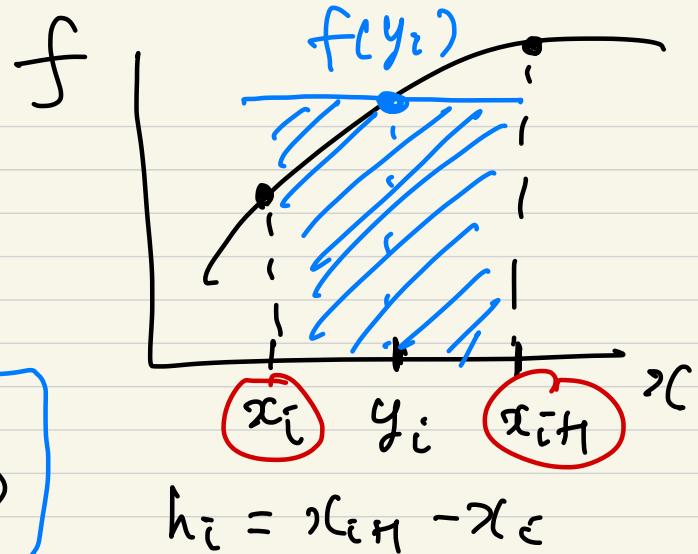


### 3.2 Error analysis

\* Rectangle (or midpoint) rule

$$y_i = \frac{1}{2}(x_i + x_{i+1}) : \text{midpoint}$$

$$\int_{x_i}^{x_{i+1}} f(x) dx = (x_{i+1} - x_i) f(y_i) = h_i f(y_i)$$



$$f(x) = f(y_i) + (x - y_i) f'(y_i) + \frac{1}{2} (x - y_i)^2 f''(y_i) + \dots$$

$$\int_{x_i}^{x_{i+1}} f(x) dx = f(y_i) \cdot h_i + \frac{1}{2} (x - y_i)^2 \Big|_{y_i}^{x_{i+1}} f'(y_i) + \frac{1}{6} (x - y_i)^3 \Big|_{y_i}^{x_{i+1}} f''(y_i) + \dots$$

$$= \boxed{f(y_i) \cdot h_i} + \frac{1}{24} h_i^3 f'''(y_i) + \frac{1}{920} h_i^5 f''''(y_i) + \dots$$

midpoint rule  
(MR)

leading error

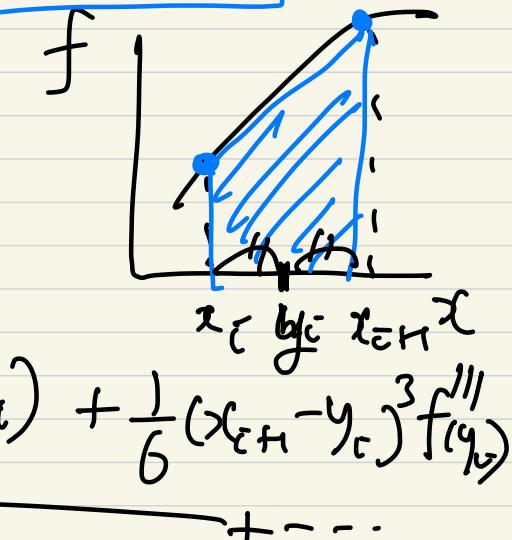
3rd-order accurate for one interval

For trapezoidal rule

$$-\frac{1}{2} h_i$$

$$\int_{x_i}^{x_{i+1}} f(x) dx = \frac{h_i}{2} [f(x_i) + f(x_{i+1})]$$

$$f(x_i) = f(y_i) + (x_i - y_i) f'(y_i) + \frac{1}{2} (x_i - y_i)^2 f''(y_i) + \dots + \frac{1}{6} (x_i - y_i)^3 f'''(y_i) + \dots$$



$$+ \left| f(x_{i+1}) = f(y_i) + (x_{i+1} - y_i) f'(y_i) + \frac{1}{2} (x_{i+1} - y_i)^2 f''(y_i) + \frac{1}{6} (x_{i+1} - y_i)^3 f'''(y_i) + \dots \right.$$

$$\frac{1}{2} (f(y_i) + f(x_{i+1})) = f(y_i) + \frac{1}{8} h_i^2 f''(y_i) + \frac{1}{384} h_i^4 f''''(y_i) + \dots$$

$$\rightarrow f(y_i) = \frac{1}{2} [f(y_i) + f(x_{i+1})] - \frac{1}{8} h_i^2 f''(y_i) - \frac{1}{384} h_i^4 f''''(y_i) + \dots$$

Substitute this into  $\textcircled{X}$

$$\rightarrow \int_{x_i}^{x_{i+1}} f(x) dx = \frac{1}{2} h_i [f(x_i) + f(x_{i+1})]$$

trapezoidal rule (TR)

$$- \frac{1}{12} h_i^3 f''(y_i) - \frac{1}{480} h_i^5 f''''(y_i) + \dots$$

leading error

3rd-order accurate for one interval

$\Rightarrow$  The error of TR is twice bigger than that of MR.

- Global interval  $[a, b]$

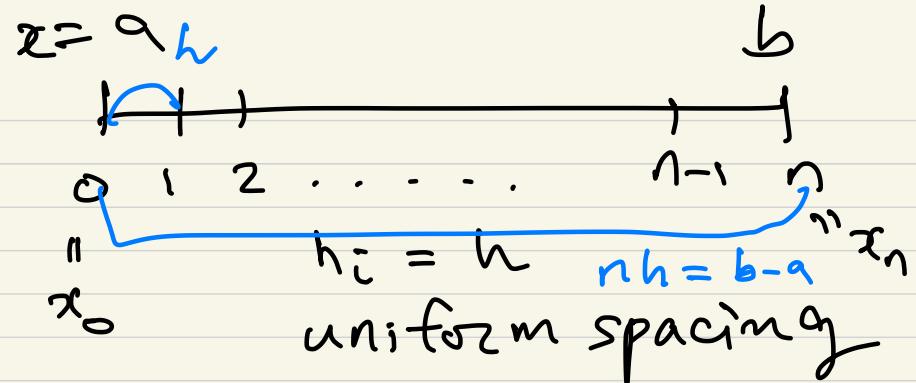
$$I = \int_a^b f(x) dx \quad TR$$

$$= \sum_{i=0}^{n-1} \int_{x_i}^{x_{i+1}} f(x) dx$$

$$= \sum_{i=0}^{n-1} \left[ \frac{1}{2} h \left( f(x_i) + f(x_{i+1}) \right) - \frac{1}{12} h_i^3 f''(q_i) - \frac{1}{480} h_i^5 f'''(q_i) + \dots \right]$$

$$= \frac{h}{2} \left[ f(a) + f(b) + 2 \sum_{i=1}^{n-1} f(x_i) \right] - \frac{1}{12} h^2 \sum_{i=0}^{n-1} f''(q_i) - \frac{1}{480} h^5 \sum_{i=0}^{n-1} f'''(q_i) + \dots$$

$(b-a)/h$



(Mean value theorem  $\sum_{i=0}^{n-1} f''(q_i) = n f''(\bar{x})$  where  $a \leq \bar{x} \leq b$ )

$\sum_{i=0}^{n-1} f'''(q_i) = n f'''(\xi)$   $a \leq \xi \leq b$

$$= \frac{h}{2} \left[ f(a) + f(b) + 2 \sum_{i=1}^{n-1} f(x_i) \right] - (b-a) \frac{1}{12} h^2 f''(\bar{x})$$

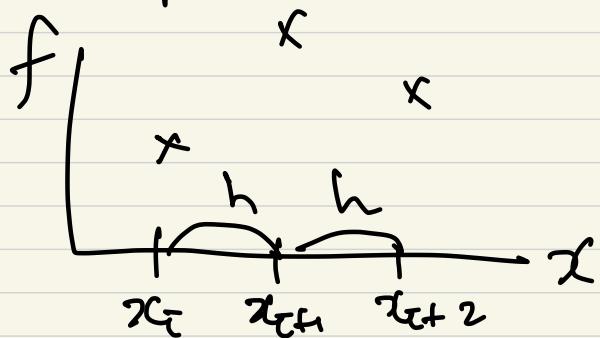
leading error

$$- (b-a) \frac{1}{480} h^4 f'''(\xi) \dots$$

$\therefore$  2nd-order accurate for entire interval.

$\Rightarrow$  TR TS second-order accurate!

Simpson's rule



$$\int_{x_c}^{x_{c+2}} f(x) dx = \frac{(2h)}{6} [f(x_c) + 4f(x_{c+1}) + f(x_{c+2})] \equiv S(f)$$

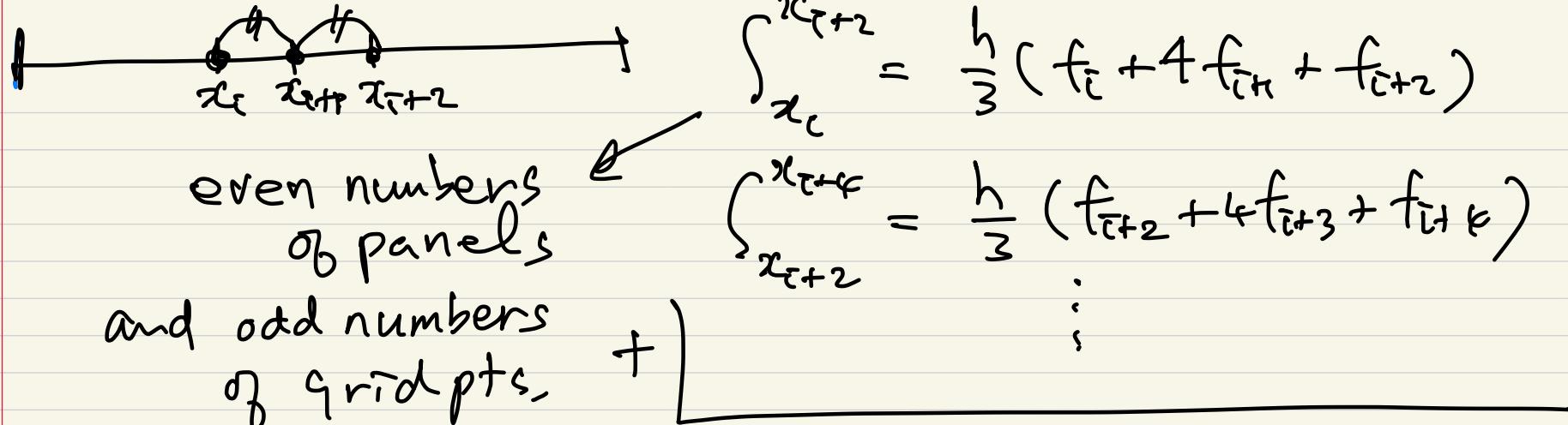
$$\int_{x_c}^{x_{c+2}} \epsilon(x) dx = \frac{(2h)}{2} [f(x_c) + f(x_{c+2})] \equiv T(f)$$

$$\int_{x_c}^{x_{c+2}} f(x) dx = (2h) f(x_{c+1}) \equiv R(f)$$

$$\rightarrow S(f) = \frac{2}{3} R(f) + \frac{1}{3} T(f)$$

Recall that the truncation error of TR is twice that of MR with opposite sign.

$\rightarrow$  Simpson's rule is 5th order accurate for one interval  
and 4th-order accurate for entire interval.



$$I = \int_a^{x_{i+2}} f(x) dx = \frac{h}{3} [f(a) + f(b) + 4 \sum_{j=1, \text{ odd}}^{n-1} f_j + 2 \sum_{j=2, \text{ even}}^{n-2} f_j]$$

$\frac{h^4}{180}$

$$-\frac{h^4}{180} (b-a) f''(\bar{x}) + \dots$$

(leading error)

4th-order accurate.

### 3.3 TR w/ end correction

$$\int_{x_i}^{x_{i+1}} f(x) dx = \frac{h}{2} (f_i + f_{i+1}) - \frac{1}{12} h_c^3 f'''(q_c) - \frac{1}{480} h_c^5 f''''(q_c) + \dots$$

TR

CD2

$$\frac{f'_{i+1} - f'_i}{h_c} - \frac{1}{6} \left( \frac{h_c}{2} \right)^2 f''''(q_c) + \dots$$

$$= \frac{h}{2} (f_i + f_{i+1}) - \frac{1}{12} h_c^2 (f'_{i+1} - f'_i) + \frac{1}{720} h_c^5 f''''(q_c) + \dots$$

∴ neglect this w/o losing accuracy.

Sum over the entire domain.

$$I = \int_a^b f(x) dx = \left[ \frac{h}{2} \sum_{j=1}^{n-1} (f_j + f_{j+n}) - \frac{1}{12} h^2 (f'(b) - f'(a)) + \frac{1}{720} (b-a) h^4 f''(\xi) \right] + \dots$$

TR w/ end correction

4<sup>th</sup> order accurate

ex)  $f(x) = e^x \quad \int_0^4 e^x dx = e^4 - 1 = 53.59815 \dots$

apts.  $I_{TR} = 54.71015 \quad \text{error} = -1.112$

$I_{SP} = 53.61622 \quad -0.01807$

$I_{TC} = 53.59352 \quad +0.00463 !$

### 3.4 Romberg integration and Richardson extrapolation

technique for obtaining an accurate sol. by combining two or more less accurate sols.

→ integral method + Richardson extrapolation

$$TR : I = \int_a^b f(x) dx = \frac{h}{2} [f(a) + f(b) + 2 \sum_{j=1}^{n-1} f_j] + c_1 h^2 + c_2 h^4 + \dots \equiv \tilde{I}$$

$$\rightarrow \tilde{I}_1 = I - c_1 h^2 - c_2 h^4 - \dots : \text{2nd-order accurate}$$

Apply TR w/  $h_1 = h/2$  → call this  $\tilde{I}_2$

$$\rightarrow \tilde{I}_2 = I - c_1 \left(\frac{h}{2}\right)^2 - c_2 \left(\frac{h}{2}\right)^4 - \dots$$

$$\text{Idea: } 4\tilde{I}_2 - \tilde{I}_1 = 3I + \frac{3}{4} c_2 h^4 + \dots$$

$$\rightarrow \frac{4\tilde{I}_2 - \tilde{I}_1}{3} = I + \frac{1}{4} c_2 h^4 + \dots : \text{4th-order accurate}$$

Combined two 2nd-order estimates of  $I \rightarrow$  4th-order accurate estimate

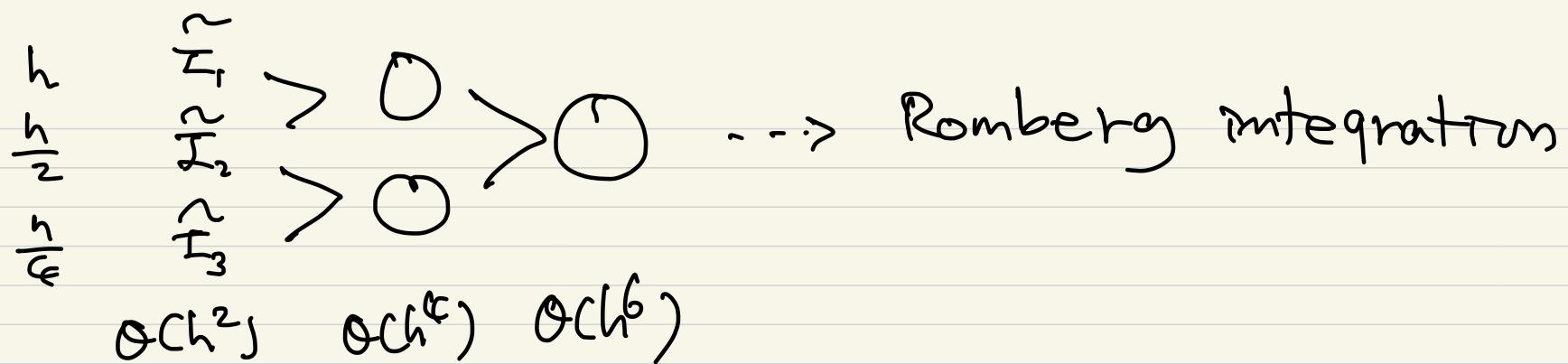
$$\text{Evaluate } I \text{ w/ } h_2 = h/4 \rightarrow \tilde{I}_3$$

$$\rightarrow \tilde{I}_3 = I - c_1 \left(\frac{h}{4}\right)^2 - c_2 \left(\frac{h}{4}\right)^4 - \dots$$

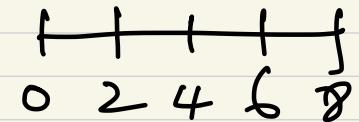
$$\rightarrow \frac{4\tilde{I}_3 - \tilde{I}_2}{3} = I + \frac{1}{64} c_2 h^4 + \frac{5}{1024} c_3 h^6 + \dots$$

$$\Rightarrow \frac{16}{15} \left( \frac{4\tilde{I}_3 - \tilde{I}_2}{3} \right) - \frac{1}{15} \left( \frac{4\tilde{I}_2 - \tilde{I}_1}{3} \right) = I + O(h^6) + \dots$$

6th-order accurate!



$$\text{ex)} \quad I = \int_0^8 \left( \frac{5}{8}x^4 - 4x^3 + 2x + 1 \right) dx = 72$$



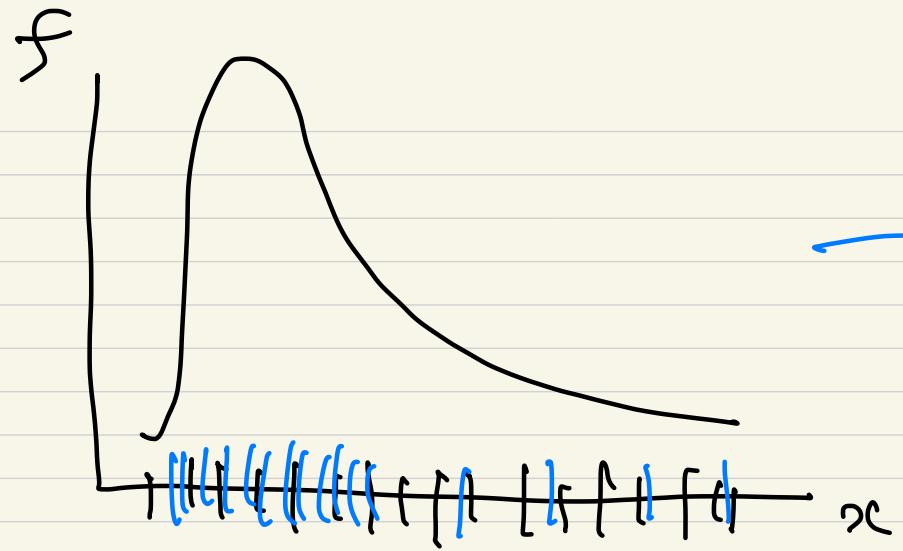
$$TR : I_1 = \frac{8-0}{2} (f(8) + f(0)) = 2120$$

$$h/2 : I_2 = \frac{8-0}{4} (f(8) + f(0) + 2f(4)) = 712$$

$$h/4 : I_3 = \frac{8-0}{8} [f(8) + f(0) + 2f(2) + 2f(4) + 2f(6)] = 240$$

$$\begin{aligned}
 h &: 2120 \xrightarrow{\quad} \frac{712 \times 4 - 2400}{3} = \frac{728}{3} \xrightarrow{\quad} 72 \leftarrow \text{exact sol.} \\
 h/2 &: 712 \\
 h/4 &: 240 \\
 &\qquad\qquad\qquad \dots = \frac{248}{3} \xrightarrow{\quad} \alpha h^6 f^{(6)}(\bar{x}) \\
 &\qquad\qquad\qquad O(h^2) \qquad O(h^4) \qquad O(h^6)
 \end{aligned}$$

**restriction**: points are evenly distributed throughout the interval of integration.



adaptive quadrature