cos x + c_2 \sin x$$

Method of Reduction of Order (1)

- How to obtain basis if one solution is known
- set

$$y_2 = uy_1$$

- Differentiating

$$y_2' = u'y_1 + uy_1'$$
$$y_2'' = u''y_1 + 2u'y_1' + uy_1''$$

- Original DE

$$u''y_1 + 2u'y_1' + uy_1'' + p\left(u'y_1 + uy_1'\right) + quy_1 = 0$$

- Rearranging in u

0

$$u''y_1 + u'\left(2y_1' + py_1\right) + u\left(y_1'' + py_1' + qy_1\right) = 0$$

Method of Reduction of Order (2)

– Divide by y_1 and set

$$u' = U, u'' = U'$$

– Then

$$u'' + u' \frac{2y_1' + py_1}{y_1} = 0$$

$$U' + \left(\frac{2y_1'}{y_1} + p \right) U = 0$$

– Separation

$$\frac{dU}{U} = -\left(\frac{2y_1'}{y_1} + p \right) dx$$

$$\ln|U| = -2\ln|y_1| - \int p dx$$

Method of Reduction of Order (3)

– Taking exponentials

$$U = \frac{1}{y_1^2} e^{-\int p dx}$$

– Desired sol. y_2

$$y_2 = u y_1 = y_1 \int U dx$$

--- where

$$\frac{y_2}{y_1} = u = \int U dx \neq \text{const.}$$

Example of Reduction of Order

– 2nd order DE

$$x^2 y'' - xy' + y = 0$$

– Standard form

$$y'' - \frac{1}{x} y' + \frac{1}{x^2} y = 0$$

– One known sol.

$$y_1 = x$$

– Formulation of the other sol.

$$U = \frac{1}{x^2} e^{\ln x} = \frac{1}{x}$$

$$y_2 = ux = x \int U dx = x \ln x$$

2nd Order DE with Const. Coeff. (1)

– 2nd order DE

$$y'' + ay' + by = 0$$

– 1st order DE and its solution

$$y' + ky = 0$$

$$y = e^{-k}x$$

– Substitution

$$y = e^{\lambda}x$$

$$y' = \lambda e^{\lambda}x, y'' = \lambda^2 e^{\lambda}x$$

$$\Rightarrow \left(\lambda^2 + a\lambda + b\right)e^{\lambda x} = 0$$

← *Characteristic equation*

2nd Order DE with Const. Coeff. (2)

- Roots of the characteristic eqn.

$$\lambda_1 = \frac{1}{2}\left(-a + \sqrt{a^2 - 4b}\right), \lambda_2 = \frac{1}{2}\left(-a - \sqrt{a^2 - 4b}\right)$$

- Bases of the original DE

$$y_1 = e^{\lambda_1} x, y_2 = e^{\lambda_2} x$$

- Three cases

  (Case I) two real roots if $a^2 - 4b > 0$

  (Case II) a real double root if $a^2 - 4b = 0$

  (Case I) complex conjugate roots if $a^2 - 4b < 0$

• Two Distinct Real Roots

- Bases of the original DE

$$y_1 = e^{\lambda_1} x, y_2 = e^{\lambda_2} x$$

Case I. Two Distinct Real Roots (1)

– General sol.

$$y = c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x}$$

– Example

$$y'' - y = 0$$

– Characteristic Eqn., its roots

$$\lambda^2 - 1 = 0,$$

$$\lambda_1 = 1, \lambda_2 = -1$$

– General sol.

$$y = c_1 e^{x} + c_2 e^{-x}$$

Case I. Two Distinct Real Roots (2)

– Initial Value Problem

$$y'' + y' - 2y = 0, y(0) = 4, y'(0) = -5$$

– Characteristic Eqn., its roots

$$\lambda^2 + \lambda - 2 = 0,$$

$$\lambda_1 = \frac{1}{2}\left(-1 + \sqrt{9}\right) = 1, \lambda_1 = \frac{1}{2}\left(-1 - \sqrt{9}\right) = -2$$

– General sol.

$$y = c_1 e^x + c_2 e^{-2x}$$

– Particular sol.

$$y = e^x + 3e^{-2x}$$

Case II. Real Double Root (1)

– Only one sol.

$$y_1 = e^{-(a/2)x}$$

– To obtain a second independent solution y_2, use reduction of order.

– Set

$$y_2 = uy_1$$

$$y_2' = u'y_1 + uy_1'$$

$$y_2'' = u''y_1 + 2u'y_1' + uy_1''$$

– Substituting and rearranging

$$u''y_1 + u'\underbrace{\left(2y_1' + py_1\right)}_{0} + u\underbrace{\left(y_1'' + py_1' + qy_1\right)}_{0} = 0$$

Since

$$2y_1' = -ae^{-ax/2} = -ay_1$$

Case II. Real Double Root (2)

– Thus

$$u''y_1 = 0, u'' = 0$$

– Double integrating

$$u = c_1 x + c_2$$

– Select an independent sol.

$$y_2 = uy_1, u = x$$

– Corresponding general solution

$$y = \left(c_1 + xc_2\right)e^{-ax/2}$$

Example of Real Double Root (1)

– DE

$$y'' + 8y' + 16y = 0$$

– Characteristic Eq.

$$\lambda^2 + 8\lambda + 16 = 0$$

– Double root

$$\lambda = -4$$

– Corresponding general solution

$$y = \left(c_1 + xc_2\right)e^{-4x}$$

Example of Real Double Root (2)

- Initial Value Problem

$$y'' - 4y' + 4y = 0,\ y(0) = 3,\ y'(0) = 1$$

- Characteristic Eq.

$$\lambda^2 + 4\lambda + 4 = 0$$

- Double root

$$\lambda = 2$$

- Corresponding general solution

$$\boxed{y = \left(c_1 + xc_2\right)e^{2x}}$$

- Particular solution

$$y = \left(3 - 5x\right)e^{2x}$$

Case III. Complex Conjugate Roots (1)

- 2nd order DE

$$y'' + ay' + by = 0$$

- Characteristic Eq.

$$\lambda^2 + a\lambda + b = 0$$

- Roots of the characteristic eqn.

$$\lambda_1 = \frac{1}{2}\left(-a + \sqrt{a^2 - 4b}\right), \lambda_2 = \frac{1}{2}\left(-a - \sqrt{a^2 - 4b}\right)$$

- (Case III) complex conjugate roots if

$$a^2 - 4b < 0$$

Fundamental of Complex Roots (1)

- 2nd order DE

$$y'' + y = 0$$

- Characteristic Eq.

$$\lambda^2 + 1 = 0$$

- Roots of the characteristic eqn.

$$\lambda = \pm\sqrt{-1} = \pm i$$

- Two complex bases

$$e^{ix}, e^{-ix}$$

Fundamental of Complex Roots (2)

– Complex exponential function

$$e^{ix} = \cos x + i \sin x$$
$$e^{-ix} = \cos x - i \sin x$$

Euler formula

– Adding

$$\cos x = \frac{1}{2}\left(e^{ix} + e^{-ix}\right)$$

– Subtracting

$$\sin x = \frac{1}{2i}\left(e^{ix} - e^{-ix}\right)$$

– The other bases:

$$\cos x, \sin x$$

Complex Exponential Function

$$e^z = e^{s+it} = e^s e^{it} = e^s\left(\cos t + i\sin t\right)$$

- Real case: $z = s$
- Maclaurin series

$$e^{it} = 1 + it + \frac{(it)^2}{2!} + \frac{(it)^3}{3!} + \frac{(it)^4}{4!} + \frac{(it)^5}{5!} + \cdots$$

$$= 1 - \frac{t^2}{2!} + \frac{t^4}{4!} - + \cdots + i\left(t - \frac{t^3}{3!} + \frac{t^5}{5!} - + \cdots\right)$$

$$= \cos t + i\sin t$$

Case III. Complex Conjugate Roots (2)

– One root of the characteristic eqn.

$$\lambda_1 = -\frac{1}{2}a + \frac{1}{2}\sqrt{a^2 - 4b} = -\frac{1}{2}a + i\sqrt{b - \frac{1}{4}a^2}$$

– Roots of the characteristic eqn.

$$\lambda_1 = -\frac{1}{2}a + i\omega, \lambda_2 = -\frac{1}{2}a - i\omega$$

where

$$\omega = \sqrt{b - \frac{1}{4}a^2}$$

Case III. Complex Conjugate Roots (3)

– Using complex exponential function

$$e^{\lambda_1 x} = e^{-(a/2)x + i\omega x} = e^{-(a/2)x}\left(\cos\omega x + i\sin\omega x\right)$$

$$e^{\lambda_2 x} = e^{-(a/2)x - i\omega x} = e^{-(a/2)x}\left(\cos\omega x - i\sin\omega x\right)$$

– Adding and divide by 2 → y_1

$$y_1 = e^{-ax/2}\cos\omega x$$

– Subtracting and divide by 2i → y_2

$$y_2 = e^{-ax/2}\sin\omega x$$

New bases

– General solution

$$y = e^{-ax/2}\left(A\cos\omega x + B\sin\omega x\right)$$

Example of Complex Conjugate Roots (1)

– Initial value problem

$$y'' + 0.2y' + 4.01y = 0,\ y(0) = 0,\ y'(0) = 2$$

– Characteristic eqn.

$$\lambda^2 + 0.2\lambda + 4.01 = 0$$

– Roots of the characteristic eqn.

$$\lambda = -0.1 \pm 2i$$

– General solution

$$y = e^{-0.1x}\left(A\cos 2x + B\sin 2x\right)$$

Example of Complex Conjugate Roots (2)

– First IC

$$y(0) = A = 0$$

– Then, remaining and its derivative

$$y = Be^{-0.1x}\sin 2x$$

$$y' = B\left(-0.1e^{-0.1x}\sin 2x + 2e^{-0.1x}\cos 2x\right)$$

– Second IC

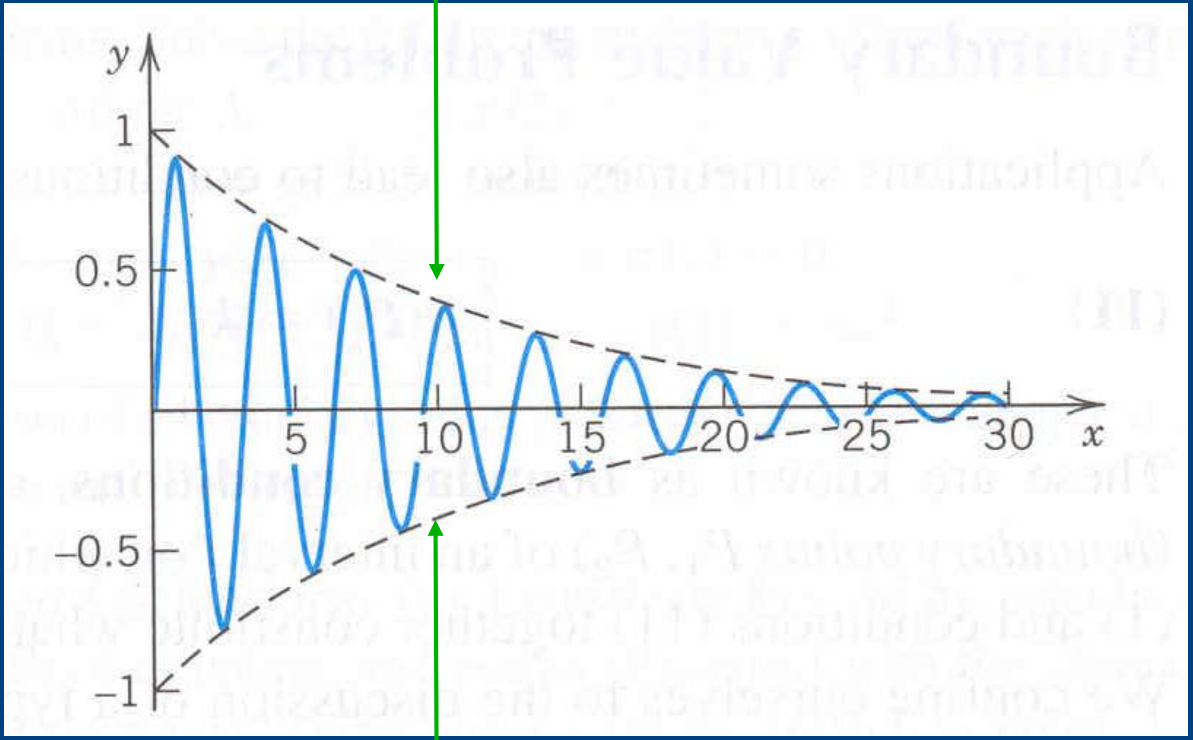
$$y'(0) = 2B = 2, B = 1$$

– Particular solution

$$y = e^{-0.1x}\sin 2x$$

Example of Complex Conjugate Roots (3)

$e^{-0.1x}$



$-e^{-0.1x}$

Damped vibrations

Example of Complex Conjugate Roots (4)

– DE

$$y'' + \omega^2 y = 0$$

– General solution

$$y = A\cos\omega x + B\sin\omega x$$

Summary

Case	Roots of Characteristic Eq.	Basis of DE	General sol.
I	Distinct real λ_1, λ_2	$e^{\lambda_1 x}, e^{\lambda_2 x}$	$y = c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x}$
II	Real double root $\lambda = -\frac{1}{2}a$	$e^{-ax/2}, xe^{-ax/2}$	$y = (c_1 + c_2 x) e^{-ax/2}$
III	Complex conjugate $\lambda_{1,2} = -\frac{1}{2}a \pm i\omega$	$e^{-ax/2}\cos\omega x,$ $e^{-ax/2}\sin\omega x$	$y = e^{-ax/2} \cdot$ $(A\cos\omega x + B\sin\omega x)$

Boundary Value Problem

– DE

$$y(P_1)=k_1,\, y(P_2)=k_2$$

Boundary conditons

– Example

$$y''+y=0,\, y(0)=3,\, y(\pi)=-3$$

– General sol.

$$y(x)=c_1\cos x+c_2\sin x$$

– Particular sol.

$$y(x)=3\cos x+c_2\sin x$$

Not determined yet