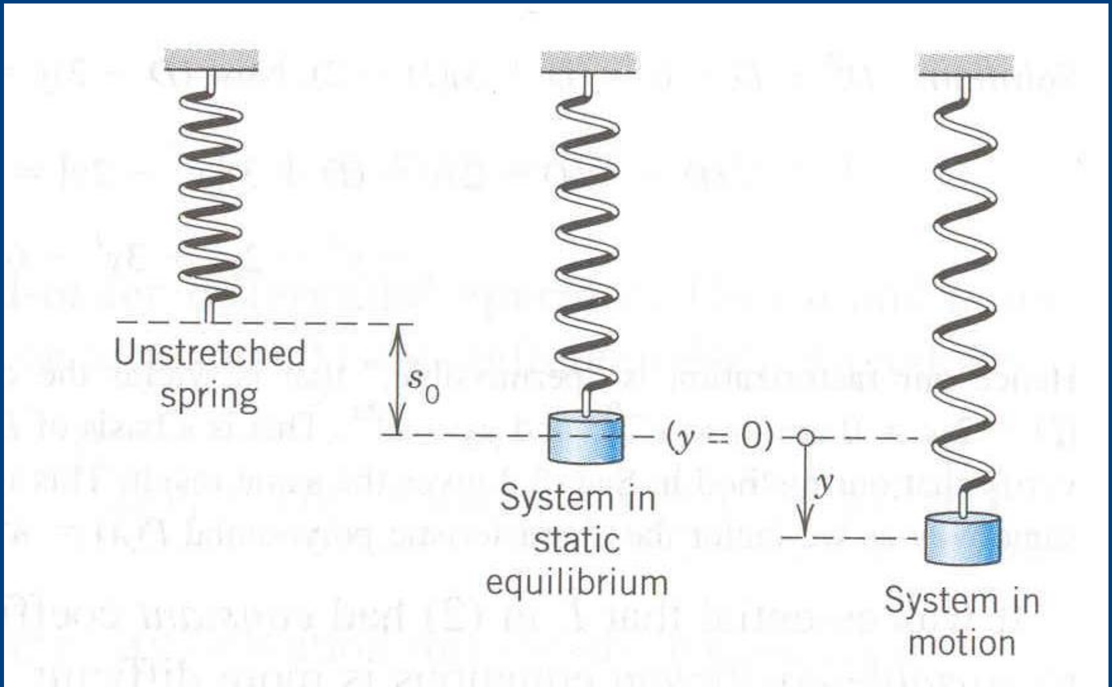
Modeling: Mass-Spring System (1)

- Newton's 2nd Law

$$my'' = F$$

- Spring-Mass System



Modeling: Mass-Spring System (2)

– Static equilibrium

$$F_0 = -ks_0$$ ← *Hooke's law*

--- F_0 balances the weight $W=mg$ ← *Datum*

$$F_0 + W = -ks_0 + mg = 0$$

– Restoring force

$$F_1 = -ky$$

- Undamped System --- assume no damping

$$my'' + ky = 0$$

Modeling: Undamped System (1)

– General sol.

$$y(t) = A\cos\omega_0 t + B\sin\omega_0 t, \left(\omega_0 = \sqrt{k/m}\right)$$

Harmonic oscillation

Or

$$y(t) = C\cos(\omega_0 t - \delta)$$

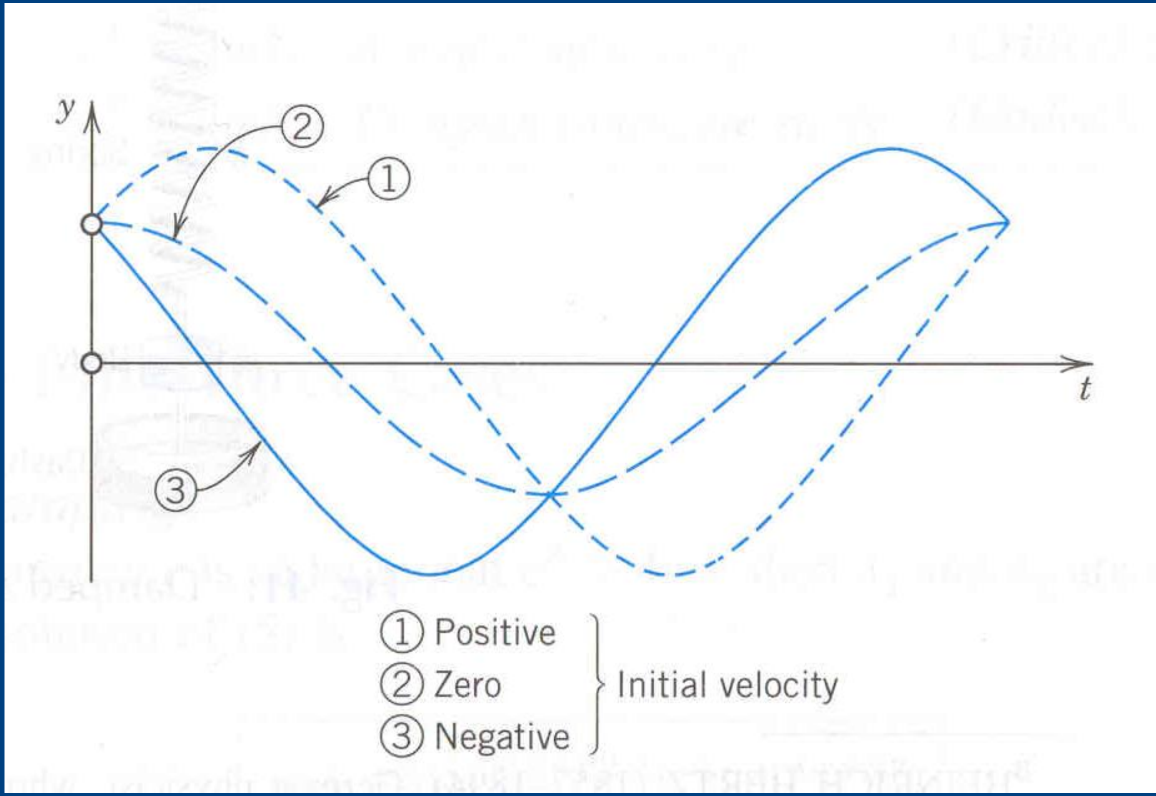
where

$$C = \sqrt{A^2 + B^2}, \tan\delta = \frac{B}{A}$$

– Period $2\pi/\omega_0$

Frequency $\omega_0/2\pi$ (cycle/sec, Hz)

Modeling: Undamped System (2)



Modeling: Damped System (1)

– Damping force existing

$$F = -cy'$$

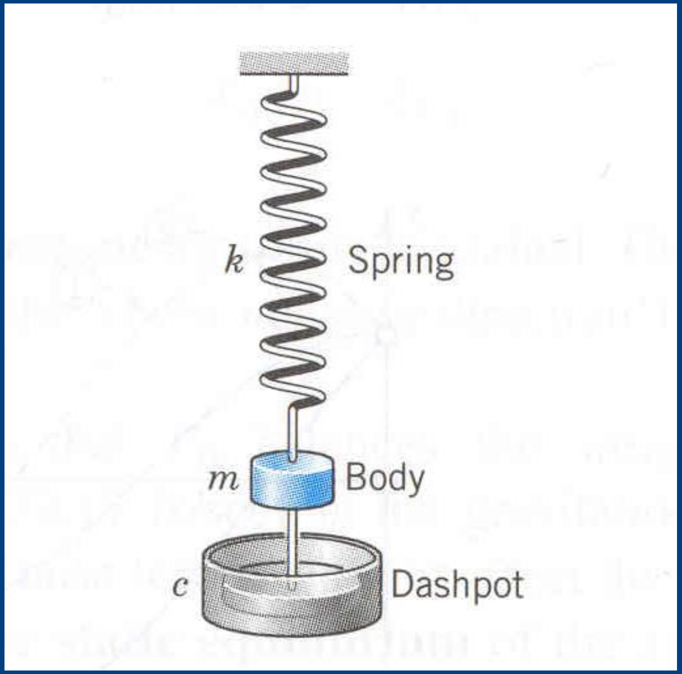
Damping constant, >0

– Governing eqn.

$$my'' + cy' + ky = 0$$

– Characteristic eqn.

$$\lambda^2 + \frac{c}{m}\lambda + \frac{k}{m} = 0$$



Modeling: Damped System (2)

– Roots of the characteristic eqn.

$$\lambda_{1,2} = -\frac{c}{2m} \pm \frac{1}{2m}\sqrt{c^2 - 4mk}$$

Or

$$\lambda = -\alpha \pm \beta$$

– Three cases

I. Distinct real roots $c^2 > 4mk$ ← *Overdamping*

II. Real double root $c^2 = 4mk$ ← *Critical damping*

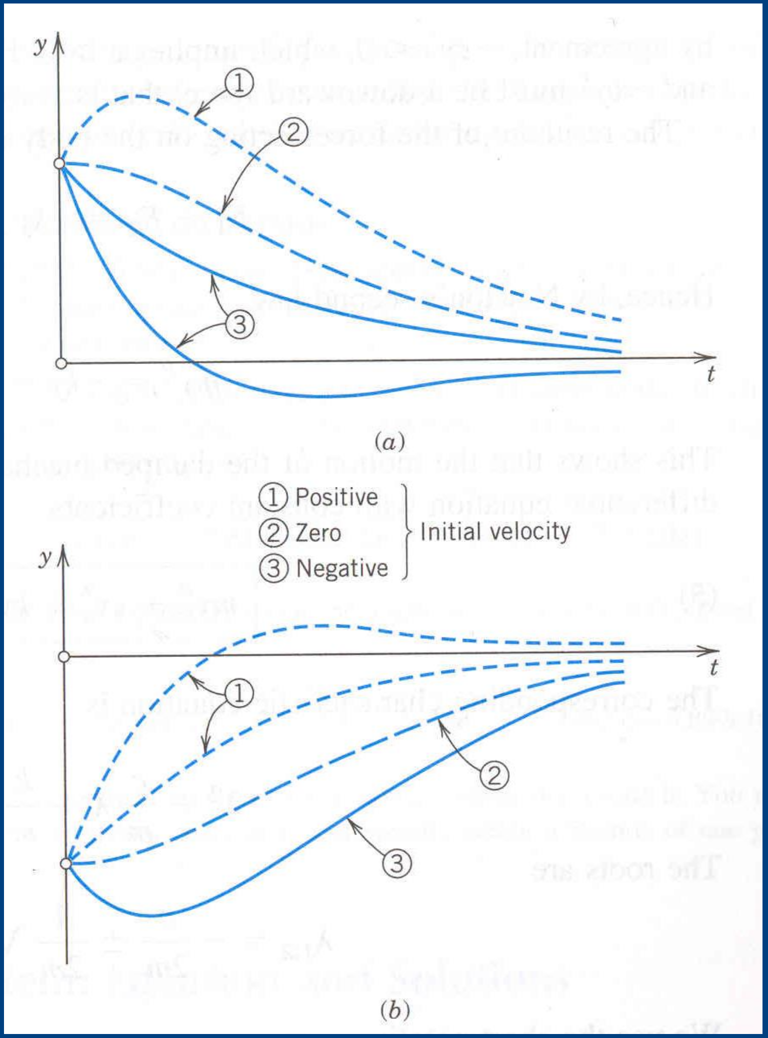
III. Complex conjugate roots $c^2 < 4mk$ ← *Underdamping*

Damped System: I. Overdamping

– General sol.

$$y(t) = c_1 e^{-(\alpha-\beta)t} + c_2 e^{-(\alpha+\beta)t}$$

--- The mass will be at rest $y = 0$ since the damping takes energy from the system.



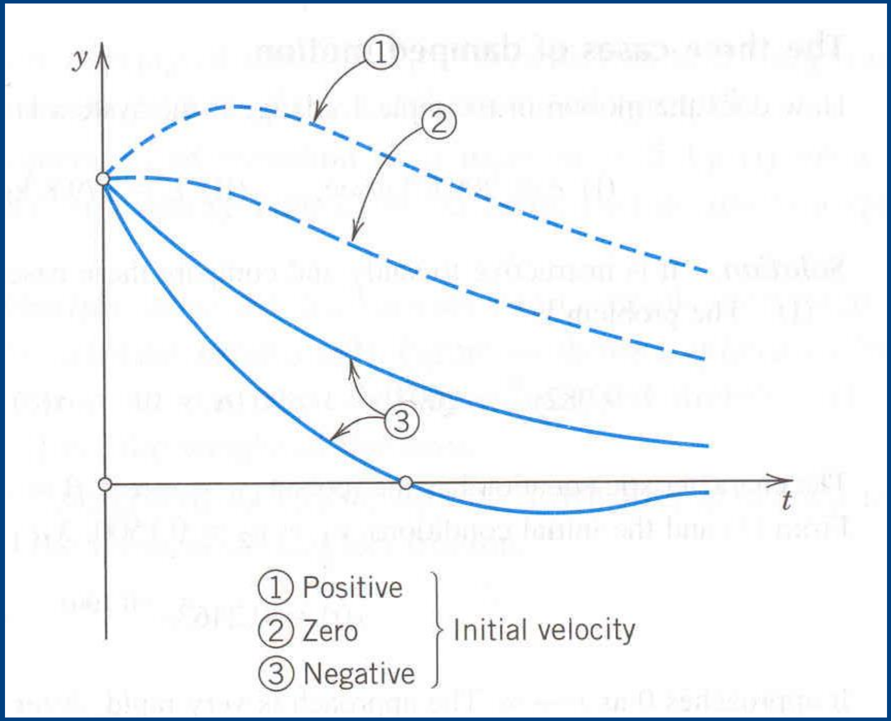
Damped System: II. Critical Damping

– General sol.

$$y(t) = (c_1 + c_2 t)e^{-\alpha t}$$

--- May or may not pass over static equilibrium y = 0.

– Border between non-oscillatory and oscillatory motion



Damped System: III. Underdamping (1)

– Pure imaginary

$$\beta = i\omega^*, \omega^* = \frac{1}{2m}\sqrt{4mk - c^2} = \sqrt{\frac{k}{m} - \frac{c^2}{4m^2}}$$

– Complex conjugate roots

$$\lambda_{1,2} = -\alpha \pm i\omega^*$$

– General sol.

$$y(t) = e^{-\alpha t}\left(A\cos\omega^* t + B\sin\omega^* t\right) = Ce^{-\alpha t}\cos\left(\omega^* t - \delta\right)$$

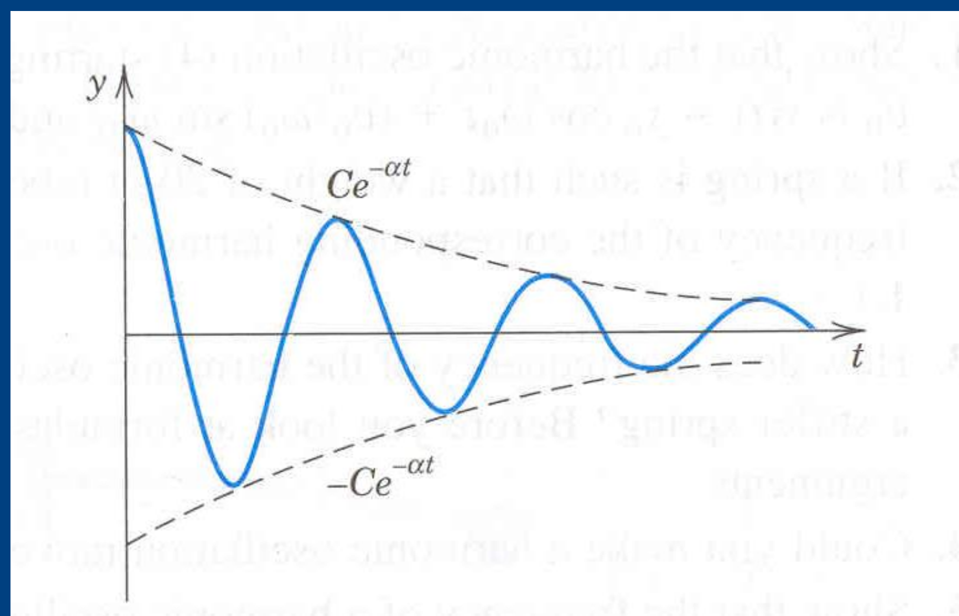
where

$$C = \sqrt{A^2 + B^2}, \tan\delta = \frac{B}{A}$$

Damped System: III. Underdamping (2)

– Damped oscillations
– Frequency: $\omega^* / 2\pi$
  will increase as *c* becomes smaller

– As *c* approaches zero, $\omega^* \approx \omega_0 = \sqrt{k/m}$



Euler-Cauchy Equation

$$x^2 y'' + axy' + by = 0$$

- Try

$$y = x^m$$

- Substituting

$$x^2 m(m-1)x^{m-2} + axmx^{m-1} + bx^m = 0$$

- Auxiliary eqn.

$$m^2 + (a-1)m + b = 0$$

- Three cases

Case I. Distinct Real Roots

- Two real roots m_1, m_2
- General sol.

$$y = c_1 x^{m_1} + c_2 x^{m_2}$$

- Example

$$x^2 y'' - 2.5xy' - 2.0y = 0$$

- Auxiliary eqn.

$$m^2 - 3.5m - 2.0 = 0$$

- Roots of the auxiliary eqn.

$$m_1 = -0.5, m_2 = 4$$

- General sol.

$$y = \frac{c_1}{\sqrt{x}} + c_2 x^4$$

Case II. Real Double Root (1)

– One double root m_1

$$m_1 = \frac{1}{2}(1-a)$$

– First basis

$$y_1 = x^{(1-a)/2}$$

– Second solution --- by the reduction of order

Set

$$y_2 = uy_1$$

– Substituting into the DE

$$x^2\left(u''y_1 + 2u'y_1' + uy_1''\right) + ax\left(u'y_1 + uy_1'\right) + buy_1 = 0$$

– Rearranging

$$u''x^2y_1 + u'x\left(2xy_1' + ay_1\right) + u\underbrace{\left(x^2y_1'' + axy_1' + by_1\right)}_{0} = 0$$

Case II. Real Double Root (2)

– 2nd parenthesis

$$2xy_1' + ay_1 = (1-a)x^{(1-a)/2} + ax^{(1-a)/2} = x^{(1-a)/2} = y_1$$

– Remaining

$$\left(u''x^2 + u'x\right)y_1 = 0$$

– Divide by y1 and separate

$$\frac{u''}{u'} = -\frac{1}{x}, \ln|u'| = -\ln x,$$

$$u' = \frac{1}{x}, u = \ln x$$

– General sol.

$$y = \left(c_1 + c_2 \ln x\right)x^{(1-a)/2}$$

Case III. Complex Conjugate Roots

– Two complex roots

$$m_1 = \mu + i\nu, m_2 = \mu - i\nu$$

– Using Euler formula and expressing

$$x^{m_1} = x^{\mu} x^{i\nu} = x^{\mu} e^{i\nu \ln x} = x^{\mu} \left[\cos(\nu \ln x) + i \sin(\nu \ln x) \right]$$

$$x^{m_2} = x^{\mu} x^{-i\nu} = x^{\mu} e^{-i\nu \ln x} = x^{\mu} \left[\cos(\nu \ln x) - i \sin(\nu \ln x) \right]$$

– Adding and subtracting

$$x^{\mu} \cos(\nu \ln x), x^{\mu} \sin(\nu \ln x)$$

– General sol.

$$y = x^{\mu} \left[A \cos(\nu \ln x) + B \sin(\nu \ln x) \right]$$

Existence and Uniqueness of Sol.

- Homogeneous Linear 2nd Order DE

$$y'' + p(x)y' + q(x) = 0$$

  - IC's

$$y(x_0) = K_0, y'(x_0) = K_1$$

  - Existence and uniqueness of the general sol.

$$y = c_1 y_1 + c_2 y_2$$

- Linear Independence of the sol.
  - Linearly independent y_1, y_2 on I

$$k_1 y_1(x) + k_2 y_2(x) = 0 \Rightarrow k_1 = 0, k_2 = 0$$

  - Linearly dependent $y_1 = ky_2, y_2 = ly_1$

Linear Dependence, Wronskian (1)

– Wronski determinant (Wronskian)

$$W(y_1, y_2) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = y_1 y_2' - y_2 y_1'$$

– y_1, y_2 are linearly dependent ⇒ $W(y_1, y_2) = 0$ for any x_0

– If y_1, y_2 are linearly dependent, then $y_1 = k y_2$

$$W(y_1, y_2) = \begin{vmatrix} k y_2 & y_2 \\ k y_2' & y_2' \end{vmatrix} = k y_2 y_2' - y_2 k y_2' = 0$$

– If $W(y_1, y_2) = 0$, consider the linear system

$$k_1 y_1(x) + k_2 y_2(x) = 0$$

$$k_1 y_1'(x) + k_2 y_2'(x) = 0$$

← *Unknowns k_1, k_2*

Linear Dependence, Wronskian (2)

– Coefficient matrix of the linear system = Wronskian

– Nontrivial sol. of k_1, k_2 must be obtained if $W = 0$

$$y = k_1 y_1(x) + k_2 y_2(x)$$

– Example

$$y'' + \omega^2 y = 0 \quad \Rightarrow \quad y_1 = \cos\omega x,\ y_2 = \sin\omega x$$

– Wronskian

$$W(\cos\omega x, \sin\omega x) = \begin{vmatrix} \cos\omega x & \sin\omega x \\ -\omega\sin\omega x & \omega\cos\omega x \end{vmatrix}$$

$$= \omega\left(\cos^2\omega x + \sin^2\omega x\right) = \omega$$

⇒ linearly independent if and only if $\omega \neq 0$

Existence (1)

- Existence Theorem

  --- If $p(x)$ and $q(x)$ continuous, general sol. exists.

  – First sol. y_1

$$y_1(x_0)=1, y_1'(x_0)=0$$

  – Second sol. y_2

$$y_2(x_0)=0, y_2'(x_0)=1$$

  – Wronskian

$$W=1$$ ⇨ linearly independent

  – General sol.

$$y=k_1y_1(x)+k_2y_2(x)$$

  – Every sol. $Y(x)$ can be expressed

$$Y(x)=C_1y_1(x)+C_2y_2(x)$$

Existence (2)

- Does not have singular sol.
- Assume x0 which gives

$$y(x_0) = Y(x_0), y'(x_0) = Y'(x_0)$$

- Matrix form

$$c_1 y_1(x_0) + c_2 y_2(x_0) = Y(x_0)$$
$$c_1 y_1'(x_0) + c_2 y_2'(x_0) = Y'(x_0)$$

Unknowns
c_1, c_2

- Coefficient matrix = Wronskian $\neq 0$

⇒ unique sol. $c_1 = C_1, c_2 = C_2$

$$y^*(x) = C_1 y_1(x) + C_2 y_2(x)$$

- Everywhere on I

$$y^*(x_0) = Y(x_0), y^{*\prime}(x_0) = Y'(x_0)$$

Non-homogeneous DE (1)

$$y'' + p(x)y' + q(x)y = r(x)$$

– Relation between sol. of corresponding homogeneous DE

(1) Difference of two sol. of non-homogeneous DE:

sol. of homogeneous DE

$$L[y] = r(x), L[\tilde{y}] = r(x)$$

$$L[y - \tilde{y}] = L[y] - L[\tilde{y}] = r(x) - r(x) = 0$$

(2) Sum of sol. To non-homogeneous DE and that to homogeneous DE: another sol. of non-homogeneous DE

$$L\left[y + y^*\right] = L[y] - L\left[y^*\right] = r(x) + 0 = r(x)$$

Non-homogeneous DE (2)

- General sol. to non-homogeneous DE

$$y(x) = y_h(x) + y_p(x)$$

  - y_h: general sol. of the corresponding homogeneous DE

$$y_h(x) = c_1 y_1(x) + c_2 y_2(x)$$

  - y_p: any sol. of the non-homogeneous DE with no arbitrary constants
  - Particular sol. can be obtained by assigning values to c_1 and c_2
- Uniqueness and Existence of the General sol.

$$y(x_0) = K_0, y'(x_0) = K_1$$

$$\tilde{y}(x_0) = K_0 - y_p(x_0), \tilde{y}'(x_0) = K_0 - y_p'(x_0)$$

Non-homogeneous DE (3)

$$y = \tilde{y} + y_p$$ ← Unique sol. to non-homogeneous DE

– Every sol. can be obtained by assigning values to c_1 and c_2 in y_h

$\tilde{y}$: any sol. of the non-homogeneous DE

$y(x) = y_h(x) + y_p(x)$: any general sol. of the same DE

$Y(x) = \tilde{y}(x) - y_p(x)$: sol. of the homogeneous DE

$$\Rightarrow \quad \tilde{y}(x) = Y(x) + y_p(x)$$

– Sol. of the non-homogeneous DE

--- find the sol. to corresponding homogeneous DE, y_h and y_p

Example of Non-homogeneous DE

- Initial Value Problem

$$y'' + 2y' + 101y = 10.4e^{x},\ y(0) = 1.1,\ y'(0) = -0.9$$

  - General sol. to the corresponding homogeneous DE

$$y_h = e^{-x}\left(A\cos 10x + B\sin 10x\right)$$

  - General sol. to the non-homogeneous DE

Try $y_p = Ce^{x}$

  - Substituting

$$\left(1+2+101\right)Ce^{x} = 10.4e^{x} \quad \Rightarrow \quad C = 0.1$$

  - General sol. of the non-homogeneous DE

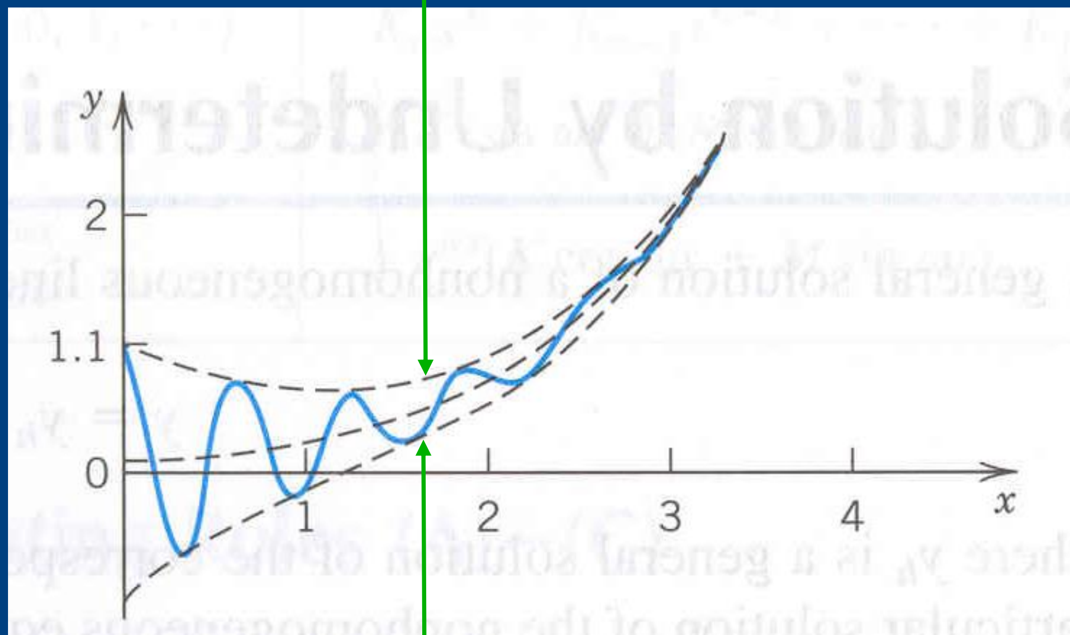
$$y = y_h + y_p = e^{-x}\left(A\cos 10x + B\sin 10x\right) + 0.1e^{x}$$

  - From IC

$$y = e^{-x}\cos 10x + 0.1e^{x}$$

Example of Non-homogeneous DE

$$e^{-x} + 0.1e^{x}$$



$$-e^{-x} + 0.1e^{x}$$

Sol. by Undetermined Coefficients

- Finding y_p in the non-homogeneous DE

$$y'' + ay' + by = r(x)$$

- Method of Undetermined Coefficients

  (A) Basic rule

  (B) Modification rule --- If your choice for y_p happens to be a sol. of the corresponding DE, then multiply by x or x^2 (if it is a double root).

  (C) Sum rule --- If r(x) is a sum of several functions in Table, then choose for y_p the sum of the corresponding trial functions.

Basic Rule

Term in $r(x)$	Choice for y_p
$ke^{\gamma x}$	$Ce^{\gamma x}$
$kx^n \left(n = 0, 1, \cdots\right)$	$K_n x^n + K_{n-1} x^{n-1} + \cdots + K_1 x + K_0$
$k \cos \omega x$	$K \cos \omega x + M \sin \omega x$
$k \sin \omega x$	$K \cos \omega x + M \sin \omega x$
$ke^{\alpha x} \cos \omega x$	$e^{\alpha x} \left(K \cos \omega x + M \sin \omega x\right)$
$ke^{\alpha x} \sin \omega x$	$e^{\alpha x} \left(K \cos \omega x + M \sin \omega x\right)$

Example of Undetermined Coefficients (1)

$$y'' + 4y' = 8x^2$$

– Choice of y_p

$$y_p = K_2x^2 + K_1x + K_0, y_p'' = 2K_2$$

– Substituting

$$2K_2 + 4\left(K_2x^2 + K_1x + K_0\right) = 8x^2$$

– Equating coefficients of x^2, x, and x^0

$$y_p = 2x^2 - 1$$

– General sol.

$$y = y_h + y_p = A\cos 2x + B\sin 2x + 2x^2 - 1$$

Example of Undetermined Coefficients (2)

$$y'' - 3y' + 2y = e^x$$

- Sol. of the corresponding homogeneous DE y_h

$$y_h = c_1 e^x + c_2 e^{2x}$$

- Choice of y_p — *Same as y_h*

$$y_p = Cxe^x, \; y_p' = C\left(e^x + xe^x\right), \; y_p'' = C\left(2e^x + xe^x\right)$$

- Substituting

$$C\left(2 + x\right)e^x - 3\left(1 + x\right)e^x + 2Ce^x = e^x, \; C = -1$$

- General sol.

$$y = y_h + y_p = c_1 e^x + c_2 e^{2x} - xe^x$$

Example of Undetermined Coefficients (3)

$$y'' + 2y' + y = (D+1)^2 y = e^{-x}, y(0) = -1, y'(0) = 1$$

– Sol. of the corresponding homogeneous DE y_h

$$y_h = (c_1 + c_2 x)e^{-x}$$

– Choice of y_p

Same as y_h, , double root

$$y_p = Cx^2 e^{-x}, y_p' = C(2x - x^2)e^{-x}, y_p'' = C(2 - 4x + x^2)e^{-x}$$

– Substituting

$$C(2 - 4x + x^2)e^{-x} + 2C(2x - x^2)e^{-x} + Cx^2 e^{-x} = e^{-x}, C = \frac{1}{2}$$

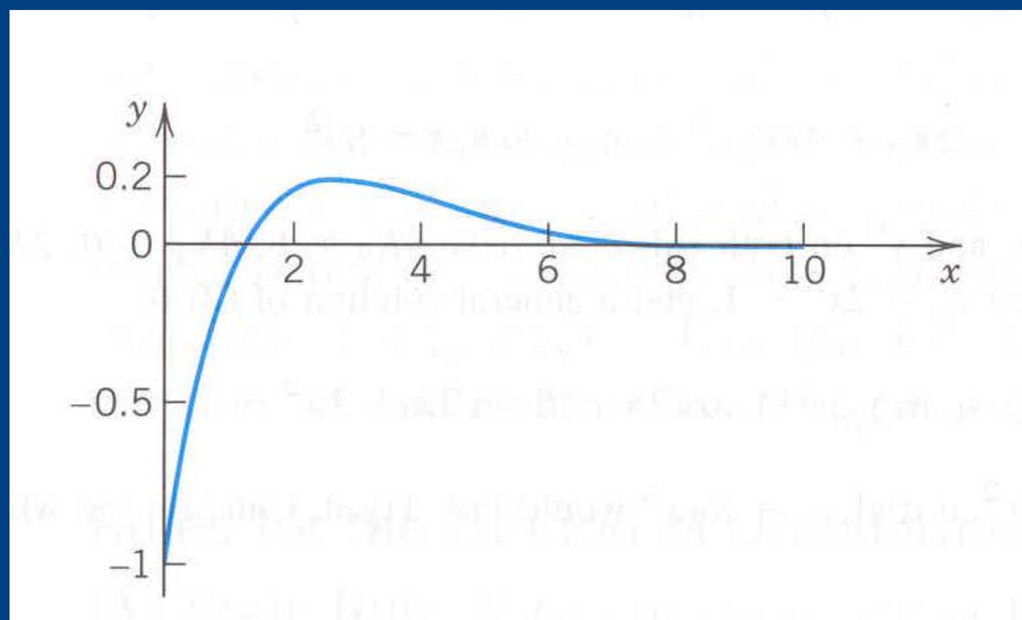
– General sol.

$$y = y_h + y_p = (c_1 + c_2 x)e^{-x} + \frac{1}{2}x^2 e^{-x}$$

Example of Undetermined Coefficients (4)

– From IC

$$y = \left(\frac{1}{2}x^2 - 1 \right) e^{-x}$$



Example of Undetermined Coefficients (5)

$$y'' + 2y' + 5y = 1.25e^{0.5x} + 40\cos 4x - 55\sin 4x,$$

$$y(0) = 0.2,\ y'(0) = 60.1$$

– Sol. of the corresponding homogeneous DE y_h

$$y_h = e^{-x}\left(A\cos 2x + B\sin 2x\right)$$

– Choice of y_p

$$y_p = Ce^{0.5x} + K\cos 4x + M\sin 4x,$$

$$y_p' = 0.5Ce^{0.5x} - 4K\sin 4x + 4M\cos 4x,$$

$$y_p'' = 0.25Ce^{0.5x} - 16K\cos 4x - 16M\sin 4x$$

Example of Undetermined Coefficients (6)

- Substituting and equating

$$C = 0.2, K = 0, M = 5$$

- General sol.

$$y = e^{-x}\left(A\cos 2x + B\sin 2x\right) + 0.2e^{0.5x} + 5\sin 4x$$

- Particular sol.

$$y = 20e^{-x}\sin 2x + 0.2e^{0.5x} + 5\sin 4x$$

Example of Undetermined Coefficients (7)