Suppl. lecture. (should watch this lecture by Next Tuesday)

HW2

Ch.4 Numerical solution of ordinary differential eqs. (ODEs)

$$y'' + \omega^2 y = f(x)$$

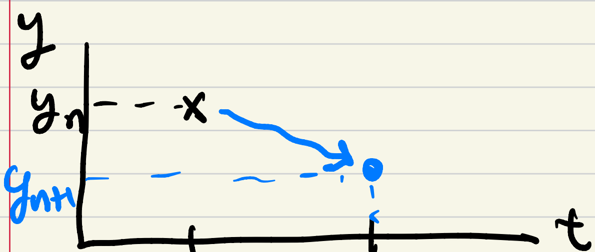
$$\begin{cases} y(0) = y_0 \\ \left.\dfrac{dy}{dx}\right|_0 = v \end{cases} \quad \text{or} \quad \begin{cases} y(0) = y_0 \\ y(L) = y_L \end{cases}$$

initial value problem (all the conditions are given at one point.)

boundary value problem (conditions are given at more than one pt.)

4.1 Initial value problems

$$\begin{cases} y' = \dfrac{dy}{dt} = f(y,t) \\ y(0) = y_0 \end{cases}$$ first-order ODE



t_n time step

$t_{n+1} = t_n + \underbrace{\Delta t}_{}$ computational time step

All methods assume that solution is known at $0 \le t \le t_n$, and use it to get sol. at $t = t_{n+1}$.

* Higher-order ODEs can be converted to a system of 1st-order ODEs

$$y'' + \omega^2 y = f$$

$$y_1 = y$$

$$y_2 = y_1' = y'$$

$$y'' = y_2' = -\omega^2 y_1 + f$$

$$\Rightarrow \begin{cases} y_1' = y_2 \\ y_2' = -\omega^2 y_1 + f \end{cases}$$

$y' = f(y, t)$

Taylor series expansion

$$y_{n+1} = y(t_{n+1}) = y(t_n + \underbrace{\Delta t}_{= h}) = y_n + h\, y_n' + \frac{h^2}{2} y_n'' + \frac{h^3}{6} y_n''' + \cdots$$

$y_n' = f(y_n, t_n)$? ?

$$y_n'' = \frac{d}{dt} y' \Big|_n = \frac{d}{dt} f(y,t) \Big|_n = \frac{\partial f}{\partial t}\Big|_n + \frac{\partial f}{\partial y} \frac{\partial y}{\partial t}\Big|_n$$

$$= f_{tn} + f_{yn} f_n$$

$$y_n''' = \frac{d}{dt}(f_t + f_y f)\Big|_n = \frac{\partial}{\partial t}(f_t + f_y f)\Big|_n + \frac{\partial}{\partial y}(f_t + f_y f)\frac{\partial y}{\partial t}\Big|_n$$

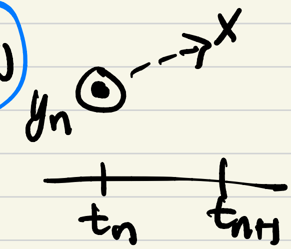
$$= f_{ttn} + f_{ytn} f_n + f_{yn} f_{tn} + f_{tyn} f_n + f_{yyn} f_n f_n + f_{yn} f_{yn} f_n$$

# of terms increases very rapidly.

Hence, it is not very practical to include higher-order terms than third order.

① Euler method (Explicit Euler or Forward Euler)

$$y' = f(y,t) \rightarrow \boxed{y_{n+1} = y_n + h f(y_n, t_n)} + O(h^2)$$



$$y' = \lim_{h\to 0} \frac{y_{n+1}-y_n}{h} \quad \text{or} \quad \boxed{\frac{y_{n+1}-y_n}{h} = f(y_n, t_n)}$$

2nd-order accurate for one time step

globally (integration from t_0 to t_f)

$$t_f - t_0 = nh \qquad \text{error} \sim \Sigma h^2 y'' = n h^2 y'' = \frac{t_f - t_0}{h} \cdot h^2 y'' \sim O(h)$$

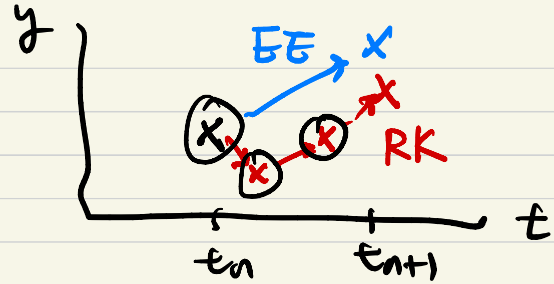
$\therefore$ 1st-order accurate!

ex) $\frac{dy}{dt} = y' = -y^2$, $y(0) = 1$ → exact sol. $y = \frac{1}{1+t}$

(h=0.1) EE ⇒ $y_{n+1} = y_n + h f(y_n, t_n) = y_n - h y_n^2$

	EE	exact sol.
$y_0 = 1$		
$y_1 = y_0 - h y_0^2 =$	0.9	0.90909
$y_2 = y_1 - h y_1^2 =$	...	...
$y_{10} = y_9 - h y_9^2 =$	0.4627	0.5

* Runge-Kutta method (RK)



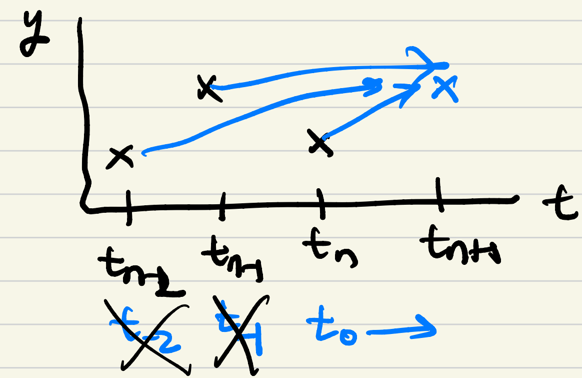
y_{n+1} is obtained in terms of y_n, $f(y_n, t_n)$ and values of f at intermediate times ($t_n \le t \le t_{n+1}$).

* Multi-step method

uses information from $t \le t_n$

y_{n-1} @ t_{n-1}, y_{n-2} @ t_{n-2}, ...

not self-starting !



* explicit vs. implicit methods

EE: $y_{n+1} = y_n + h f(y_n, t_n)$

but provides better numerical stability. ←

involves $f(y_{n+1}, t_{n+1})$

ex) implicit Euler method

$y_{n+1} = y_n + h f(y_{n+1}, t_{n+1})$

may involve solving a nonlinear algebraic eq.

4.2 Numerical stability

It is possible for numerical sol. of a diff'l eq. to blow up (grow unbounded) even though the exact sol. is well behaved.

→ We seek parameters of the numerical method such as h ($=\Delta t$) so that numerical sol. is well behaved.

- Stable numerical method (A-stable numerical method) (absolutely stable " ")

→ Numerical sol. does not blow up with any choice of parameters.

- Unstable numerical method

→ Numerical sol. always blows up irrespective of the choice of parameters.

- Conditionally stable numerical method

→ Numerical sol. is well behaved with some choice of parameters.

$$\boxed{y' = f(y,t)}$$

accuracy ≠

- Linear stability analysis

$$y' = f(y,t) = f(y_0,t_0) + (t-t_0)\underbrace{\frac{\partial f}{\partial t}\Big|_{y_0,t_0}}_{\alpha_2} + (y-y_0)\underbrace{\frac{\partial f}{\partial y}\Big|_{y_0,t_0}}_{\lambda} + \underset{(y^2,\,t^2,\,\cdots)}{HOT}$$

$$= \boxed{\lambda y} + \underbrace{\alpha_1 + \alpha_2 t}_{\text{a particular sol. (which doesn't affect stability)}} + HOT$$

HOT → usually provides a decaying sol.

ex) $y' = \gamma y^2$

↓

$y = \frac{1}{a-\gamma t}$

Model problem $\boxed{y' = \lambda y}$ given $y(0) = y_0$

$\lambda = \lambda_R + i\lambda_I$: complex number

exact sol. $y = y_0 e^{\lambda t} = y_0 e^{\lambda_R t} e^{i\lambda_I t}$

$\boxed{\lambda_R \le 0 \text{ for stability}}$

Midterm after finishing Ch. 4 (maybe someday in Nov.)

Model problem : $\boxed{y' = \lambda y}$ $\qquad y' = f(y,t)$

$\lambda = \lambda_R + i\lambda_I$

4.3 stability analysis for Euler method

$$y' = f(y,t) \xrightarrow{EE} \frac{y_{n+1} - y_n}{h} = f(y_n, t_n)$$

$$y_{n+1} = y_n + h f(y_n, t_n)$$

Apply EE to model problem $y' = \lambda y = f$

$\rightarrow y_{n+1} = y_n + h(\lambda y_n) = (1+\lambda h) y_n$

$\rightarrow y_n = (1+\lambda h)^n y_0 \qquad (t = nh)$

$= (1 + \lambda_R h + i\lambda_I h)^n y_0$

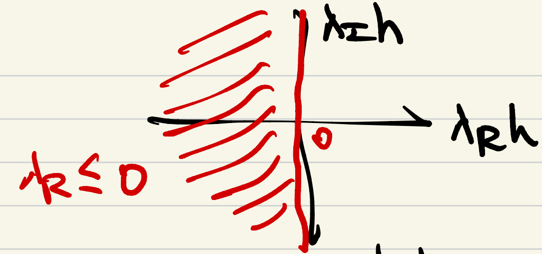
$y_1 = (1+\lambda h) y_0$

$y_2 = (1+\lambda h)^2 y_0$

$\vdots$

whether sol. remains bounded depends on $\lambda_R h$ & $\lambda_I h$

For exact sol. to be well behaved

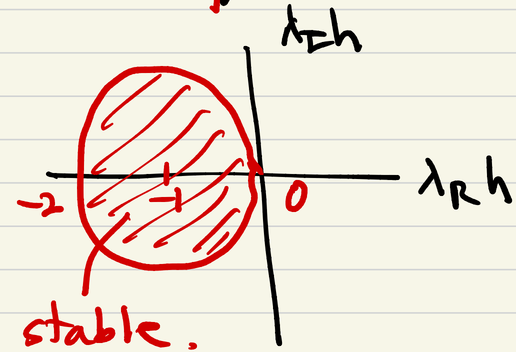


EE : $y_n = (1+\lambda_R h + i\lambda_I h)^n y_0$

$= \sigma^n y_0$

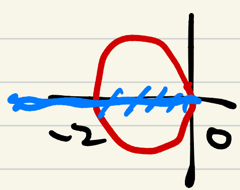
to be stable, $|\sigma| \le 1$

$|\sigma|^2 = (1+\lambda_R h)^2 + (\lambda_I h)^2 \le 1$



$\therefore$ EE is conditionally stable.

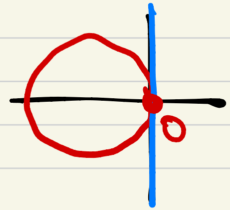
- when λ is real & negative $(\lambda_I = 0)$



$|\lambda h| \le 2 \rightarrow h_{max} = 2/|\lambda|$

$y' = -2y \xrightarrow{EE} h_{max} = \frac{2}{|-2|} = 1$ for stability

• when λ is purely imaginary ($\lambda_R = 0$) ($y' = i\omega y$)



EE is unstable for $\lambda = i\omega$. → sol. blows up.

$$|\sigma|^2 = (1+\cancel{\lambda_R h})^2 + (\lambda_I h)^2 > 1$$

• Accuracy of EE

$y' = \lambda y$ exact sol. $y = y_0 e^{\lambda t} = y_0 e^{\lambda n h} = y_0\left(1+\lambda h+\tfrac{1}{2}\lambda^2 h^2+\cdots\right)^n$

EE: $y_n = (1+\lambda h)^n y_0$

∴ <u>EE is 1st-order accurate.</u>

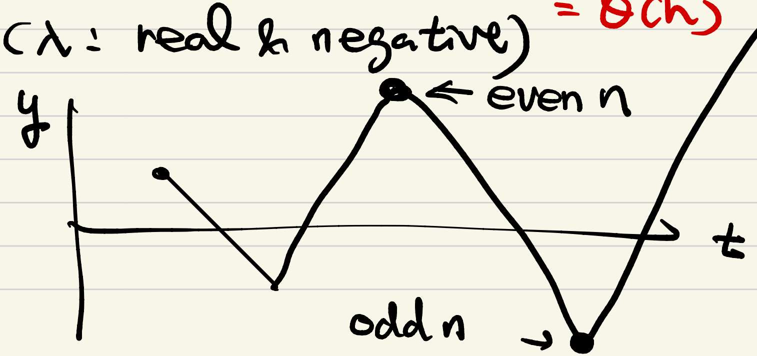
leading error

$\frac{1}{2}\lambda^2 h^2 \cdot n = \frac{1}{2}\lambda^2 h^2 \cdot \frac{T}{h} = O(h)$

* Signal for instability (λ: real & negative)

EE: $y_n = (1+\lambda h)^n y_0$

$|1+\lambda h| > 1$



4.4 Implicit Euler or backward Euler method

$$y' = f(y,t) \xrightarrow{\text{IE}} \boxed{\begin{aligned} &\frac{y_{n+1}-y_n}{h} = f(y_{n+1}, t_{n+1}) \\ &y_{n+1} = y_n + h f(y_{n+1}, t_{n+1}) \end{aligned}}$$

cost/timestep is higher than EE

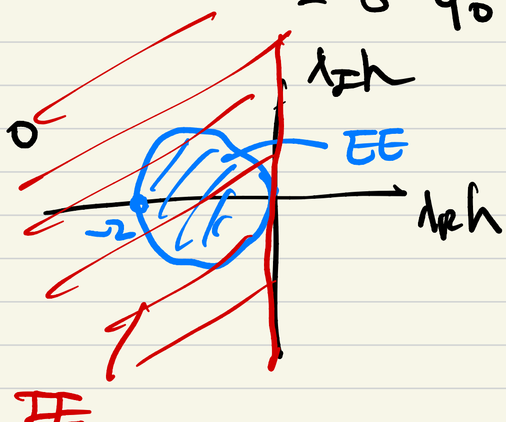
model prob. $y' = \lambda y$

$$\text{IE}: \quad y_{n+1} = y_n + h\lambda y_{n+1} \rightarrow y_{n+1} = \frac{1}{1-\lambda h} y_n \rightarrow y_n = \left(\frac{1}{1-\lambda h}\right)^n y_0$$

$$= \sigma^n y_0$$

$$\sigma = \frac{1}{1-\lambda h} = \frac{1}{1-\lambda_R h - i\lambda_I h}$$

$$|\sigma|^2 = \frac{1}{(1-\lambda_R h)^2 + (\lambda_I h)^2} \leq 1 \quad \text{for } \lambda_R \leq 0$$



$\therefore$ IE is unconditionally stable

A-stable

4.5 Numerical accuracy

IE : $\sigma = \frac{1}{1-\lambda h} = 1 + \lambda h + (\lambda h)^2 + (\lambda h)^3 + \cdots$

exact sol. : $e^{\lambda h} = 1 + \lambda h + \frac{1}{2}(\lambda h)^2 + \frac{1}{6}(\lambda h)^3 + \cdots$

IE : $y_n = \sigma^n y_0 = (1 + \lambda h + (\lambda h)^2 + (\lambda h)^3 + \cdots)^n y_0$

error $\quad \lambda^2 h^2 \cdot n = \lambda^2 h^2 \cdot \frac{T}{h} = O(h)$

IE is 2nd-order accurate for one time step

but is 1st-order accurate overall.

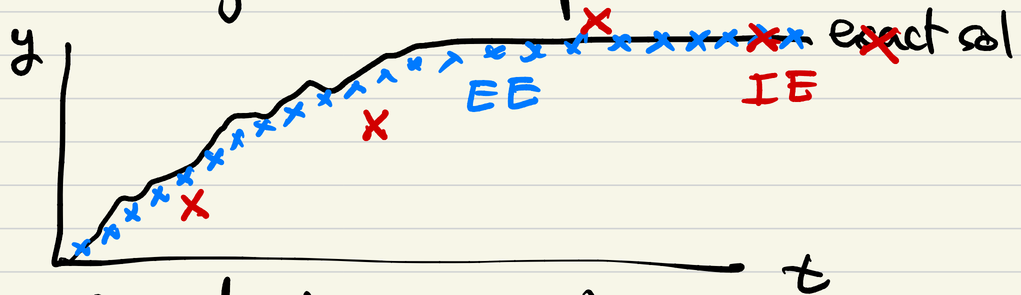
IE : $e^{\lambda h} - \sigma = e^{\lambda h} - \frac{1}{1-\lambda h} = -\frac{1}{2}\lambda^2 h^2 + \cdots$

EE : $e^{\lambda h} - \sigma = e^{\lambda h} - (1+\lambda h) = +\frac{1}{2}\lambda^2 h^2 + \cdots$

$\Rightarrow$ stability is nothing to do with accuracy.

From stability point of view,
our objective is to take largest time step h.

$y' = f(y,t)$



- Accuracy for $\lambda = i\omega$ (purely imaginary)

$$y' = \lambda y = i\omega y$$

exact sol. $\quad y = y_0 e^{i\omega t} = y_0 (\cos\omega t + i \sin\omega t)$

EE : $\quad y_{n+1} = y_n + \lambda h y_n = (1+\lambda h) y_n$

$\rightarrow y_n = (1+\lambda h)^n y_0 = \sigma^n y_0 \quad . \ \sigma = 1 + i\omega h$

exact sol. : amplitude $|e^{i\omega t}| = 1$

EE : amplitude $|\sigma| = \sqrt{1+\omega^2 h^2} > 1$ unstable

$$\sigma = |\sigma| e^{i\theta} = 1 + i\omega h$$

$$\text{phase } \theta = \tan^{-1} \frac{\mathrm{Im}(\sigma)}{\mathrm{Re}(\sigma)} = \tan^{-1} \frac{\omega h}{1} = \tan^{-1} \omega h$$

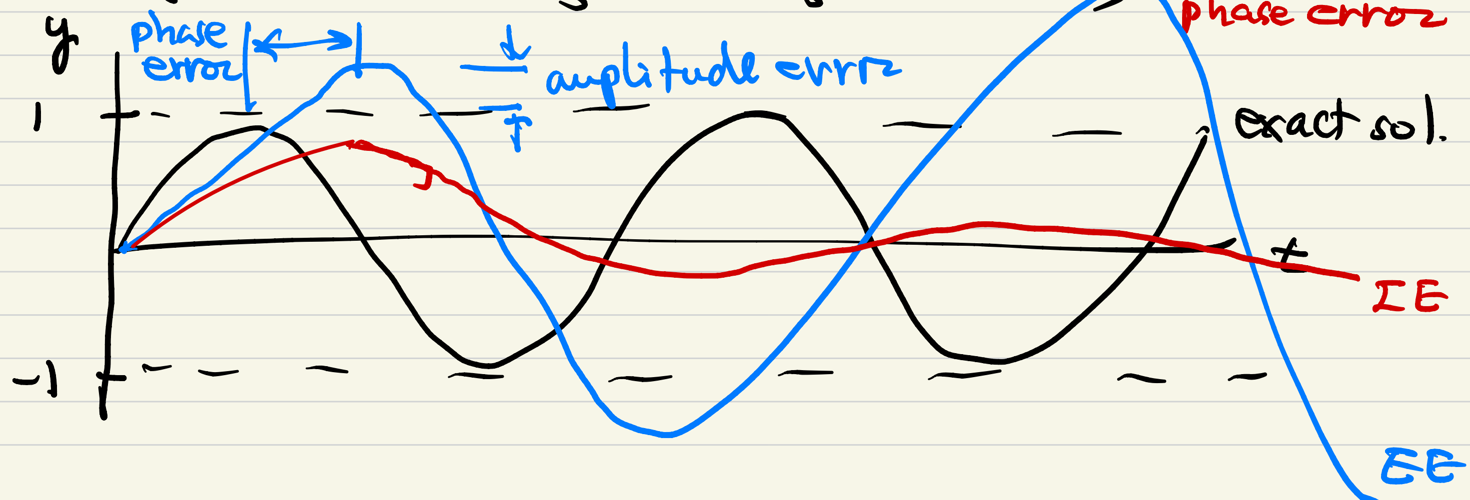
exact sol. : $y = y_0 e^{i\omega t} = y_0 e^{i\omega h n} = y_0 \cdot 1 \cdot e^{i\omega h n}$

EE : $y_n = \sigma^n y_0 = y_0 |\sigma|^n e^{i\theta n}$

phase error $= \omega h - \theta = \omega h - \tan^{-1} \omega h = \boxed{\frac{1}{3} (\omega h)^3} + \cdots$

$$\left(\tan^{-1} \omega h = \omega h - \frac{1}{3} (\omega h)^3 + \frac{1}{5} (\omega h)^5 + \cdots\right)$$

leading phase error



IE : $y_{n+1} = \dfrac{1}{1-\lambda h} y_n = \dfrac{1}{1-i\omega h} y_n \rightarrow y_n = \left(\dfrac{1}{1-i\omega h}\right)^n y_0$

$= \sigma^n y_0$

$\sigma = \dfrac{1}{1-i\omega h} = |\sigma| e^{i\theta}$

$|\sigma|^2 = \dfrac{1}{1+\omega^2 h^2} < 1$ stable $\rightarrow$ but decaying sol.

$\sigma = \dfrac{1}{1-i\omega h} = \dfrac{1+i\omega h}{1+\omega^2 h^2}$

$\theta = \tan^{-1} \dfrac{\omega h}{1} = \tan^{-1} \omega h$: same as that of EE.

phase error $= \omega h - \theta = \frac{1}{3}(\omega h)^3 + \cdots$