Modeling: Forced Oscillations (1)

- Free Motion of a Mass-Spring-Damper System

$$\boxed{my'' + cy' + ky = 0}$$

- Forced Motion of a Mass-Spring-Damper System

$$\boxed{my'' + cy' + ky = r(t)}$$

Input, driving force

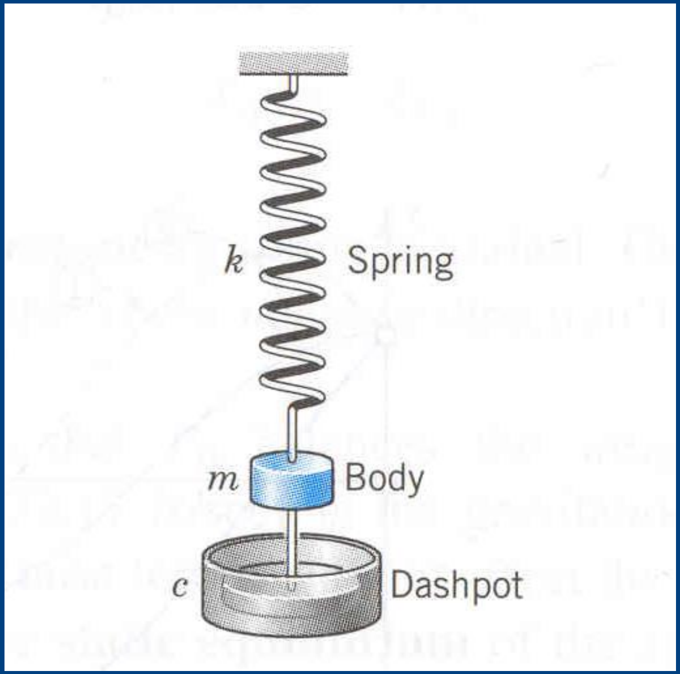
  - Periodic input

$$my'' + cy' + ky = F_0 \cos \omega t$$

  - Particular sol. y_p

$$\boxed{y_p(t) = a \cos \omega t + b \sin \omega t}$$

Modeling: Forced Oscillations (2)



Modeling: Forced Oscillations (3)

– Differentiating

$$y_p' = -\omega a \sin \omega t + \omega b \cos \omega t$$

$$y_p'' = -\omega^2 a \sin \omega t - \omega^2 b \sin \omega t$$

– Substituting and rearranging

$$\left[\left(k - m\omega^2\right)a + \omega c b\right]\cos \omega t + \left[-\omega c a + \left(k - m\omega^2\right)b\right]\sin \omega t = F_0 \cos \omega t$$

– Equating sine and cosine terms

$$\left(k - m\omega^2\right)a + \omega c b = F_0$$

$$-\omega c a + \left(k - m\omega^2\right)b = 0$$

Unknowns a, b

Modeling: Forced Oscillations (4)

– Solve for *a*, *b*

$$a = F_0 \frac{m\left(\omega_0^2 - \omega^2\right)}{m^2\left(\omega_0^2 - \omega^2\right)^2 + \omega^2 c^2}, b = F_0 \frac{\omega c}{m^2\left(\omega_0^2 - \omega^2\right)^2 + \omega^2 c^2}$$

where

$$\omega_0 = \sqrt{k/m}$$

– General sol.

$$y(t) = y_h(t) + y_p(t)$$

– Two cases --- no damping (*c*=0), damping existing (*c*>0)

Undamped Forced Oscillations (1)

– no damping (c=0), assume $\omega \neq \omega_0$

$$y_p(t) = \frac{F_0}{m\left(\omega_0^2 - \omega^2\right)} \cos \omega t = \frac{F_0}{k\left(1 - \left(\omega/\omega_0\right)^2\right)} \cos \omega t$$

– General sol.

$$y(t) = C \cos\left(\omega_0 t - \delta\right) + \frac{F_0}{m\left(\omega_0^2 - \omega^2\right)} \cos \omega t$$

Natural frequency,
ω_0 /2π (Hz)

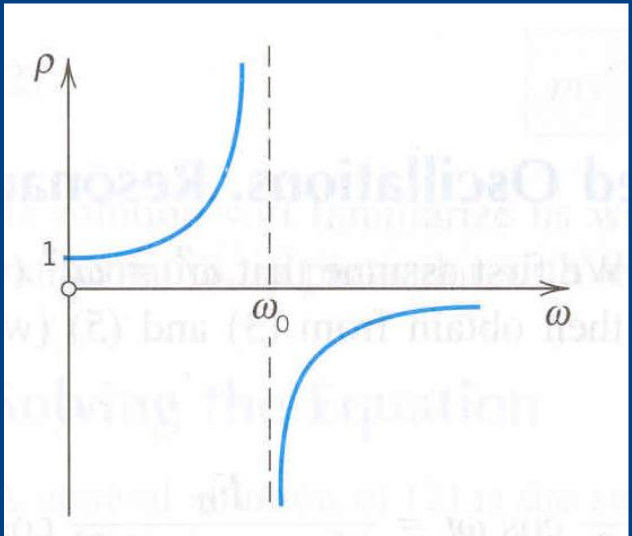
Input

– Maximum amplitude of y_p

$$a_0 = \frac{F_0}{k} \rho, \rho = \frac{1}{1 - \left(\omega/\omega_0\right)^2}$$

Resonance (1)

- Resonance
  - when $\omega \to \omega_0$, ρ and a_0 tends to infinity, excitation of large oscillations. ρ: resonance factor.



  - In the case of resonance

$$y'' + \omega_0^2 y = \frac{F_0}{m} \cos \omega_0 t$$

Resonance (2)

– Modification rule

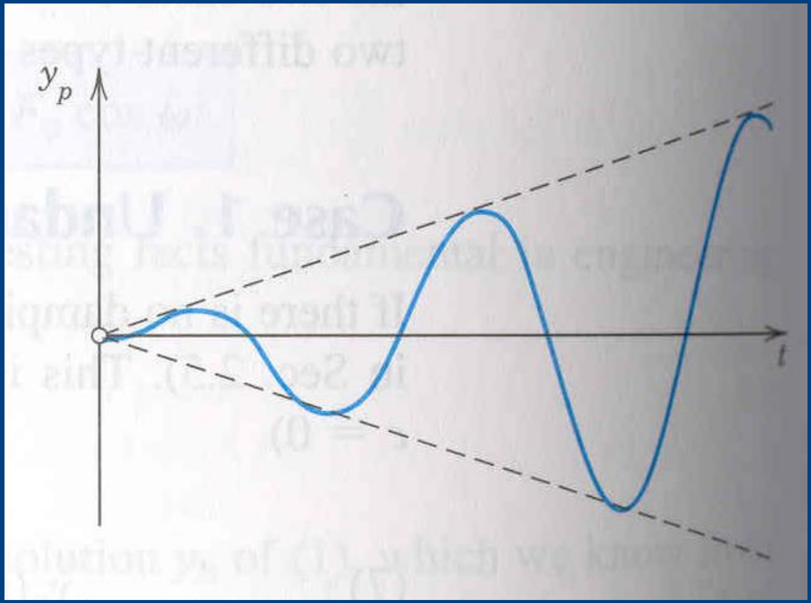
$$y_p(t) = t\left(a\cos\omega_0 t + b\sin\omega_0 t\right)$$

– Substituting

$$a = 0, b = F_0/2m\omega_0$$

– Particular sol.

$$\boxed{y_p(t) = \frac{F_0}{2m\omega_0} t\cos\omega_0 t}$$



Undamped Forced Oscillations (2)

- "Beat" phenomenon
  - when ω is close to ω_0, particular sol.

$$y(t)=\frac{F_0}{m\left(\omega_0^2-\omega^2\right)}\left(\cos\omega t-\cos\omega_0 t\right)$$

with IC

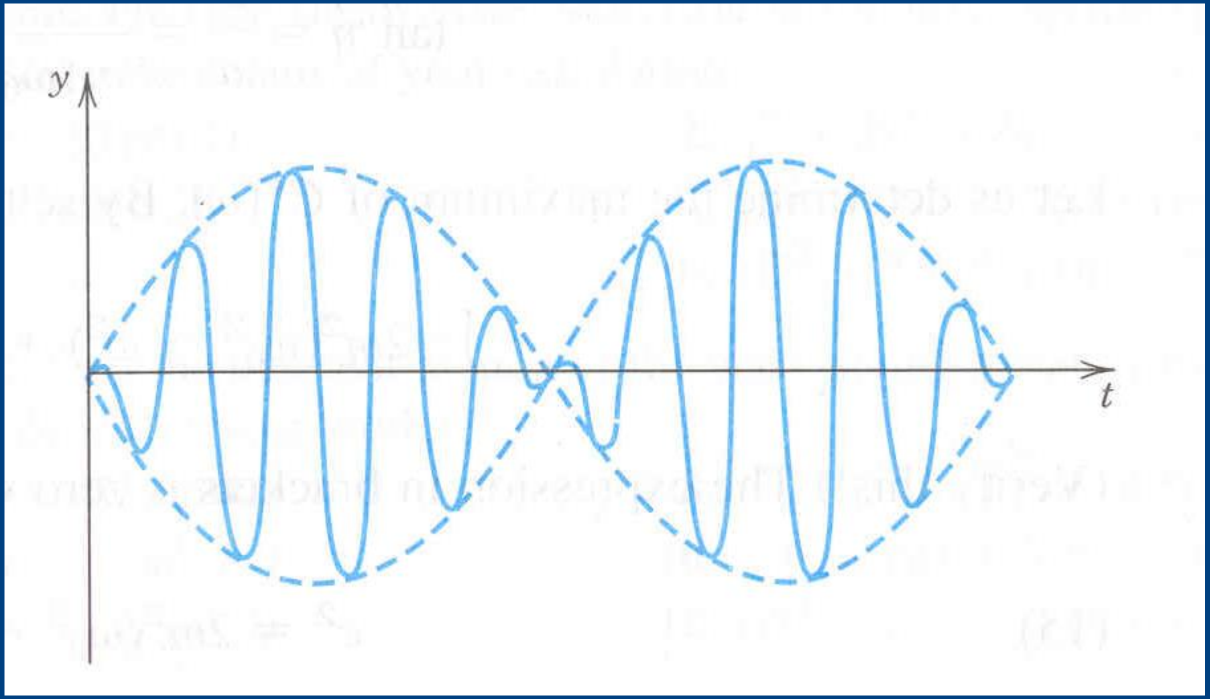
$$y(0)=0, y'(0)=0$$

  - Rewriting

$$y(t)=\frac{2F_0}{m\left(\omega_0^2-\omega^2\right)}\sin\left(\frac{\omega_0+\omega}{2}t\right)\sin\left(\frac{\omega_0-\omega}{2}t\right)$$

Large period

Undamped Forced Oscillations (3)



Beat” phenomenon

Damped Forced Oscillations (1)

– Damping existing (c>0), homogeneous sol. y_h

$$y_h(t) = e^{-\alpha t}\left(A\cos\omega^* t + B\sin\omega^* t\right), \alpha = \frac{c}{2m}$$

Transient sol.

– Solution approaches to y_p (steady-state solution)

– How about resonance in the damped system?

--- when $\omega \to \omega_0$, finite amplitude occurs (“practical resonance”)

$$y_p(t) = C^* \cos(\omega t - \eta)$$

where

$$C^*(\omega) = \sqrt{a^2 + b^2} = \frac{F_0}{\sqrt{m^2\left(\omega_0^2 - \omega^2\right)^2 + \omega^2 c^2}}$$

Damped Forced Oscillations (2)

$$\tan\eta = \frac{b}{a} = \frac{\omega c}{m\left(\omega_0^2 - \omega^2\right)}$$

- Maximum amplitude of $C^*(\omega)$ when $dC^* \big/ d\omega = 0$

$$\left[-2m^2\left(\omega_0^2 - \omega^2\right) + c^2\right]\omega = 0$$

- Bracket is zero when

$$c^2 = 2m^2\left(\omega_0^2 - \omega^2\right)$$

- Sufficiently large damping $\left(c^2 > 2m^2\omega_0^2 = 2mk\right)$

  --- no real sol., $C^*(\omega)$ decreases in a monotone way
- Smaller damping $\left(c^2 \le 2mk\right)$

  --- One real sol. $\omega = \omega_{max}$, increases as c decreases,

  $\rightarrow \omega_0$ as $c \rightarrow 0$

Damped Forced Oscillations (3)

– Maximum amplitude of $C^*(\omega)$ at $\omega = \omega_{max}$

$$C^*(\omega_{\max}) = \frac{2mF_0}{c\sqrt{4m^2\omega_0^2 - c^2}}$$

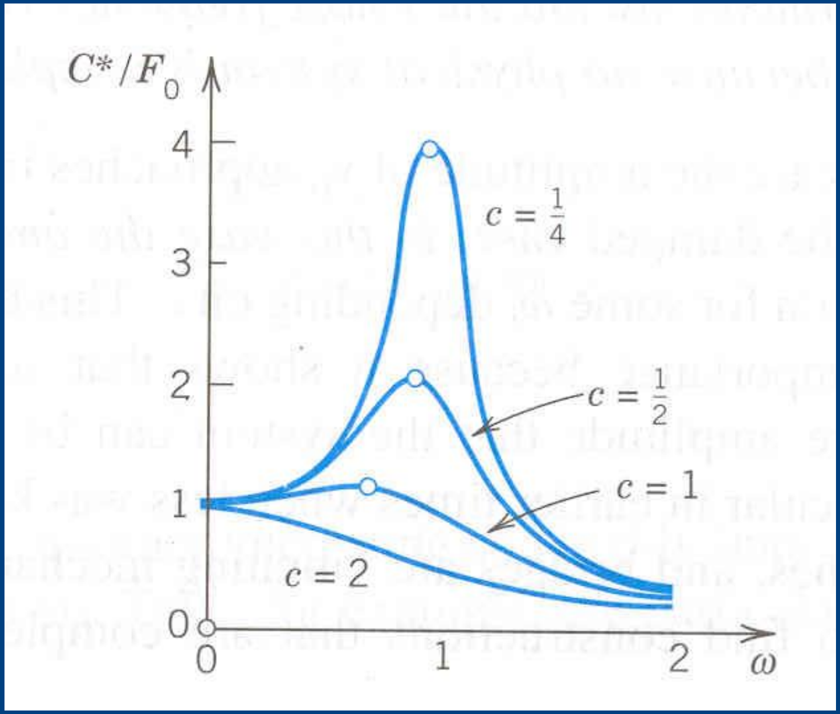
– $C^*(\omega_{max})$ is finite when $c>0$

– Smaller damping $\left(c^2 < 2mk\right)$

--- $C^*(\omega_{max})$ increases

as c decreases,

$\rightarrow \infty$ as $c \rightarrow 0$



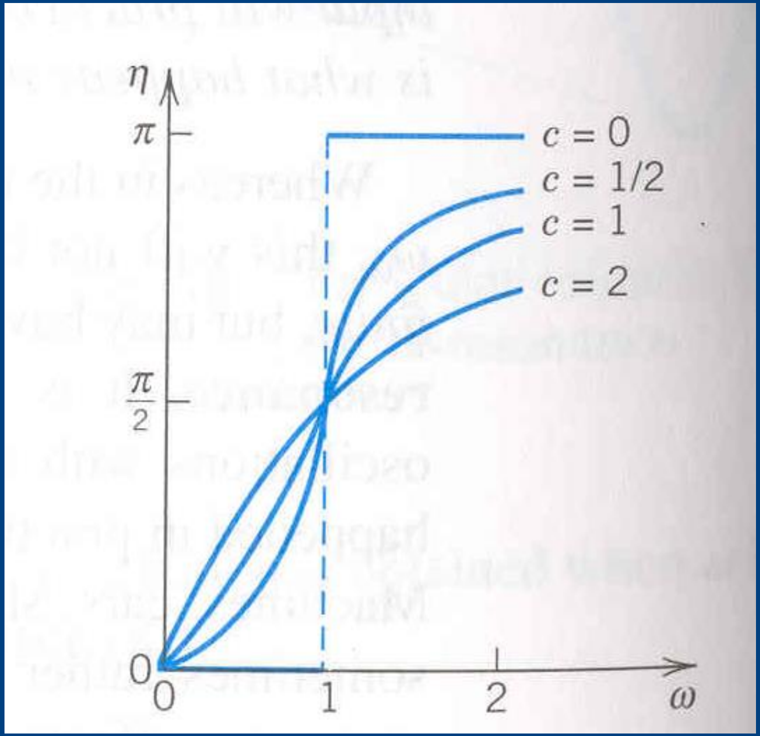
Damped Forced Oscillations (4)

– η: phase angle, phase lag --- lag of the output behind the input

if $\omega < \omega_0$, then $\eta < \pi/2$,

if $\omega = \omega_0$, then $\eta = \pi/2$,

if $\omega > \omega_0$, then $\eta > \pi/2$.



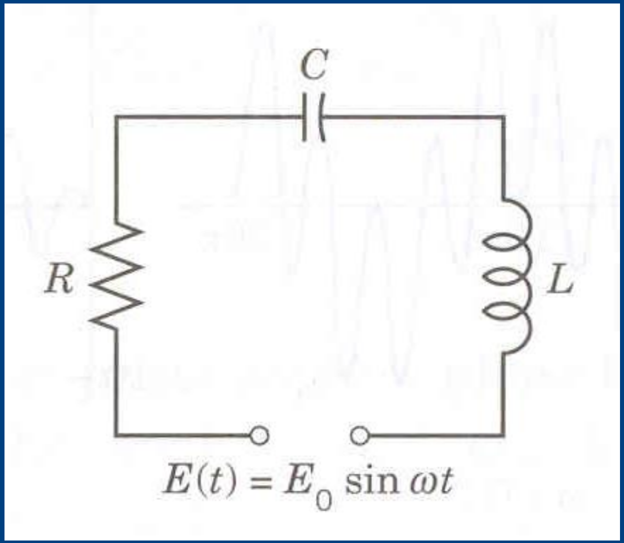
Electric Circuit (1)

- RLC circuit
  - Voltage drop across each element

    Inductor $E_L = LI'$

    Resistor $E_R = RI$

    Capacitor $E_C = \dfrac{1}{C}\int I(t)\,dt$



  - Kirchhoff's voltage law

$$LI' + RI + \frac{1}{C}\int I\,dt = E(t) = E_0 \sin \omega t$$

  - Differentiating

$$\boxed{LI'' + RI' + \frac{1}{C} I = E_0 \omega \cos \omega t}$$

Electric Circuit (2)

Electrical System	Mechanical System
Inductance *L*	Mass *m*
Resistance *R*	Damping constant *c*
Capacitance 1/*C*	Spring constant *k*
Electromotive force (derivative) $E_0 \omega \cos \omega t$	Driving force $F_0 \cos \omega t$
Current *I*(*t*)	Displacement *y*(*t*)

– Another expression using charge *Q*

$$I = Q', I' = Q'', \int I dt = Q$$

$$\Rightarrow \quad LQ'' + RQ' + \frac{1}{C} Q = E_0 \sin \omega t$$

Electric Circuit (3)

– Particular sol.

$$I_p = a\cos\omega t + b\sin\omega t$$

– Substituting and equating sine and cosine terms

$$L\omega^2\left(-a\right) + R\omega b + a/C = E_0\omega$$

$$L\omega^2\left(-b\right) + R\omega\left(-a\right) + b/C = 0$$

– Solve for *a*, *b*

$$a = \frac{-E_0 S}{R^2 + S^2}, b = \frac{E_0 R}{R^2 + S^2}$$

where

$$\boxed{S = \omega L - \frac{1}{\omega C}}$$

reactance

Electric Circuit (4)

– Another form of particular sol.

$$I_p(t) = I_0 \sin(\omega t - \theta)$$

where

$$I_0 = \sqrt{a^2 + b^2} = \frac{E_0}{\sqrt{R^2 + S^2}}, \tan\theta = -\frac{b}{a} = \frac{S}{R}$$

Impedance

– Homogeneous sol.

$$I_h = c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t}$$

– Characteristic eqn.

$$\lambda^2 + \frac{R}{L}\lambda + \frac{1}{LC} = 0$$

Electric Circuit (5)

– Root of the characteristic eqn.

$$\lambda_1 = -\alpha + \beta, \lambda_2 = -\alpha - \beta$$

where

$$\alpha = \frac{R}{2L}, \beta = \frac{1}{2L}\sqrt{R^2 - \frac{4L}{C}}$$

– If $R > 0$, $I_h(t)$ approaches to zero.

– After some time, the output will be a harmonic oscillation with the frequency of the input.

Variation of Parameters (1)

$$\boxed{y'' + p(x)y' + q(x)y = r(x)}$$

--- Arbitrary and continuous $p(x)$, $q(x)$, $r(x)$ on I

– Sol. For y_p by variation of parameters

$$\boxed{y_p = -y_1 \int \frac{y_2 r}{W} dx + y_2 \int \frac{y_1 r}{W} dx}$$

--- where y_1, y_2 are basis of the corresponding homogeneous DE

$$y'' + p(x)y' + q(x)y = 0$$

– Wronskian

$$W = y_1 y_2' - y_2 y_1'$$

Variation of Parameters (2)

- General sol. of the corresponding homogeneous DE

$$y_h(x) = c_1 y_1(x) + c_2 y_2(x)$$

- Replacing c_1, c_2 (“parameters”) by $u(x)$, $v(x)$ to determine y_p

$$y_p(x) = u(x)y_1(x) + v(x)y_2(x)$$

- Differentiating

$$y_p' = u'y_1 + uy_1' + v'y_2 + vy_2'$$

- y_p satisfies the given DE $\Rightarrow$ first condition
- Second arbitrary condition

$$u'y_1 + v'y_2 = 0$$

- Then

$$y_p' = uy_1' + vy_2'$$

Variation of Parameters (3)

– Differentiating

$$y_p'' = u'y_1' + uy_1'' + v'y_2' + vy_2''$$

– Substituting and rearranging

$$u\left(\cancelto{0}{y_1'' + py_1' + qy_1}\right) + v\left(\cancelto{0}{y_2'' + py_2' + qy_2}\right) + u'y_1' + v'y_2' = r$$

– Reduces to

$$u'y_1' + v'y_2' = r$$
$$u'y_1 + v'y_2 = 0$$

Unknowns
u', v'

– Solve for *u', v'*

$$u' = -\frac{y_2 r}{W}, v' = \frac{y_1 r}{W}$$

Variation of Parameters (4)

– Integrating

$$u = -\int \frac{y_2 r}{W} dx, v = \int \frac{y_1 r}{W} dx$$

– Example

$$y'' + y = \sec x$$

– Basis of the corresponding homogeneous DE

$$y_1 = \cos x, y_2 = \sin x$$

– Wronskian of Basis

$$W\left(y_1, y_2\right) = \cos x \cos x - \sin x \left(\sin x\right) = 1$$

– y_p by variation of parameters

$$y_p = -\cos x \int \sin x \sec x dx + \sin x \int \cos x \sec x dx$$

Variation of Parameters (5)

$$y_p = -\cos x \ln\left|\cos x\right| + x\sin x$$

– General sol.

$$y = y_h + y_p = \left[c_1 + \ln\left|\cos x\right|\right]\cos x + \left(c_1 + x\right)\sin x$$