Seoul National University
Dept. of Civil and Environmental Engineering

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457.646 Topics in Structural Reliability
In-Class Material: Class 03

3. Random Variables, Prob. Functions & Partial Descriptors:

Tools to associate uncertain q_________ with probabilities

◎ Random variables

: a ~~variable~~ _________ that takes on one of the values in a specified set according to the assigned probabilities

Example: X = the random number one can get from throwing a fair dice

Specified set:

Assigned probabilities:

◎ Prob. Functions (mapping b/w   &   )

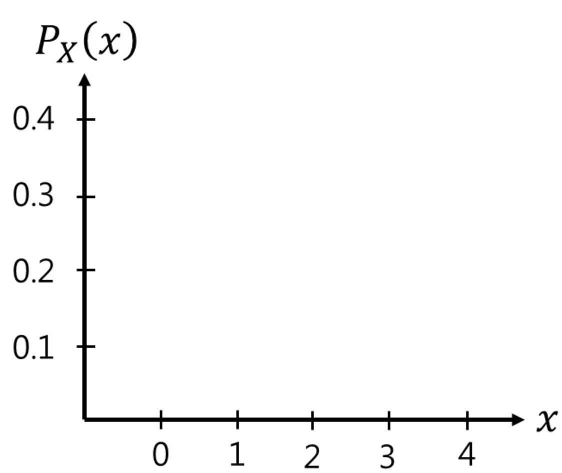
Functions for discrete random variables

① Probability ________ Function (   ) of X

$P_X(x) \equiv$ $x \rightarrow \boxed{P_X(\;)} \rightarrow$

e.g. # of land falls of hurricanes/year

x	$P_X(x)$
0	0.10
1	0.40
2	0.30
3	0.15
4	0.05



※ $\quad \le P_X(x) \le$

$\sum_{\text{all } x} P_X(x) =$

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$P(a < X \leq b) = \sum$

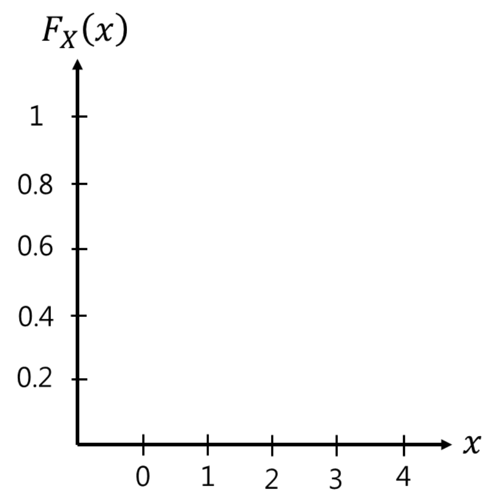
e.g. $P(0 < X \leq 2) =$

② Cumulative ________ Function ( ) of X

$F_X(x) \equiv \qquad = \sum$

$x \rightarrow \boxed{F_X(\ )} \rightarrow$

x	$P_X(x)$	$F_X(x)$
0	0.10	
1	0.40	
2	0.30	
3	0.15	
4	0.05	



※ $F_X(a) = \sum$

$F_X(-\infty)$

$F_X(\infty)$

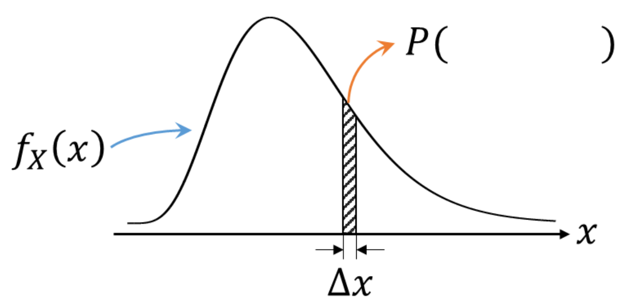
$P(a < X \leq b) = \qquad -$

Functions for continuous r.v.

③ Probability ________ Function ( ) of X

$f_X(x) = \lim_{\Delta x \to 0}$

"Density" of Probability at $X = x$



Normalization rule Area =

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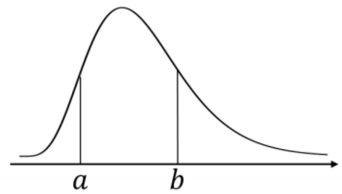
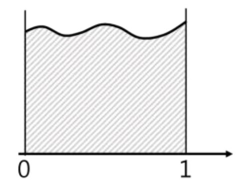
※ $\leq f_X(x)$

$\int f_X(x)dx = P(\qquad\qquad) =$

$P(a < X \leq b) = \int f_X(x)dx$



e.g. X ∈ [0,1]



$-\infty$ in
the integral
=

④ Cumulative ___________ Function (   ) of X

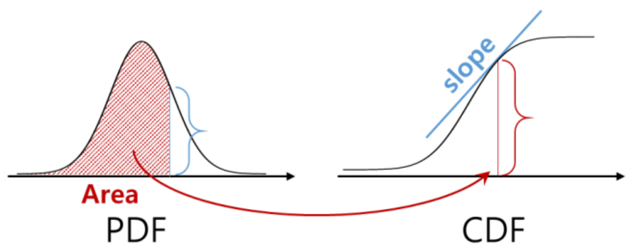
$F_X(x) \equiv P(X \quad x) = \int \qquad dx$

※ $\dfrac{dF_X(x)}{dx} =$

non-   ing

$F_X(-\infty)$

$F_X(\infty)$



◎ Partial Descriptors of a r.v. :

(a) “Complete” description by probability functions:

(b) “Partial” descriptors: measures of key characteristics; can derive from (   )

Note:

- Expectation: $E[\cdot] = \int_{-\infty}^{\infty} (\cdot) f_X(x)dx$ (continuous) or $\sum_{\text{all } x} (\cdot) p_X(x)$ (discrete)
- Moment: $E[\cdot X^n] = \int_{-\infty}^{\infty} x^n f_X(x)dx$ or $\sum_{\text{all } x} x^n p_X(x)$
- Central Moment, $E[(X - \mu_X)^n] = \int_{-\infty}^{\infty} (x - \mu_X)^n f_X(x)dx$ or $\sum_{\text{all } x} (x - \mu_X)^n p_X(x)$

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`	Name	Definition	Meaning (PDF/CDF)
Measure of Central Location	Mean, μ_X	First moment, $E[X]$	Location of the ( ) of an area underneath ( )
	Median, $x_{0.5}$	$F_X(x_{0.5}) = 0.5$ $F_X^{-1}(0.5)$	The value of a r.v. at which values above and below it are _______lly probable. If symmetric?
	Mode, $\tilde{x}$	$\arg\max_x f_X(x)$	The outcome that has the _______est probability mass or density
Measure of Dispersion	Variance, σ_X^2	Second-order central moment $E[(X-\mu_X)^2]$ $= E[X^2] - E[X]^2$	Average of squared deviations
	Standard Deviation, σ_X	$\sqrt{\sigma_X^2}$	Radius of ( )
	Coefficient of Variation (C.O.V.), δ_X	$\dfrac{\sigma_X}{\lvert\mu_X\rvert}$	__________ed radius of ( )
Asymmetry	Coefficient of Skewness, γ_X	Third-order central moment normalized by σ_X^3, $\dfrac{E[(X-\mu_X)^3]}{\sigma_X^3}$	Behavior of two tails > 0 = 0 < 0
Flatness	Coefficient of Kurtosis, κ_X	Fourth-order central moment normalized by σ_X^4, $\dfrac{E[(X-\mu_X)^4]}{\sigma_X^4}$	"Peakedness" - more of the variance is due to infrequent extreme deviations, as opposed to frequent modestly-sized deviations.

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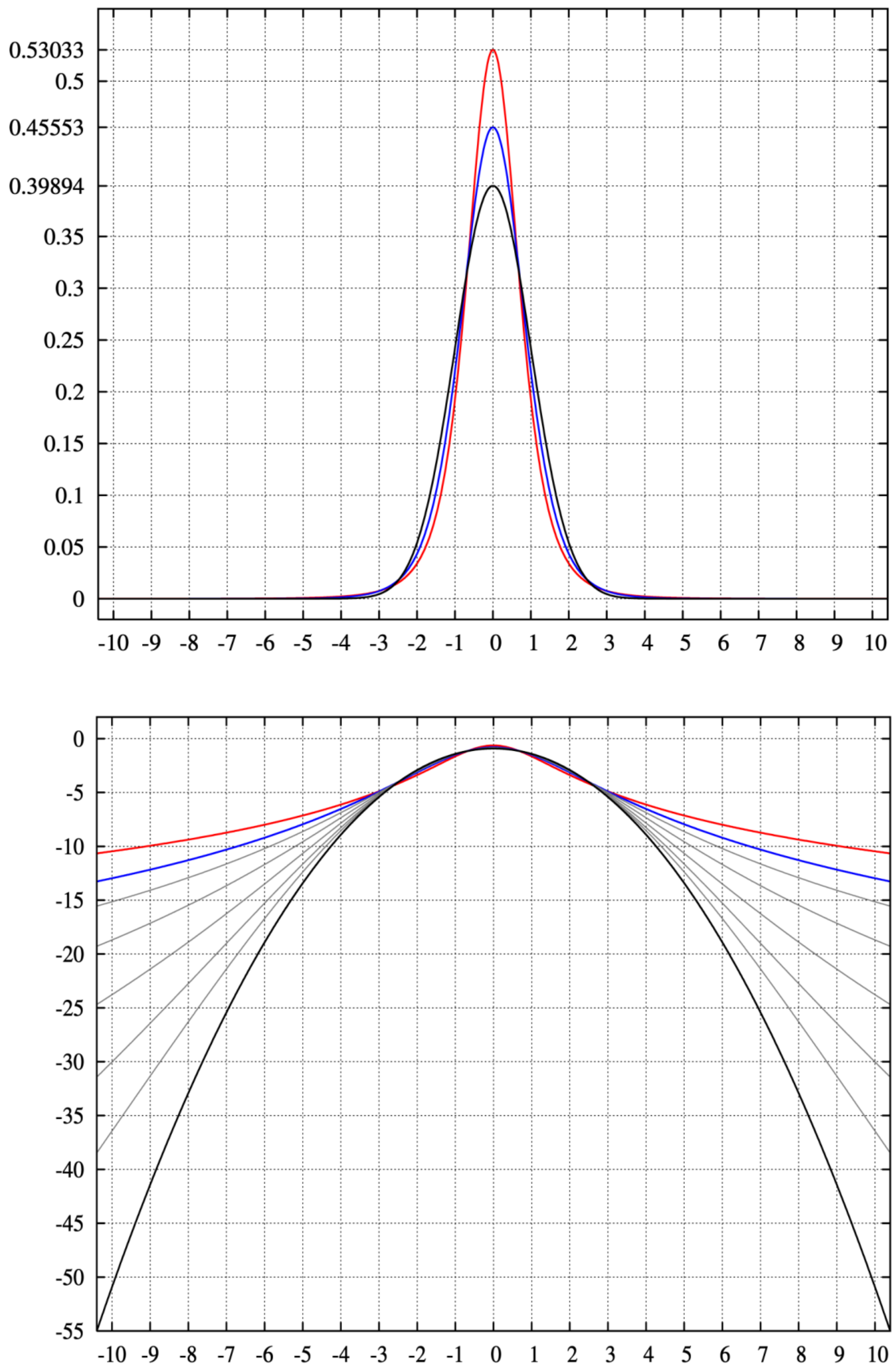
Example: PDF and Log-PDF of Pearson type VII distribution with kurtosis of infinity (red), 2 (blue), and 0 (black) (source: Wikipedia)



4. Probability Distribution Models

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Normal (Gaussian) Distribution

1. **Normal** distribution

- Best known and most widely used. Also known as _______________ distribution.
- According to _________________________, the sum of random variables converges to a normal random variable as the number of the variables increases, no matter what distributions the variables are subjected to.
- Completely defined by the ______________ and the ____________________ of the random variable.

(a) PDF: $X \sim N(\mu, \sigma^2)$

$$f_X(x) = \frac{1}{\sqrt{2\pi}\sigma} \exp\left[-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2\right], \ -\infty < x < \infty$$

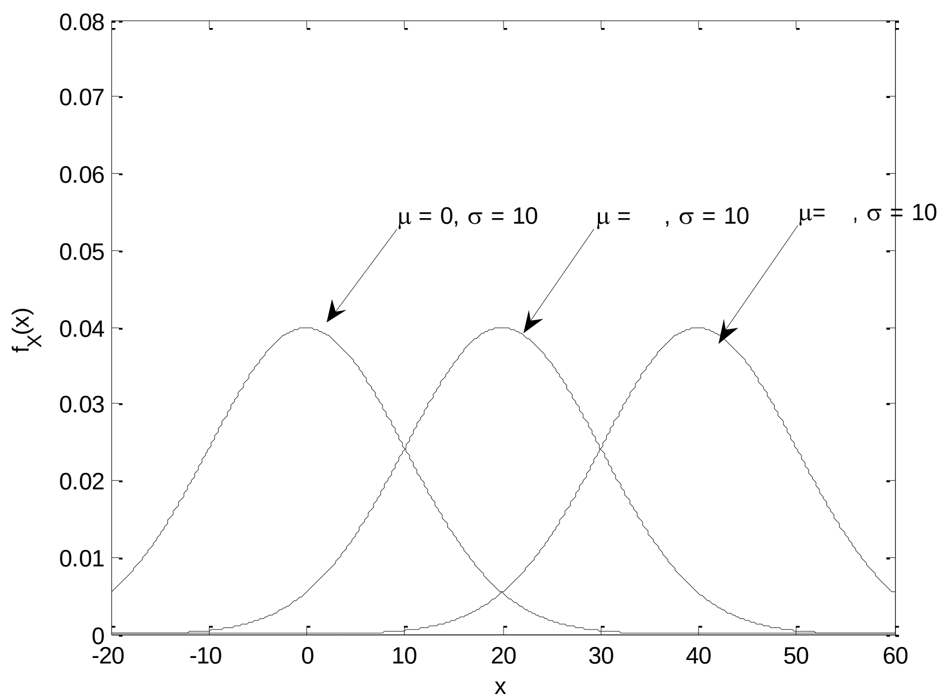

Figure 1. PDF's of normal random variables with different values of μ

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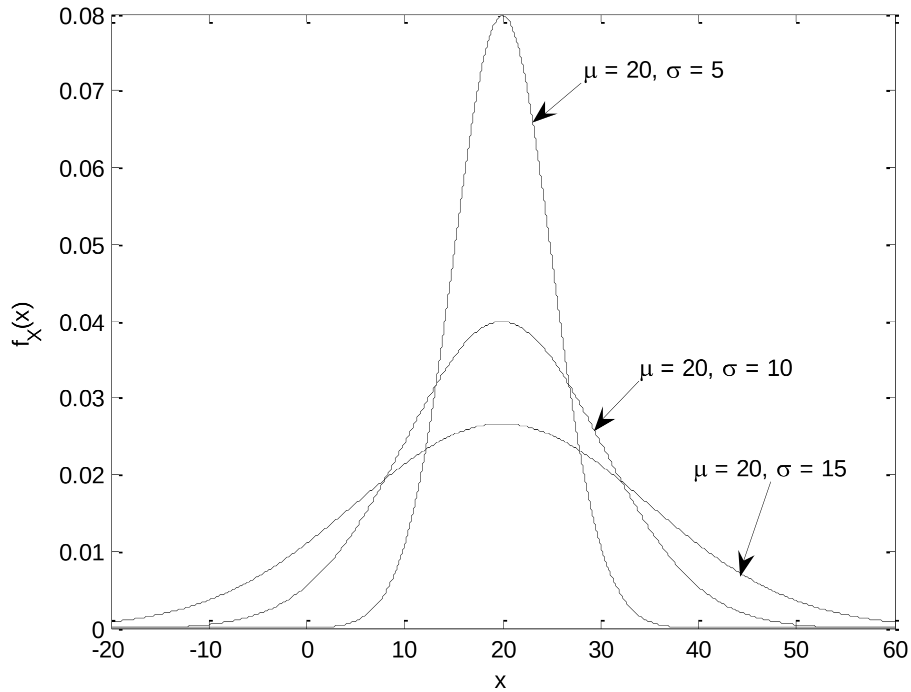

Figure 2. PDF's of normal random variables with different values of σ

(b) CDF: no closed-form expression available

$$F_X(x) = \int_{-\infty}^{x} f_X(x)dx, \; -\infty < x < \infty$$

(c) Parameters: μ, σ

- μ: __________ of the random variable, i.e. $\mu = \mu_X \equiv \mathrm{E}[X]$
- σ: ______________________ of the random variable, i.e. $\sigma = \sigma_X \equiv \left\{\mathrm{E}[(X-\mu_X)^2]\right\}^{0.5}$

(d) Shape of the PDF plots

- Symmetric around $x =$
- A change in μ_X ________________ the PDF horizontally by the same amount.
- The larger the value of σ_X gets, the more _______________ the PDF becomes around the central axis.

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1a. **Standard** normal distribution

- A special case of the normal distribution: $\mu_X =$, $\sigma_X =$.
- The CDF of the standard normal distribution can be used for computing the CDF of any general normal random variable.

(a) PDF: $U \sim N(\quad,\quad^2)$

$$\varphi(u) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{1}{2}u^2\right), \quad -\infty < u < \infty$$

(b) CDF:

$$\Phi(u) = \int_{-\infty}^{u} \varphi(u)du, \quad -\infty < u < \infty$$

→ no closed-form expression available, but the table of the standard normal CDF $\Phi(\cdot)$ can be found in books or computer software (e.g. See Appendix A of A&T)

(c) Inverse CDF of standard normal distribution: $\Phi^{-1}(\cdot)$

$$\Phi(u_p) = p \quad \Leftrightarrow \quad u_p = \Phi^{-1}(p)$$

(d) Symmetry around $u =$:

$$\Phi(-u) = 1 - \Phi(u)$$
$$u_{1-p} = -u_p$$

→ The table of the standard normal CDF is often provided for positive *u* values only, but using the symmetry one can find the CDF for negative values as well.

(e) One can compute the CDF of a general normal random variable $X \sim N(\mu, \sigma^2)$ by use of the CDF of the standard normal random variable $U \sim N(0, 1^2)$ as follows.

$$\begin{aligned} F_X(a) &= P(X \le a) \\ &= \int_{-\infty}^{a} \frac{1}{\sqrt{2\pi}\sigma} \exp\left[-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2\right] dx \\ &= \int_{-\infty}^{\left(\frac{a-\mu}{\sigma}\right)} \frac{1}{\sqrt{2\pi}\sigma} \exp\left(-\frac{1}{2}u^2\right) \sigma du \\ &= \Phi\left(\frac{\qquad}{}\right) \end{aligned}$$

Hence, $P(a < X \le b) = F_X(\quad) - F_X(\quad) = \Phi\left(\frac{\qquad}{}\right) - \Phi\left(\frac{\qquad}{}\right)$

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Example 1: Given a standard normal distribution, find the area under the curve that lies

(a) to the right of $u = 1.84$

(b) between $u = -1.97$ and $u = 0.86$

Example 2: The drainage demand during a storm (in mgd: million gallons/day): $X \sim N(1.2, 0.4^2)$. The maximum drain capacity is 1.5 mgd.

(a) Probability of flooding?

(b) Probability that the drainage demand during a storm will be between 1.0 and 1.6 mgd?

(c) The 90-percentile drainage demand?

x	PHI(x)	x	PHI(x)	x	PHI(x)	x	PHI(x)	x	PHI(x)
0.00	0.5	0.90	0.81593987	1.80	0.96406968	2.70	0.99653303	3.60	0.999840891
0.01	0.50398936	0.91	0.81858875	1.81	0.96485211	2.71	0.99663584	3.61	0.999846901
0.02	0.50797831	0.92	0.82121362	1.82	0.9656205	2.72	0.9967359	3.62	0.999852698
0.03	0.51196647	0.93	0.82381446	1.83	0.96637503	2.73	0.99683328	3.63	0.999858289
0.04	0.51595344	0.94	0.82639122	1.84	0.96711588	2.74	0.99692804	3.64	0.999863681
0.05	0.51993881	0.95	0.82894387	1.85	0.96784323	2.75	0.99702024	3.65	0.99986888
0.06	0.52392218	0.96	0.83147239	1.86	0.96855724	2.76	0.99710993	3.66	0.999873892
0.07	0.52790317	0.97	0.83397675	1.87	0.96925809	2.77	0.99719719	3.67	0.999878725
0.08	0.53188137	0.98	0.83645694	1.88	0.96994596	2.78	0.99728206	3.68	0.999883383
0.09	0.53585639	0.99	0.83891294	1.89	0.97062102	2.79	0.9973646	3.69	0.999887873
0.10	0.53982784	1.00	0.84134475	1.90	0.97128344	2.80	0.99744487	3.70	0.9998922
0.11	0.54379531	1.01	0.84375235	1.91	0.97193339	2.81	0.99752293	3.71	0.99989637
0.12	0.54775843	1.02	0.84613577	1.92	0.97257105	2.82	0.99759882	3.72	0.999900389
0.13	0.55171679	1.03	0.848495	1.93	0.97319658	2.83	0.9976726	3.73	0.99990426
0.14	0.55567	1.04	0.85083005	1.94	0.97381016	2.84	0.99774432	3.74	0.99990799
0.15	0.55961769	1.05	0.85314094	1.95	0.97441194	2.85	0.99781404	3.75	0.999911583
0.16	0.56355946	1.06	0.8554277	1.96	0.9750021	2.86	0.99788179	3.76	0.999915043
0.17	0.56749493	1.07	0.85769035	1.97	0.97558081	2.87	0.99794764	3.77	0.999918376
0.18	0.57142372	1.08	0.85992891	1.98	0.97614824	2.88	0.99801162	3.78	0.999921586
0.19	0.57534543	1.09	0.86214343	1.99	0.97670453	2.89	0.99807379	3.79	0.999924676
0.20	0.57925971	1.10	0.86433394	2.00	0.97724987	2.90	0.99813419	3.80	0.999927652
0.21	0.58316616	1.11	0.86650049	2.01	0.97778441	2.91	0.99819286	3.81	0.999930517
0.22	0.58706442	1.12	0.86864312	2.02	0.97830831	2.92	0.99824984	3.82	0.999933274
0.23	0.59095412	1.13	0.87076189	2.03	0.97882173	2.93	0.99830519	3.83	0.999935928
0.24	0.59483487	1.14	0.87285685	2.04	0.97932484	2.94	0.99835894	3.84	0.999938483
0.25	0.59870633	1.15	0.87492806	2.05	0.97981778	2.95	0.99841113	3.85	0.999940941
0.26	0.60256811	1.16	0.8769756	2.06	0.98030073	2.96	0.9984618	3.86	0.999943306
0.27	0.60641987	1.17	0.87899952	2.07	0.98077383	2.97	0.998511	3.87	0.999945582
0.28	0.61026125	1.18	0.88099989	2.08	0.98123723	2.98	0.99855876	3.88	0.999947772
0.29	0.61409188	1.19	0.8829768	2.09	0.9816911	2.99	0.99860511	3.89	0.999949878
0.30	0.61791142	1.20	0.88493033	2.10	0.98213558	3.00	0.9986501	3.90	0.999951904
0.31	0.62171952	1.21	0.88686055	2.11	0.98257082	3.01	0.99869376	3.91	0.999953852
0.32	0.62551583	1.22	0.88876756	2.12	0.98299698	3.02	0.99873613	3.92	0.999955726
0.33	0.62930002	1.23	0.89065145	2.13	0.98341419	3.03	0.99877723	3.93	0.999957527
0.34	0.63307174	1.24	0.8925123	2.14	0.98382262	3.04	0.99881711	3.94	0.999959259
0.35	0.63683065	1.25	0.89435023	2.15	0.98422239	3.05	0.99885579	3.95	0.999960924
0.36	0.64057643	1.26	0.89616532	2.16	0.98461367	3.06	0.99889332	3.96	0.999962525
0.37	0.64430875	1.27	0.89795768	2.17	0.98499658	3.07	0.99892971	3.97	0.999964064
0.38	0.64802729	1.28	0.89972743	2.18	0.98537127	3.08	0.998965	3.98	0.999965542
0.39	0.65173173	1.29	0.90147467	2.19	0.98573788	3.09	0.99899922	3.99	0.999966963
0.40	0.65542174	1.30	0.90319952	2.20	0.98609655	3.10	0.9990324	**4.00***	3.16712E-05
0.41	0.65909703	1.31	0.90490208	2.21	0.98644742	3.11	0.99906456	**4.05**	2.56088E-05
0.42	0.66275727	1.32	0.90658249	2.22	0.98679062	3.12	0.99909574	**4.10**	2.06575E-05
0.43	0.66640218	1.33	0.90824086	2.23	0.98712628	3.13	0.99912597	**4.15**	1.66238E-05
0.44	0.67003145	1.34	0.90987733	2.24	0.98745454	3.14	0.99915526	**4.20**	1.33457E-05
0.45	0.67364478	1.35	0.91149201	2.25	0.98777553	3.15	0.99918365	**4.25**	1.06885E-05
0.46	0.67724189	1.36	0.91308504	2.26	0.98808937	3.16	0.99921115	**4.30**	8.53991E-06
0.47	0.68082249	1.37	0.91465655	2.27	0.98839621	3.17	0.99923781	**4.35**	6.80688E-06
0.48	0.6843863	1.38	0.91620668	2.28	0.98869616	3.18	0.99926362	**4.40**	5.41254E-06
0.49	0.68793305	1.39	0.91773556	2.29	0.98898934	3.19	0.99928864	**4.45**	4.29351E-06
0.50	0.69146246	1.40	0.91924334	2.30	0.98927589	3.20	0.99931286	**4.50**	3.39767E-06
0.51	0.69497427	1.41	0.92073016	2.31	0.98955592	3.21	0.99933633	**4.55**	2.68230E-06
0.52	0.69846821	1.42	0.92219616	2.32	0.98982956	3.22	0.99935905	**4.60**	2.11245E-06
0.53	0.70194403	1.43	0.92364149	2.33	0.99009692	3.23	0.99938105	**4.65**	1.65968E-06
0.54	0.70540148	1.44	0.9250663	2.34	0.99035813	3.24	0.99940235	**4.70**	1.30081E-06
0.55	0.70884031	1.45	0.92647074	2.35	0.99061329	3.25	0.99942297	**4.75**	1.01708E-06
0.56	0.71226028	1.46	0.92785496	2.36	0.99086253	3.26	0.99944294	**4.80**	7.93328E-07
0.57	0.71566115	1.47	0.92921912	2.37	0.99110596	3.27	0.99946226	**4.85**	6.17307E-07
0.58	0.71904269	1.48	0.93056338	2.38	0.99134368	3.28	0.99948096	**4.90**	4.79183E-07
0.59	0.72240468	1.49	0.93188788	2.39	0.99157581	3.29	0.99949906	**4.95**	3.71068E-07
0.60	0.72574688	1.50	0.9331928	2.40	0.99180246	3.30	0.99951658	**5.00**	2.86652E-07
0.61	0.7290691	1.51	0.93447829	2.41	0.99202374	3.31	0.99953352	**5.10**	1.69827E-07
0.62	0.73237111	1.52	0.93574451	2.42	0.99223975	3.32	0.99954991	**5.20**	9.96443E-08
0.63	0.73565271	1.53	0.93699164	2.43	0.99245059	3.33	0.99956577	**5.30**	5.79013E-08
0.64	0.7389137	1.54	0.93821982	2.44	0.99265637	3.34	0.99958111	**5.40**	3.33204E-08
0.65	0.74215389	1.55	0.93942924	2.45	0.99285719	3.35	0.99959594	**5.50**	1.89896E-08
0.66	0.74537309	1.56	0.94062006	2.46	0.99305315	3.36	0.99961029	**5.60**	1.07176E-08
0.67	0.7485711	1.57	0.94179244	2.47	0.99324435	3.37	0.99962416	**5.70**	5.99037E-09
0.68	0.75174777	1.58	0.94294657	2.48	0.99343088	3.38	0.99963757	**5.80**	3.31575E-09
0.69	0.75490291	1.59	0.9440826	2.49	0.99361285	3.39	0.99965054	**5.90**	1.81751E-09
0.70	0.75803635	1.60	0.94520071	2.50	0.99379033	3.40	0.99966307	**6.00**	9.86588E-10
0.71	0.76114793	1.61	0.94630107	2.51	0.99396344	3.41	0.99967519	**6.10**	5.30342E-10
0.72	0.7642375	1.62	0.94738386	2.52	0.99413226	3.42	0.99968689	**6.20**	2.82316E-10
0.73	0.76730491	1.63	0.94844925	2.53	0.99429687	3.43	0.99969821	**6.30**	1.48823E-10
0.74	0.77035	1.64	0.94949742	2.54	0.99445738	3.44	0.99970914	**6.40**	7.76885E-11
0.75	0.77337265	1.65	0.95052853	2.55	0.99461385	3.45	0.99971971	**6.50**	4.01600E-11
0.76	0.77637271	1.66	0.95154277	2.56	0.99476639	3.46	0.99972991	**6.60**	2.05579E-11
0.77	0.77935005	1.67	0.95254032	2.57	0.99491507	3.47	0.99973977	**6.70**	1.04210E-11
0.78	0.78230456	1.68	0.95352134	2.58	0.99505998	3.48	0.99974929	**6.80**	5.23093E-12
0.79	0.78523612	1.69	0.95448602	2.59	0.9952012	3.49	0.99975849	**6.90**	2.60014E-12
0.80	0.7881446	1.70	0.95543454	2.60	0.99533881	3.50	0.99976737	**7.00**	1.27987E-12
0.81	0.79102991	1.71	0.95636706	2.61	0.99547289	3.51	0.99977595	**7.10**	6.23834E-13
0.82	0.79389195	1.72	0.95728378	2.62	0.99560351	3.52	0.99978423	**7.20**	3.01092E-13
0.83	0.79673061	1.73	0.95818486	2.63	0.99573076	3.53	0.99979222	**7.30**	1.43885E-13
0.84	0.79954581	1.74	0.95907049	2.64	0.9958547	3.54	0.99979994	**7.40**	6.80567E-14
0.85	0.80233746	1.75	0.95994084	2.65	0.99597541	3.55	0.99980738	**7.50**	3.18634E-14
0.86	0.80510548	1.76	0.9607961	2.66	0.99609297	3.56	0.99981457	**7.60**	1.47660E-14
0.87	0.8078498	1.77	0.96163643	2.67	0.99620744	3.57	0.99982151	**7.70**	6.77236E-15
0.88	0.81057035	1.78	0.96246202	2.68	0.99631889	3.58	0.9998282	**7.80**	3.10862E-15
0.89	0.81326706	1.79	0.96327304	2.69	0.9964274	3.59	0.99983466	**7.90**	0.00000E+00

* Note: For x>=4.0, 1-PHI(x) is given instead.

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Probability Distribution Models in Matlab® Statistics Toolbox

Full Name	Short	Parameters	Probability Density/Mass Function	Mean	Variance
Binomial	*bino*	$0<p<1$, n integer	$\binom{n}{x}p^x(1-p)^{(n-x)},\quad x=0,1,\ldots,n$	np	$np(1-p)$
Geometric	*geo*	$0<p<1$	$p(1-p)^x,\quad x=0,1,2,\ldots$	$(1-p)/p$	$(1-p)/p^2$
Hypergeometric	*hyge*	$0<K,N\le M$; K,N,M integers	$\binom{K}{x}\binom{M-K}{N-x}\binom{M}{N}^{-1},\quad K+N-M\le x\le K$	$\frac{NK}{M}$	$N\frac{K}{M}\frac{M-K}{M}\frac{M-N}{M-1}$
Negative Binomial	*nbin*	$0<p<1$, r integer	$\binom{r+x-1}{x}p^r(1-p)^x,\quad x=0,1,\ldots$	$r(1-p)/p$	$r(1-p)/p^2$
Poisson	*poiss*	$0<\lambda$	$\frac{\lambda^x}{x!}e^{-\lambda},\quad x=0,1,\ldots$	λ	λ
Beta	*beta*	$0<a,b$	$B(a,b)^{-1}x^{a-1}(1-x)^{b-1},\quad 0\le x\le 1$	$a/(a+b)$	$ab/(a+b+1)/(a+b)^2$
Chisquare	*chi2*	$0<\nu$	$x^{(\nu-2)/2}e^{-x/2}2^{-\nu/2}\Gamma(\nu/2)^{-1},\quad 0<x$	ν	2ν
Exponential	*exp*	$0<\mu$	$\mu^{-1}e^{-x/\mu},\quad 0<x$	μ	μ^2
F	*f*	$0<\nu_1,\nu_2$	$\frac{\Gamma[(\nu_1+\nu_2)/2](\nu_1/\nu_2)^{\nu_1/2}x^{\nu_1/2-1}}{\Gamma(\nu_1/2)\Gamma(\nu_2/2)[1+(\nu_1/\nu_2)x]^{(\nu_1+\nu_2)/2}},\quad 0<x$	$\nu_2/(\nu_2-2)$	$\frac{2\nu_2^2(\nu_1+\nu_2-2)}{\nu_1(\nu_2-2)^2(\nu_2-4)}$
Gamma	*gam*	$0<a,b$	$b^{-a}\Gamma(a)^{-1}x^{a-1}e^{-x/b},\quad 0<x$	ab	ab^2
Lognormal	*logn*	$\lambda, 0<\zeta$	$x^{-1}\xi^{-1}(2\pi)^{-1/2}\exp[-(\ln x-\lambda)^2/2\xi^2],\quad 0<x$	$e^{(\lambda+0.5\xi^2)}$	$e^{(2\lambda+2\xi^2)}-e^{(2\lambda+\xi^2)}$
Normal	*norm*	$\mu, 0<\sigma$	$\sigma^{-1}(2\pi)^{-1/2}\exp[-(x-\mu)^2/2\sigma^2]$	μ	σ^2
Rayleigh	*rayl*	$0<b$	$xb^{-2}\exp(-x^2/2b^2),\quad 0<x$	$b\sqrt{\pi/2}$	$(4-\pi)b^2/2$
T	*t*	$0<\nu$	$(\nu\pi)^{-1/2}\Gamma((\nu+1)/2)\Gamma(\nu/2)^{-1}(1+x^2/\nu)^{-(\nu+1)/2}$	0	$\nu/(\nu-2)$
Uniform	*unif*	$a<b$	$(b-a)^{-1},\quad a\le x\le b$	$(a+b)/2$	$(b-a)^2/12$
Weibull	*weib*	$0<a,b$	$abx^{b-1}e^{-ax^b},\quad 0<x$	$a^{-1/b}\Gamma(1+b^{-1})$	$a^{-2/b}[\Gamma(1+2b^{-1})-\Gamma^2(1+b^{-1})]$

Use *shortname***pdf**( ) to compute the probability density/mass function; *shortname***cdf**( ) to compute cumulative distribution function; *shortname***fit**( ) to estimate parameters from data; *shortname***rnd**( ) to generate random numbers; *shortname***stat**( ) to compute mean and variance for specified parameters; and *shortname***inv**( ) to compute the inverse cumulative probability. Use Matlab® help to learn more about these commands.

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457.646 Topics in Structural Reliability
Lognormal Distribution

4-2. **Lognormal** distribution

- Closely related to the ________ distribution.
- Defined for _________ values only.

(a) PDF: $X \sim LN(\lambda, \zeta^2)$

$$f_X(x) = \frac{1}{\sqrt{2\pi}\zeta x} \exp\left[-\frac{1}{2} \left(\frac{\ln x - \lambda}{\zeta} \right)^2 \right], \quad 0 < x < \infty$$

(b) CDF:

$$F_X(x) = \int_{-\infty}^{x} f_X(x) dx, \quad 0 < x < \infty$$

→ no closed-form expression available, but can be computed by use of the table of the standard normal CDF $\Phi(\cdot)$ (as shown below)

(c) Parameters: λ, ζ

- λ: mean of _____________, i.e. $\lambda = \lambda_X \equiv \mathrm{E}[\ln X]$
- ζ: standard deviation of______________, i.e. $\zeta^2 = \zeta_X^2 = \sigma_{\ln X}^2$

(d) Shape of the PDF plots

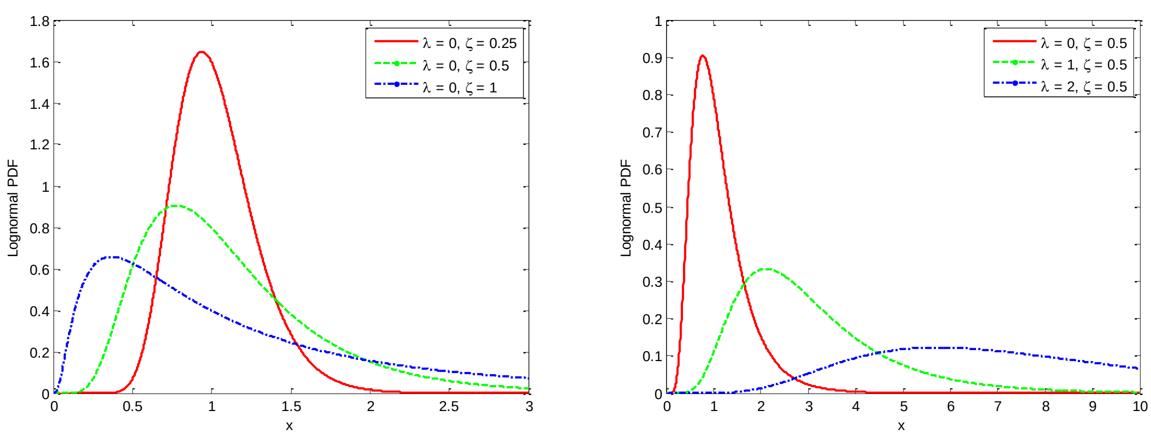


Figure 3. PDF's of lognormal random variables.

(e) Relationship between normal and lognormal distribution:

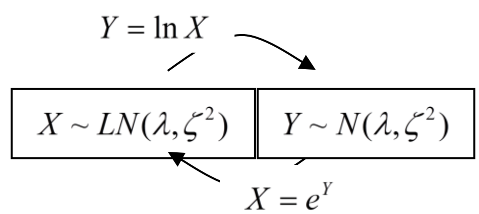
"The logarithm of a ___________ random variable is a __________ random variable."

$$X \sim LN(\lambda,\zeta^2) \Rightarrow \ln X \sim N(\lambda,\zeta^2)$$

(f) Can obtain the CDF of lognormal $X \sim LN(\lambda,\zeta^2)$ from the CDF of standard normal:

$$\begin{aligned} F_X(a) &= P(X \le a) \\ &= P(\ln X \le \ln a) \qquad \text{Since } \ln X \sim N(\lambda,\zeta^2), \\ &= \Phi\left(\frac{\ln a - \lambda}{\zeta}\right) \end{aligned}$$

(g) "The exponential function of a _____________ random variable is a ______________ random variable."



(h) $(\lambda,\zeta) \rightarrow (\mu,\delta)$: Find the mean and c.o.v. from the distribution parameters

$$\mu = \mathrm{E}[X] = \exp(\lambda + 0.5\zeta^2)$$
$$\delta = \sigma/\mu = \sqrt{\exp(\zeta^2) - 1} \quad (\cong \zeta \text{ for } \zeta << 1)$$

(i) $(\mu,\delta) \rightarrow (\lambda,\zeta)$: Find the distribution parameters from the mean and c.o.v.

$$\zeta = \sqrt{\ln(1+\delta^2)} \quad (\cong \delta \text{ for } \delta << 1)$$
$$\lambda = \ln\mu - 0.5\ln(1+\delta^2)$$

(j) $(x_{0.5}) \leftrightarrow (\lambda)$: Relationship between the median and λ

$$\lambda = \ln x_{0.5}, \quad x_{0.5} = e^{\lambda}$$

(k) $(\mu,\delta) \rightarrow (x_{0.5})$: Find the median from the mean and c.o.v.

$$x_{0.5} = \frac{\mu}{\sqrt{1+\delta^2}}$$

Note: $x_{0.5} < \mu$ for the lognormal distribution.

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Example 1: The drainage demand during a storm (in mgd: million gallons/day) is assumed to follow the lognormal distribution with the same mean and standard deviation as Example 1 (mean 1.2, standard deviation 0.4). The maximum drain capacity is 1.5 mgd.

(a) Distribution parameters, i.e. λ and ζ ?

(b) Probability of the flooding?

(c) Probability that the drainage demand during a storm will be between 1.0 and 1.6 mgd?

(d) The 90-percentile drainage demand?

Example 2: Consider a bridge whose uncertain capacity against "complete damage" limit-state caused by earthquake events is defined in terms of peak ground acceleration (PGA; unit: g) that the bridge can sustain. Suppose the median of the capacity is 1.03g and the coefficient of variation is 0.50. It is assumed that the capacity follows a lognormal distribution.

(a) Distribution parameters of the lognormal distribution, i.e. λ and ζ ?

(b) The mean and standard deviation of the uncertain capacity, i.e. μ and σ?

(c) Suppose the peak ground acceleration from an earthquake event is 0.5g. What is the probability that the structure will exceed "complete damage" limit state?

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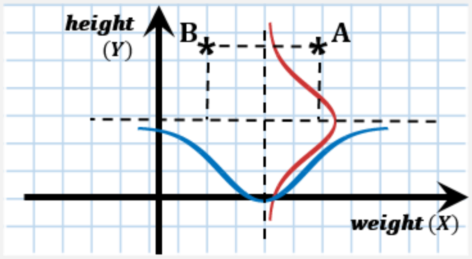
457.646 Topics in Structural Reliability
In-Class Material: Class 04



Question: Which one more likely?

Case A: Heavy & Tall

Case B: Light & Tall

Ⅱ-5. Multiple Random Variables

◎ "Joint" Probability Functions

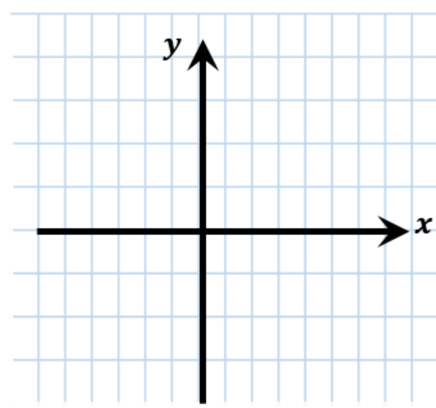
e.g. $P(X \le 20)$ = $\int \qquad dx$

=

$P(\quad \cap \quad) = ?$

Need more information than (   ) and (   )

① Joint Cumulative Distribution Function (CDF) (Discrete/Continuous) ↔ cf. __________ CDF



$F_{XY}(x, y) \equiv P(\qquad\qquad)$

- $F_{XY}(-\infty, -\infty) =$
- $F_{XY}(\infty, \infty) =$
- $F_{XY}(-\infty, y)$
- $F_{XY}(\infty, y) = P(\quad \cap \quad) = P(\quad)$
  $=$

② Joint Probability Mass Function (discrete r.v's) ↔ cf. __________ PMF

.

(a) Definition : $P_{XY}(x, y) \equiv P(\qquad , \qquad)$

(b) $F_{XY}(a, b) = \sum$

(c) Conditional PMF

$P_{X|Y}(x|y) \equiv \qquad = \frac{\qquad}{\qquad} =$

(d) $P_{XY}(x, y) \to P_X(x),\ P_Y(y)?$

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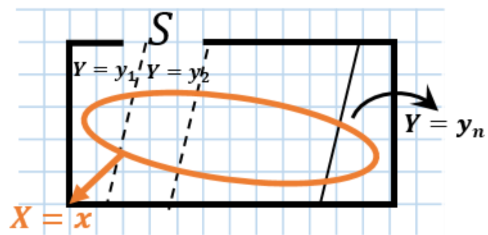
$P_X(x) = \sum$

$= \sum$



$\Rightarrow$( ) rule

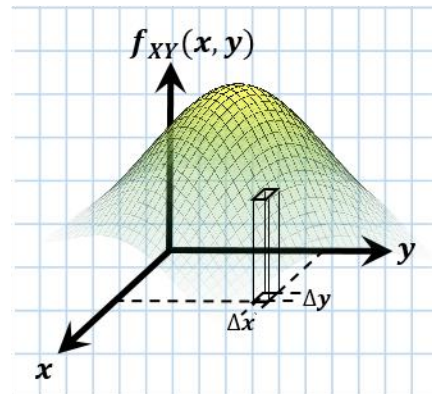
(e) If X & Y are statically independent,

$P_{X|Y}(x|y) =$

$\Leftrightarrow P_{Y|X}(y|x) \qquad P_Y(y)$

$\Leftrightarrow P_{XY}(x,y)$

* See supplementary material on Joint PMF



③ **Joint <u>PDF</u> (continuous r.v's)**

$f_{XY}(x,y) = \lim_{\Delta x, \Delta y \to 0} \overline{\qquad\qquad\qquad}$

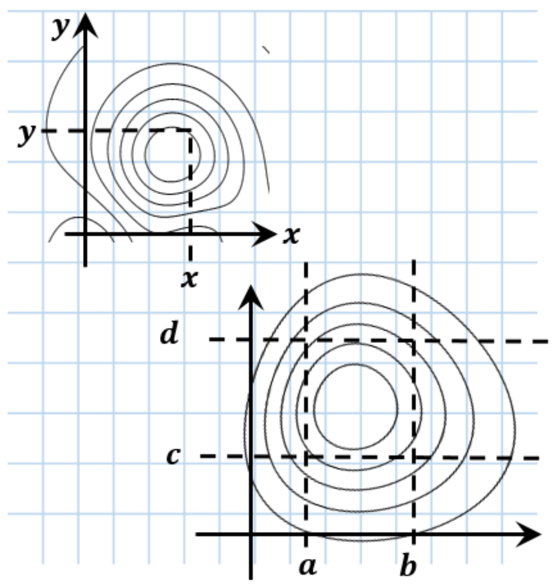
(a) Joint cumulative distribution function (CDF)

$F_{XY}(x,y) \equiv P(X \le x, Y \le y)$

$= \int$

$f_{XY}(x,y) =$

(b) $P(a < X \le b, c < Y \le d) =$



(c) Conditional PDF

$f_{X|Y}(x|y)$

$= \lim_{\Delta x \to 0} \dfrac{P(x < X \le x + \Delta x \qquad\qquad)}{\Delta x}$

Can show

$=$

※ Multiplication rule $f_{XY}(x,y) =$

(s.i $f_{XY}(x,y) =$)

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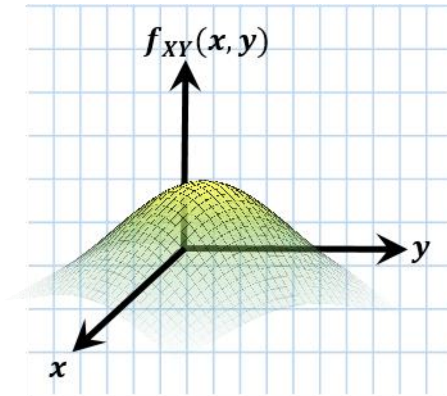
(d) Joint PDF→marginal PDF?



$$f_X(x) = \int$$

$$= \int$$