Higher Order Linear DE (1)

- Non-homogeneous DE

$$y^{(n)} + p_{n-1}(x)\,y^{(n-1)} + \cdots + p_1(x)\,y' + p_0(x)\,y = r(x)$$

  - Homogeneous DE

$$y^{(n)} + p_{n-1}(x)\,y^{(n-1)} + \cdots + p_1(x)\,y' + p_0(x)\,y = 0$$

  - Superposition principle (or linearity principle)

$$y(x) = c_1 y_1(x) + \cdots + c_n y_n(x)$$

  - Linear independent

$$k_1 y_1(x) + \cdots + k_n y_n(x) = 0 \quad \Rightarrow \quad k_1 = 0, \cdots, k_n = 0$$

Higher Order Linear DE (2)

- Linearly dependent

$$y_1 = -\frac{1}{k_1}\left(k_2 y_2 + \cdots + k_n y_n\right)$$

- Example

$$y^{iv} - 5y'' + 4y = 0$$

- Characteristic eqn.

$$\lambda^4 - 5\lambda^2 + 4 = 0$$

- Roots of the characteristic eqn.

$$\lambda = \pm 1, \pm 2$$

- General sol.

$$y = c_1 e^{-2x} + c_2 e^{-x} + c_3 e^{x} + c_4 e^{2x}$$

Higher Order Linear DE (3)

- Initial Value Problem

$$y(x_0)=K_0, y'(x_0)=K_1, \cdots, y^{(n-1)}(x_0)=K_{n-1}$$

  - Uniqueness of sol.

    --- $p_0(x)$, …, $p_{n-1}(x)$ are continuous on I, IVP has a unique sol.
  - Example (Euler-Cauchy DE)

$$x^3y'''-3x^2y''+6xy'-6y=0, y(1)=2, y'(1)=1, y''(1)=-4$$

  - Substituting $y=x^m$

$$m(m-1)(m-2)x^m-3m(m-1)x^m+6mx^m-6x^m=0$$

  - General sol.

$$y=c_1x+c_2x^2+c_3x^3$$

  - Particular sol.

$$y=2x+x^2-x^3$$

Higher Order Linear DE (4)

- Wronskian

$$W\left(y_1,\cdots,y_n\right)=\begin{vmatrix} y_1 & y_2 & \cdots & y_n \\ y_1' & y_2' & \cdots & y_n' \\ . & . & \cdots & . \\ y_1^{(n-1)} & y_2^{(n-1)} & \cdots & y_n^{(n-1)} \end{vmatrix}_1$$

$W = 0$ on I, linearly dependent.

$W \neq 0$ on I, linearly independent.

- – Existence of sol.

  --- If $p_0(x)$, …, $p_{n\text{-}1}(x)$ are continuous on I, DE has a general sol.

  has no singular sol.

- – Every sol.

$$Y\left(x\right)=C_1 y_1\left(x\right)+\cdots+C_n y_n\left(x\right)$$

Higher Order DE w/ Const. Coefficients (1)

- Homogeneous DE

$$y^{(n)} + a_{n-1}y^{(n-1)} + \cdots + a_1 y' + a_0 y = 0$$

  - Characteristic eqn.

$$\lambda^n + a_{n-1}\lambda^{n-1} + \cdots + a_1\lambda + a_0 = 0$$

- Case I: Distinct Real Roots
  - Basis

$$y_1 = e^{\lambda_1 x}, \cdots, y_n = e^{\lambda_n x}$$

  - General sol.

$$y(x) = c_1 e^{\lambda_1 x} + \cdots + c_n e^{\lambda_n x}$$

Higher Order DE w/ Const. Coefficients (2)

– Linear independence

$$W = \begin{vmatrix} e^{\lambda_1 x} & e^{\lambda_2 x} & \cdots & e^{\lambda_n x} \\ \lambda_1 e^{\lambda_1 x} & \lambda_2 e^{\lambda_2 x} & \cdots & \lambda_n e^{\lambda_n x} \\ \cdot & \cdot & \cdots & \cdot \\ \lambda_1^{n-1} e^{\lambda_1 x} & \lambda_2^{n-1} e^{\lambda_2 x} & \cdots & \lambda_n^{n-1} e^{\lambda_n x} \end{vmatrix}$$

$$= e^{(\lambda_1 + \cdots + \lambda_n)x} \begin{vmatrix} 1 & 1 & \cdots & 1 \\ \lambda_1 & \lambda_2 & \cdots & \lambda_n \\ \cdot & \cdot & \cdots & \cdot \\ \lambda_1^{n-1} & \lambda_2^{n-1} & \cdots & \lambda_n^{n-1} \end{vmatrix} = (-1)^{\frac{n(n-1)}{2}} V \neq 0$$

Vandemonde,
Cauchy Determinant

Higher Order DE w/ Const. Coefficients (3)

– Vandemonde determinant

$$V = (\lambda_1 - \lambda_2)(\lambda_1 - \lambda_3)(\lambda_2 - \lambda_3)$$

- Case II: Simple Complex Roots
  – Complex conjugate roots

$$\lambda = \gamma \pm i\omega$$

  – Corresponding bases

$$y_1 = e^{\gamma x} \cos \omega x, y_2 = e^{\gamma x} \sin \omega x$$

- Case III: Multiple Real Roots
  – Real double root

$$\lambda_1 = \lambda_2, y_1 = y_2 \quad \Rightarrow \quad y_2 = xy_1$$

Higher Order DE w/ Const. Coefficients (4)

– Real triple root

$$\lambda_1 = \lambda_2 = \lambda_3, y_1 = y_2 = y_3 \quad \Rightarrow \quad y_2 = xy_1, y_3 = x^2 y_1$$

– Root of order *m*

$$y_1 = e^{\lambda_1 x}, y_2 = xe^{\lambda_1 x} \cdots, y_m = x^{m-1} e^{\lambda_1 x}$$

– Proof

$$L\left[y\right] = \left[D^n + a_{n-1} D^{n-1} + \cdots + a_0\right] y$$

– For $y = e^{\lambda x}$

$$L\left[e^{\lambda x}\right] = \left[\lambda^n + a_{n-1} \lambda^{n-1} + \cdots + a_0\right] e^{\lambda x}$$

– Product form if a root of order *m*

$$L\left[e^{\lambda x}\right] = \left(\lambda - \lambda_1\right)^m h\left(\lambda\right) e^{\lambda x}$$

Higher Order DE w/ Const. Coefficients (5)

– Differentiating with respect to λ

$$\frac{\partial}{\partial\lambda}L\left[e^{\lambda x}\right]=m\left(\lambda-\lambda_1\right)^{m-1}h\left(\lambda\right)e^{\lambda x}+\left(\lambda-\lambda_1\right)^{m}\frac{\partial}{\partial\lambda}\left[h\left(\lambda\right)e^{\lambda x}\right]$$

– Switching the order of derivative by x and λ

$$\frac{\partial}{\partial\lambda}L\left[e^{\lambda x}\right]=L\left[\frac{\partial}{\partial\lambda}e^{\lambda x}\right]=L\left[xe^{\lambda x}\right]$$

- Case III: Multiple Complex Roots
  – Complex double root (conjugate)

$$\lambda=\gamma\pm i\omega$$

  – Corresponding bases

$$e^{\gamma x}\cos\omega x, e^{\gamma x}\sin\omega x, xe^{\gamma x}\cos\omega x, xe^{\gamma x}\sin\omega x$$

Higher Order DE w/ Const. Coefficients (6)

– General sol.

$$y = e^{\gamma x}\left[\left(A_1 + xA_2\right)\cos\omega x + \left(B_1 + \partial x B_2\right)\sin\omega x\right]$$

– Complex triple roots (conjugate)

$$x^2 e^{\gamma x}\cos\omega x, x^2 e^{\gamma x}\sin\omega x, \cdots$$

Higher Order Non-homogeneous DE (1)

- M-th Order DE

$$y^{(n)} + p_{n-1}(x)\,y^{(n-1)} + \cdots + p_1(x)\,y' + p_0(x)\,y = r(x)$$

  - General sol.

$$y(x) = y_h(x) + y_p(x)$$

- Method of Undetermined Coeff.
  - Basic rule
  - Modification rule: k the smallest positive integer so no terms is a sol.

$$x^k y_p(x)$$

  - Sum rule

Higher Order Non-homogeneous DE (2)

- Method of Variation of Parameters

$$y_p(x) = y_1(x)\int \frac{W_1(x)}{W} r(x)\,dx + y_2(x)\int \frac{W_2(x)}{W} r(x)\,dx + \cdots + y_n(x)\int \frac{W_n(x)}{W} r(x)\,dx$$

W_j: replacing j-th column of W by $[0\ 0\ \ldots\ 0\ 1]^T$

- When n=2

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}, W_1 = \begin{vmatrix} 0 & y_2 \\ 1 & y_2' \end{vmatrix} = -y_2, W_2 = \begin{vmatrix} y_1 & 0 \\ y_1' & 1 \end{vmatrix} = y_1$$

Application: Elastic Beams (1)

- Elasticity

$$M(x) = EIk$$

$$= EI\frac{y''}{\left(1+y'^2\right)^{3/2}} \approx EIy''$$

$$M''(x) = f(x)$$

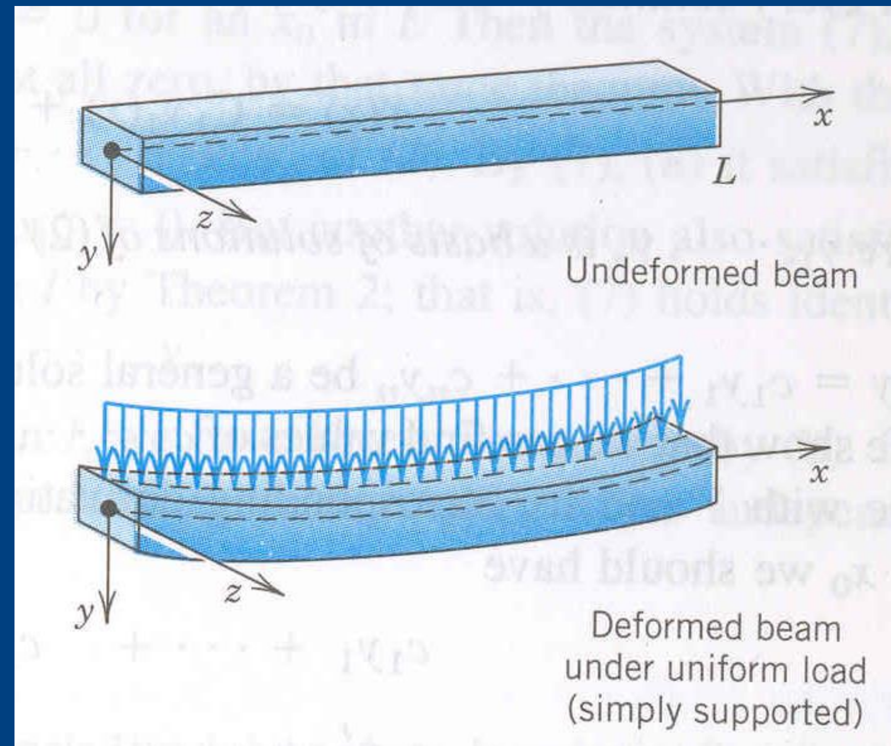
M(*x*): bending moment

k: curvature of deflection

E: Young's modulus

I: Area moment of inertia

f(*x*): load per unit length



$$\Rightarrow \quad EIy^{iv} = f(x)$$

Application: Elastic Beams (2)

- Boundary conditions
  - (A) Simply supported

$$y = y'' = 0$$

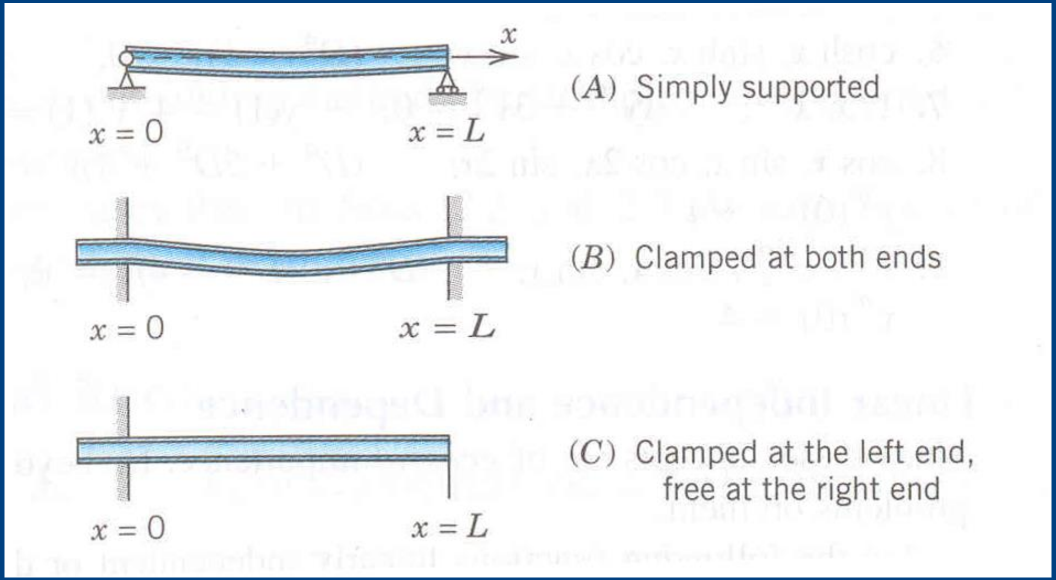
  - (B) Clamped at both ends

$$y = y' = 0$$

  - (C) Clamped, free

$$y(0) = y'(0) = 0;$$

$$y''(L) = y'''(L) = 0$$



– Example: uniformly loaded simply supported beam

$$y^{iv} = k, k = \frac{f_0}{EI}$$

Application: Elastic Beams (3)

– Integrating twice

$$y'' = \frac{k}{2}x^2 + c_1 x + c_2$$

– Two BC's gives: $y''(0) = y''(L) = 0$

$$y'' = \frac{k}{2}\left(x^2 - Lx\right)$$

– Integrating twice

$$y = \frac{k}{2}\left(\frac{1}{12}x^4 - \frac{L}{6}x^3 + c_3 x + c_4\right)$$

– Two BC's gives: $y(0) = y(L) = 0$

$$\Rightarrow \quad y = \frac{f_0}{24EI}\left(x^4 - 2Lx^3 + L^3 x\right)$$