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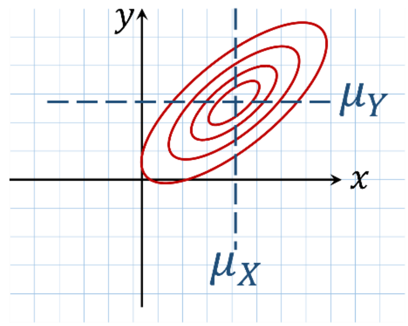
457.646 Topics in Structural Reliability
In-Class Material: Class 05

※ **See supplementary material on bivariate normal joint PDF**

◎ Covariance & Correlation Coefficient

– Partial descriptors or measures for __________ dependence

① Covariance



(a) Definition:

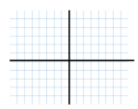
$Cov[X,Y] \equiv \mathrm{E}[\qquad\qquad\qquad\qquad]$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \qquad\qquad f_{XY}(x,y)dydx$$

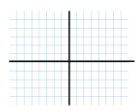
c.f. c.o.v. $\delta =$

(b) $Cov[X,Y] = \qquad\qquad -$

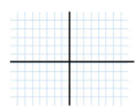
(c) $Cov[X,Y] \;\; > 0$ _______ linear dependence



$= 0$ _______ linear dependence



< 0 _______ linear dependence



⇒ **Not useful to measure/compare the strength of the linear dependence. Why?**

② Correlation Coefficient

(a) Dimensionless measure of linear dependence

$\rho_{XY} \;\; \equiv \underline{\qquad\qquad\qquad}$

(b) $\qquad \le \rho_{XY} \le$

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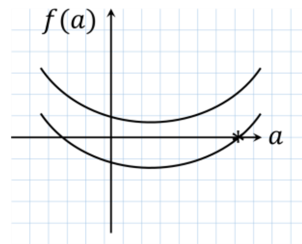
Proof: Consider



$$f(a) = \iint [a(x-\mu_X) - (y-\mu_Y)]^2 f_{XY}(x,y)dxdy$$

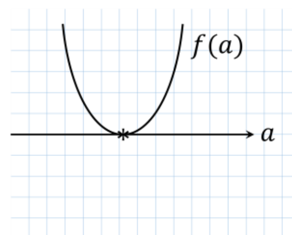
$$= a^2 Var[X] - 2a \cdot Cov[X,Y] + Var[Y] \qquad 0$$

$$\therefore D/4 = (Cov[X,Y])^2 - Var[X] \cdot Var[Y] \qquad 0$$

$$\therefore \frac{[Cov(X,Y)]^2}{Var[X] \cdot Var[Y]} \leq$$

$$\leq \rho_{XY} \leq$$

(c) What does $\rho_{XY} =$ & $\rho_{XY} =$ mean?



Consider the case D=

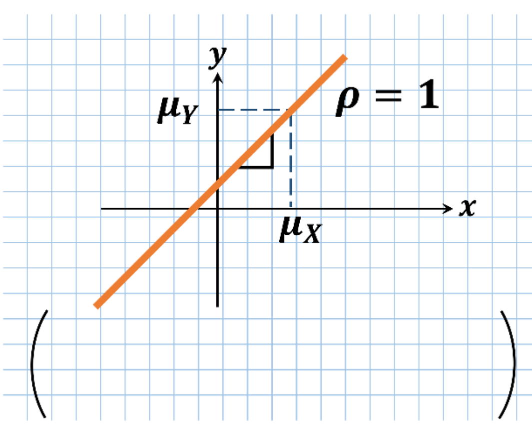
$$f(a) = Var[X]\left(a - \frac{Cov[X,Y]}{Var[X]}\right)^2 + \ldots$$

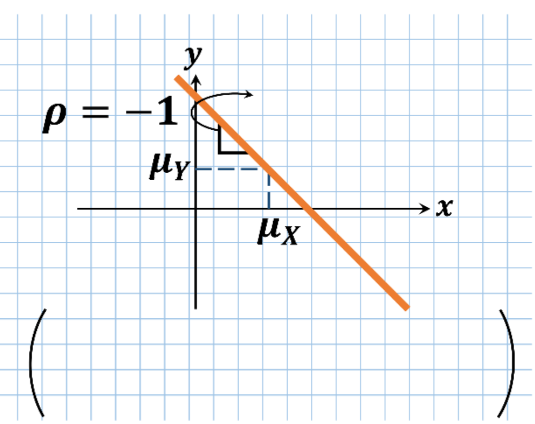
$$f(a) = 0 \text{ at } a = \frac{Cov[X,Y]}{Var[x]} = a^*$$

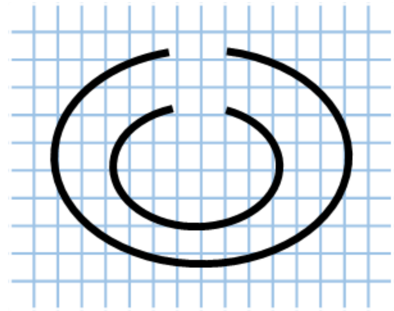
Substituting this into $f(a)$,

$$f(a^*) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} [\qquad (x-\mu_X) - (y-\mu_Y)]^2 f_{XY}(x,y)dxdy = 0$$

$\therefore$ for $\forall(x,y)$, the following (deterministic/probabilistic) and (linear/nonlinear) relationship between X and Y holds:







(d) $\rho_{XY} = 0 \quad \Leftrightarrow \quad Cov[X,Y] = 0$

"No linear dependence"
"Un   "

(e) "Uncorrelated" vs "Statistical Independence"

$\rho_{XY} = 0 \qquad \rightarrow$
$(E[XY] = \qquad) \leftarrow \qquad f_{XY}(x,y) =$

$\rightarrow ?$

Suppose $Y = X^2$ and X has a symmetric distribution in $[-a, a]$

$E[XY] =$
$E[X] =$
$Cov[X,Y] =$

$\leftarrow ?$

※ Vector/matrix formulation for multiple RVs

$$\mathbf{X} = \begin{Bmatrix} X_1 \\ \vdots \\ X_n \end{Bmatrix} \qquad \boldsymbol{\mu}_{\mathbf{X}} = \begin{Bmatrix} \mu_{X_n} \\ \vdots \\ \mu_{X_n} \end{Bmatrix} \qquad \boldsymbol{\Sigma}_{\mathbf{XX}} = \begin{bmatrix} \sigma_1^2 & & & \\ & \sigma_2^2 & & \\ & & \ddots & \vdots \\ sym & & \cdots & \sigma_n^2 \end{bmatrix}$$

(   ) vector   (   ) vector= E[**X**]   (   ) matrix

$$\boldsymbol{\Sigma}_{\mathbf{XX}} = \mathrm{E}[(\mathbf{X} - \mathbf{M}_{\mathbf{X}})(\mathbf{X} - \mathbf{M}_{\mathbf{X}})^{\mathrm{T}}] = \mathrm{E}[\mathbf{X}\mathbf{X}^{\mathrm{T}}] - \mathbf{M}_{\mathbf{X}}\mathbf{M}_{\mathbf{X}}^{\mathrm{T}}$$
$$= \mathbf{DRD}$$

where

$$\mathbf{D} = \begin{bmatrix} & & \\ & \ddots & \\ & & \end{bmatrix} = [\quad] \text{ diagonal matrix of } \_\_\_\_\_\_\_\_\_\_\_\_$$

$$\mathbf{R} = \begin{bmatrix} 1 & \rho_{12} & \rho_{13} & \\ & 1 & & \\ & & \ddots & \\ sym & & & 1 \end{bmatrix} = [\quad] \; \_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_ \text{ matrix}$$

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※ $\mathbf{\Sigma}_{\text{XX}}$ and $\mathbf{R}_{\text{XX}}$ are _______ and _______ ________

- $\mathbf{a}^{\text{T}}\mathbf{\Sigma}_{\text{XX}}\mathbf{a} > 0$ $(\forall \mathbf{a} \neq \mathbf{0})$ If no perfect linear dependence
  (a simple proof: $Y = \mathbf{a}^{\text{T}}\mathbf{X}$, $\sigma_y^2 = \mathbf{a}^{\text{T}}\mathbf{\Sigma}_{\text{XX}}\mathbf{a} > 0$)

- $\mathbf{a}^{\text{T}}\mathbf{\Sigma}_{\text{XX}}\mathbf{a} = \mathbf{0}$ for $\exists \mathbf{a}$ if there exist linear dependence among **X**

  e.g. $X_1 = 2X_2$, $Y = 1 \cdot X_1 - 2X_2 = \begin{bmatrix} 1 & -2 \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} = 0$

  $\sigma_Y^2 = \mathbf{a}^{\text{T}}\mathbf{\Sigma}_{\text{XX}}\mathbf{a} = 0$

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II-6. Functions of Random Variables (See Supp. 03)

Consider $\mathbf{Y} = \mathbf{g}(\mathbf{X})$

(1) For input $\mathbf{X}$: distribution model $f_{\mathbf{X}}(\mathbf{x})$ or expectations ($\mathbf{M}_{\mathbf{X}}$, $\boldsymbol{\Sigma}_{\mathbf{XX}}$) available

(2) For output $\mathbf{Y}$: distribution model (   ) or expectations (,   ) ?

<u>Examples:</u>

(1) Regional/inventory loss: $L = \sum_{i=1}^{n} V_i D_i$ → linear function

(2) Wind-induced pressure: $P = \frac{1}{2} C_\rho \rho V^2$

◎ Mathematical expectation of linear functions

$$Y_k = a_{k,0} + \sum_{i=1}^{n} a_{k,i} X_i, \quad k = 1, \ldots, m$$

① Algebraic formula $(n \leq 3)$: See Supp.3

② Matrix formula:

For $\mathbf{Y} = \mathbf{A}_0 + \mathbf{A}\mathbf{X}$

where

$$\mathbf{Y} = \begin{Bmatrix} Y_1 \\ Y_2 \\ \vdots \\ Y_m \end{Bmatrix}, \quad \mathbf{A}_0 = \begin{Bmatrix} a_{1,0} \\ a_{2,0} \\ \vdots \\ a_{m,0} \end{Bmatrix}, \quad \mathbf{A} = \begin{bmatrix} a_{1,1} & a_{1,2} & \cdots & a_{1,n} \\ a_{2,1} & a_{2,2} & \cdots & a_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m,1} & a_{m,2} & \cdots & a_{m,n} \end{bmatrix} \text{ and } \mathbf{X} = \begin{Bmatrix} X_1 \\ X_2 \\ \vdots \\ X_n \end{Bmatrix}$$

$\mathbf{M}_{\mathbf{Y}} =$

$\boldsymbol{\Sigma}_{\mathbf{YY}} =$

❖ Proof of Positive-definiteness of $\boldsymbol{\Sigma}_{\mathbf{XX}}$

Consider $Y = \mathbf{a}^{\mathrm{T}}\mathbf{X}$ ($\mathbf{A}_0 =$, $\mathbf{A} =$)

Using the formula above,

$\boldsymbol{\Sigma}_{\mathbf{YY}} = \sigma_Y^2 =$

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❖ **Linear transformation for standardization, i.e., &**

Suppose **X** has and

Find $\mathbf{Y} = \mathbf{A}_0 + \mathbf{A}\mathbf{X}$

such that $\mathbf{M_Y} =$ and $\mathbf{\Sigma_{YY}} =$

$\mathbf{M_Y} = \mathbf{A_0} + \mathbf{A}\mathbf{M_X} =$ (1)

$\mathbf{\Sigma_{YY}} = \mathbf{A}\mathbf{\Sigma_{XX}}\mathbf{A}^T =$ (2)

Since $\mathbf{\Sigma_{XX}}$ is positive semi-definite, $\mathbf{\Sigma_{XX}} = \mathbf{L_\Sigma}\mathbf{L}_\Sigma^T$ (e.g. by ____________ decomposition)

Therefore, $= \mathbf{I}$ and

$\mathbf{A} =$ → Substitute to ( )

$\mathbf{A_0} =$

In summary,

$\mathbf{Y} =$
$=$

Alternatively,

$$\mathbf{\Sigma}_{XX} = \mathbf{D}_X \mathbf{R}_{XX} \mathbf{D}_X$$
$$=$$
$$= \mathbf{L}_\Sigma \mathbf{L}_\Sigma^T$$

Therefore, $\mathbf{L_\Sigma} =$ and $\mathbf{L}_\Sigma^{-1} =$

$\mathbf{Y} =$

→ This version is preferred because of numerical stability in decomposition ($|\rho| \le 1$).

◎ **Mathematical expectation of nonlinear functions**

$Y_k = g_k(x),\ k = 1, \cdots, m$

Taylor series expansion around the mean point, $\mathbf{x} = \mathbf{M_X}$

$$Y_k \cong g_k(\boldsymbol{M_X}) + \left.\frac{\partial g_k}{\partial \mathbf{x}}\right|_{\mathbf{x}=\mathbf{M_X}} (\mathbf{x} - \mathbf{M_X}) + \cdots$$

Matrix form

$$\mathbf{Y} \cong \mathbf{g}(\mathbf{M}_X) + \mathbf{J}_{Y,X}\big|_{\mathbf{x}=\mathbf{M}_X} (\mathbf{X} - \mathbf{M}_X)$$

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① First-order approximation

(Scalar: See supp.)

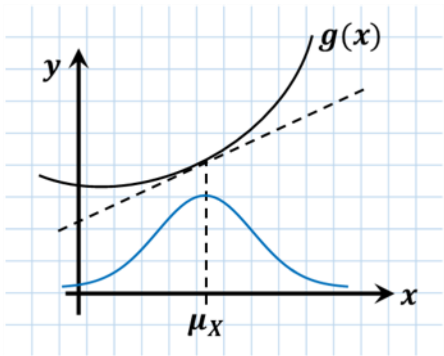
$$\mathbf{M}_{\mathbf{Y}}^{FO} = \mathbf{g}(\qquad)$$
$$\boldsymbol{\Sigma}_{\mathbf{YY}}^{FO} =$$

② Second-order approximation

⇒ Can use 2nd order approximation from Taylor series expansion

⇒ Not useful because higher-order moments are needed $(\gamma, \kappa, \cdots)$

③ Accuracy of FO/SO approximation



Sources of large errors in approx.

- σ_X

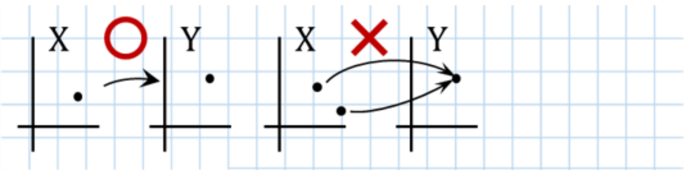
- Nonlinearity in $g(x)$

Example : $\mathbf{U} = \mathbf{K}^{-1}\mathbf{P}$ (Frame structure)

◎ Derived Distribution of Functions

Consider $\mathbf{Y} = \mathbf{g}(\mathbf{X})$ where $\mathbf{Y} = \{Y_1, \cdots, Y_m\}$ and $\mathbf{X} = \{X_1, \cdots, X_n\}$

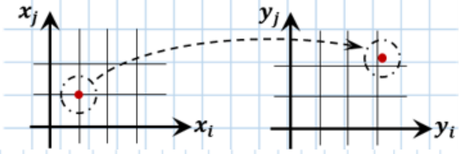
Given: $f_{\mathbf{X}}(\mathbf{x})$ → $f_{\mathbf{Y}}(\mathbf{y})$?



① $m = n$, one-to-one mapping

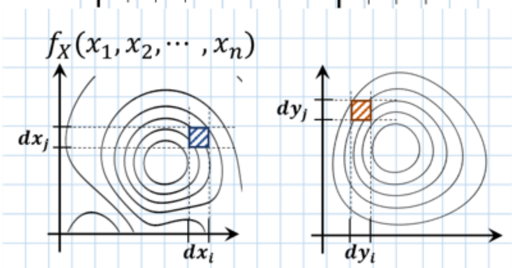
a) Discrete

$P_{\mathbf{Y}}(y_1, \cdots, y_n) \qquad P_{\mathbf{X}}(x_1, \cdots, x_n)$



b) Continuous

$f_{\mathbf{Y}}(y_1, \cdots, y_n) \qquad f_{\mathbf{X}}(x_1, \ldots, x_n)$



$$f_{\mathbf{Y}}(\mathbf{y}) = f_{\mathbf{X}}(\mathbf{x}) \cdot |\det \qquad |$$
$$= f_{\mathbf{X}}(\mathbf{x}) \cdot |\det \qquad |^{-1}$$

"Jacobian" $\mathbf{J}_{\mathbf{y},\mathbf{x}} = \begin{bmatrix} \frac{\partial y_1}{\partial x_1} & \frac{\partial y_1}{\partial x_2} & \cdots & \frac{\partial y_1}{\partial x_n} \\ \vdots & & & \vdots \\ \vdots & & & \vdots \\ \frac{\partial y_n}{\partial x_1} & \cdots & \cdots & \frac{\partial y_n}{\partial x_n} \end{bmatrix}$

Consider $\mathbf{y} = \mathbf{g}(\mathbf{x}),\ \mathbf{x} = \mathbf{h}(\mathbf{y})$

※ $f_{\mathbf{Y}}(\mathbf{y}) = f_{\mathbf{X}}(\mathbf{h}(\mathbf{y}))\left|\det \mathbf{J}_{\mathbf{y},\mathbf{x}}(\mathbf{h}(\mathbf{y}))\right|^{-1}$

※ $m = n = 1$

$f_Y(y) = f_X(x \qquad\qquad = f_X(\quad)\left|\frac{dh(y)}{dy}\right|$

Example: $X \sim N(0, 1^2)$

a) $Y = g(X) = aX + b$

One-to-one mapping?

$f_Y(y) = f_X(x) \cdot$

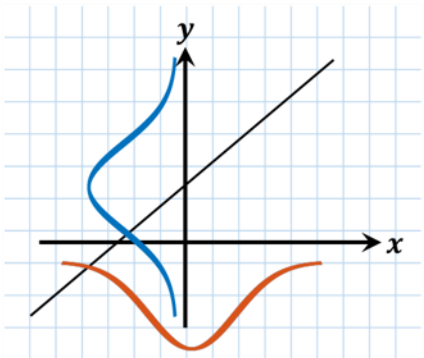
$=$

$=$

__________ Distribution

$\mu_Y =$

$\sigma_Y =$



b) T_1, T_2 ~ exponential r.v.'s (See supplement on "Other Distribution Models")

$f_{T_1}(t_1) = \alpha \cdot \exp(-\alpha t_1),\ t_1 > 0$

$f_{T_2}(t_2) = \beta \cdot \exp(-\beta t_2),\ t_2 > 0$

T_1, T_2 : statistically independent

Joint PDF of $\begin{cases} Y_1 = T_1 + T_2 \\ Y_2 = T_1 - T_2 \end{cases}$?

$f_{\mathbf{Y}}(\mathbf{y}) = f_{\mathbf{T}}(\mathbf{t})\left|\det \mathbf{J}_{y,t}\right|^{-1}$

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$$\mathbf{J}_{\mathbf{y},\mathbf{t}} = \begin{bmatrix} \frac{\partial y_1}{\partial t_1} & \frac{\partial y_1}{\partial t_2} \\ \frac{\partial y_2}{\partial t_1} & \frac{\partial y_2}{\partial t_2} \end{bmatrix} = \begin{bmatrix} \quad & \quad \\ \quad & \quad \end{bmatrix}$$

$$\left|\det \mathbf{J}_{\mathbf{y},\mathbf{t}}\right|^{-1} =$$

$$\therefore \quad f_{\mathbf{Y}}(\mathbf{y}) =$$

Inverse relationship

$$\begin{cases} T_1 = \frac{1}{2}(Y_1 + Y_2) \\ T_2 = \frac{1}{2}(Y_1 - Y_2) \end{cases}$$

$$\therefore \quad f_{\mathbf{Y}}(\mathbf{y}) = \frac{\alpha\beta}{2}\exp[-\frac{\alpha+\beta}{2}y_1 - \frac{\alpha-\beta}{2}y_2], \quad y_1 > 0, -y_1 < y_2 < y_1$$

- Range of **Y** derived from the condition $t_1, t_2 > 0 \;\; \& \;\; \mathbf{t} = \mathbf{h}(\mathbf{y})$