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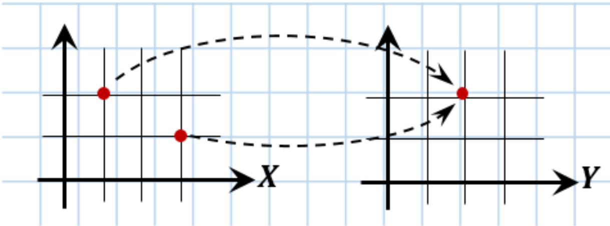
457.646 Topics in Structural Reliability
In-Class Material: Class 07

II-6. Functions of Random Variables (contd.)

◎ Derived Distribution of Functions (contd.)



② $m = n$, **but NOT one-to-one mapping**

a) Discrete

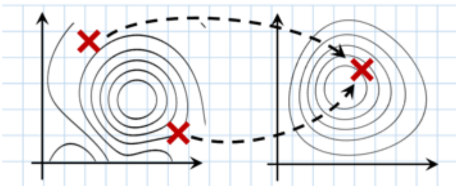
$$P_{\mathbf{Y}}(y_1, \cdots, y_n) =$$

$$P_{\mathbf{X}}(x_1, \cdots, x_n)$$

roots of $\mathbf{y} = \mathbf{g}(\mathbf{x})$

b) Continuous

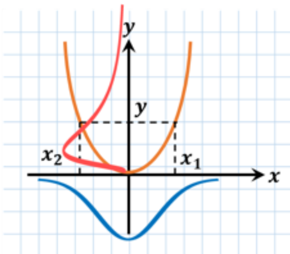
$$f_Y(y) = \sum_{\substack{\text{all roots of} \\ \mathbf{y}=\mathbf{g}(\mathbf{x})}}$$



Example c)

$$Y = g(X) = X^2, \qquad X \sim N(0, 1^2)$$

$$\begin{cases} x_1 = h_1(y) = \\ x_2 = h_2(y) = \end{cases}$$



$$f_Y(y) = f_X(x_1)\left|\frac{dx_1}{dy}\right| + f_X(x_2)\left|\frac{dx_2}{dy}\right|$$

$$= \frac{1}{\sqrt{2\pi}}\exp\left(-\frac{1}{2}x_1^2\right) \qquad + \frac{1}{\sqrt{2\pi}}\exp\left(-\frac{1}{2}x_2^2\right) \qquad =$$

③ $m < n$, **one-to one mapping**

$$\mathbf{Y}' \begin{cases} \mathbf{Y} \begin{cases} Y_1 = g_1(X_1, \cdots, X_n) \\ \vdots \quad\quad \vdots \\ Y_m = g_m(X_1, \cdots, X_n) \end{cases} \\ Y_{m+1} = \\ \vdots \\ Y_n = \end{cases}$$

$$\mathbf{Y}' = \mathbf{g}'(\mathbf{X})$$

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Discrete

$$P_{\mathbf{Y}'}(\mathbf{y}') = P_{\mathbf{X}}(\mathbf{x})$$

Then,

$$P_{\mathbf{Y}}(\mathbf{y}) = \sum \cdots \sum P_{\mathbf{X}}(\mathbf{x})$$

a) Continuous

$$f_{\mathbf{Y}'}(\mathbf{y}')dy_1 \cdots dy_m \qquad = f_{\mathbf{X}}(\mathbf{x})dx_1 \cdots dx_m dx_{m+1} \cdots dx_n$$

$$f_{\mathbf{Y}'}(\mathbf{y}') = f_{\mathbf{X}}(\mathbf{x}) \left| \det J_{\mathbf{Y}',\mathbf{X}} \right|^{-1}$$

$$= f_{\mathbf{X}}(\mathbf{x}) \left| \det J_{\mathbf{Y},\mathbf{X}} \right|^{-1}_{mxm}$$

$$J_{Y',X} = \begin{bmatrix} \frac{\partial y_1}{\partial x_1} & \frac{\partial y_1}{\partial x_2} & \cdots & \frac{\partial y_1}{\partial x_m} \\ \frac{\partial y_m}{\partial x_1} & & \cdots & \frac{\partial y_m}{\partial x_m} \end{bmatrix}$$

$$\therefore \quad f_{\mathbf{Y}}(\mathbf{y}) = \int_{x_{m+1}} \cdots \int_{x_n} f_{\mathbf{X}}(\mathbf{x}) \left| \det J_{\mathbf{Y},\mathbf{X}} \right|^{-1}_{m\times m} dx_{m+1} \cdots dx_n$$

Example d)

$Y = T_1 + T_2$ **←contd. From Example b)**

$f_Y(y)?$

$$\mathbf{Y}' \begin{cases} Y_1 = T_1 + T_2 \\ Y_2 = \quad T_2 \end{cases}$$

$$f_{\mathbf{Y}'}(\mathbf{y}') = f_{\mathbf{T}}(\mathbf{t}) \left| \det J_{\mathbf{Y}',\mathbf{T}} \right|^{-1} \qquad \left| \det J_{\mathbf{Y},\mathbf{T}} \right|^{-1}_{1\times 1} =$$

$$= f_{\mathbf{T}}(\mathbf{t}) \left| \det J_{\mathbf{Y}',\mathbf{T}} \right|^{-1}_{1\times 1}$$

$$=$$

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$$f_{\mathbf{Y}}(\mathbf{y}) = f_{Y_1}(y_1) = \int \qquad\qquad dt_2$$

$$= \int f_{T_1}(\qquad) f_{T_2}(\qquad) dt_2$$

$$= \frac{\alpha\beta}{\alpha - \beta}[\exp(-\beta y) - \exp(-\alpha y)],\ y > 0$$

When $\alpha = \beta$, using l'Hopitals rule,

$$\lim_{\beta \to \alpha} f_Y(y) = \lim_{\beta \to \alpha} \frac{\dfrac{\partial(\ )}{\partial \beta}}{\dfrac{\partial(\ )}{\partial \beta}} = \alpha^2 y \exp(-\alpha y),\ y > 0$$

④ $m < n$, **NOT one-to one mapping**

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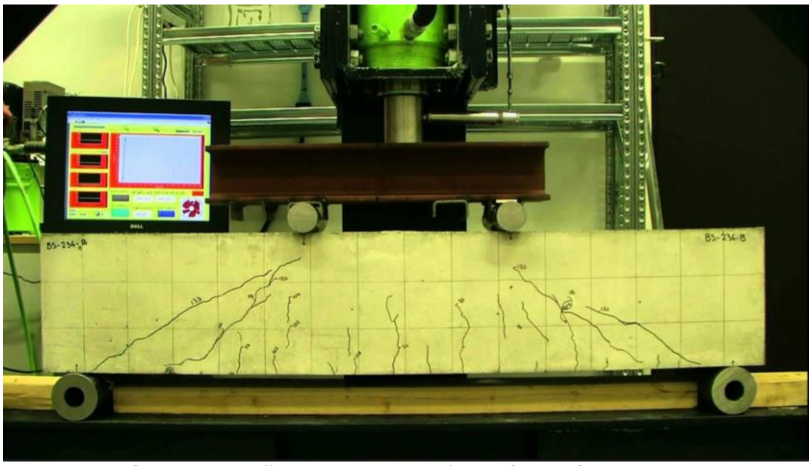
Ⅲ. Structural Reliability (Component)

◎ Structural Reliability Analysis (contd.)

e.g. Shear failure of RC beam w/o stirrups



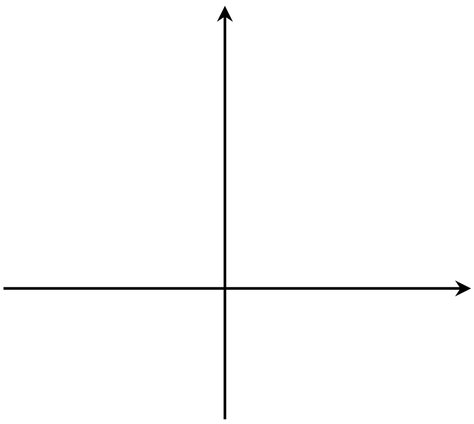
Source: https://www.youtube.com/watch?v=DPQIpT1ZvXY

"Limit-state" function

$$g(\mathbf{X}) = V_c - V_d$$

$$= \frac{1}{6}\sqrt{f_c^{'}}b_w d + \varepsilon - V_d \le 0$$

where $X = \{f_c^{'}, b_w, d, \varepsilon, V_d, \cdots\}$ random variables



Failure Probability

$P_f = P(g(\mathbf{x}) \quad\quad)$

$=$

"Structural Reliability Analysis"

(Anatomical + Systematic)

Three important tasks for structural reliability analysis:

1)

2)

3)

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◎ Joint Probability Distribution Models

① Joint Normal $\mathbf{X} \sim N(\mathbf{M}_\mathbf{X}, \mathbf{\Sigma}_\mathbf{XX})$

a) Joint PDF

$$f_\mathbf{X}(\mathbf{x}) = \frac{1}{(2\pi)^{n/2}\sqrt{\det \mathbf{\Sigma}}} \exp\left[-\frac{1}{2}(\mathbf{x}-\mathbf{M}_\mathbf{X})^\mathrm{T} \mathbf{\Sigma}_\mathbf{XX}^{-1} (\mathbf{x}-\mathbf{M}_\mathbf{X})\right]$$

$n=1 \quad f_X(x) = \frac{1}{\sqrt{2\pi}\sigma} \exp\left[-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2\right]$ Uni-variate normal PDF (See supp.)

$n=2 \quad f_{X_1X_2}(x_1, x_2) = f(\qquad\qquad\qquad)$ Bi-variate normal PDF (See supp.)

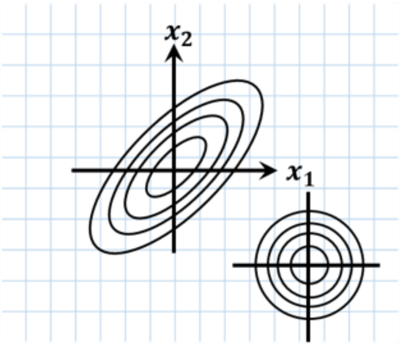
b) Properties

- Joint distribution completely defined by
- All lower order distribution are
- $\mathbf{X} = \left\{ \quad \right\} \quad \mathbf{M}_\mathbf{X} = \left\{ \quad \right\} \quad \sum_\mathbf{XX} = \left\{ \qquad \right\}$

Given $\mathbf{X}_2 = \mathbf{x}_2$, then $\mathbf{X}_1 \sim N(\mathbf{M}_{1|2}, \mathbf{\Sigma}_{1,1|2})$

Conditional mean and covariance

$$\begin{cases} \mathbf{M}_{1|2} = \mathbf{M}_1 + \mathbf{\Sigma}_{1,2}\mathbf{\Sigma}_{2,2}^{-1}(\mathbf{x}_2 - \mathbf{M}_2) \\ \mathbf{\Sigma}_{1,1|2} = \mathbf{\Sigma}_{1,1} - \mathbf{\Sigma}_{1,2}\mathbf{\Sigma}_{2,2}^{-1}\mathbf{\Sigma}_{2,1} \end{cases}$$



e.g. $n=2$, i.e. $\mathbf{X} = \begin{Bmatrix} \mathbf{X}_1 \\ \mathbf{X}_2 \end{Bmatrix} = \begin{Bmatrix} X_1 \\ X_2 \end{Bmatrix}$

$X_1 \sim N(\mu_{1\|2}, \sigma_{1\|2}^2)$	if $\rho = 0$ ("　　　　")
$\mu_{1\|2} = \mu_1 + \rho\sigma_1\left(\frac{x_2-\mu_2}{\sigma_2}\right)$	$\mu_{1\|2} =$
$\sigma_{1\|2}^2 = \sigma_1^2(1-\rho^2)$	$\sigma_{1\|2}^2 =$

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- Uncorrelated (   ) s.i for jointly normal
  (in general, $\rho = 0$ □ $s.i$)

- Linear functions of $\mathbf{X} \sim N(\mathbf{M}, \mathbf{\Sigma})$ → follow ______________

  $\mathbf{Y} = \mathbf{AX} + \mathbf{A_0}$

  $f_{\mathbf{Y}}(\mathbf{y}) = f_{\mathbf{X}}(\mathbf{x}) \cdot$ $J_{\mathbf{Y,X}} =$ $\therefore \det =$

  $$f_{\mathbf{Y}}(\mathbf{y}) \propto \exp\left[-\frac{1}{2}(\mathbf{x}-\mathbf{M}_{\mathbf{x}})^{\mathrm{T}}\mathbf{\Sigma}_{\mathbf{xx}}^{-1}(\mathbf{x}-\mathbf{M}_{\mathbf{x}})\right]$$

  In summary, $\mathbf{X} \sim N(\mathbf{M}_{\mathbf{X}}, \mathbf{\Sigma}_{\mathbf{XX}})$

  $\Rightarrow \mathbf{Y} \sim N(\mathbf{M}_{\mathbf{Y}}, \mathbf{\Sigma}_{\mathbf{YY}})$

  $\mathbf{M}_{\mathbf{Y}} =$

  $\mathbf{\Sigma}_{\mathbf{YY}} =$

c) Standard Normal

For univariate, 'standard normal' means, $\mu =$ $, \sigma =$

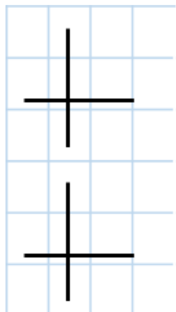
$\therefore$ For jointly normal,

$\mathbf{M}_{\mathbf{X}} =$

$\mathbf{\Sigma}_{\mathbf{XX}} =$

$$\mathbf{Z} \sim N(\mathbf{0},\quad) \qquad \varphi_n(\mathbf{z},\quad) = \frac{1}{(2\pi)^{n/2}\sqrt{\det\qquad}} \cdot \exp\left[-\frac{1}{2}\mathbf{z}^{\mathrm{T}}\mathbf{R}_{\mathbf{XX}}\mathbf{z}\right]$$

$$\mathbf{U} \sim N(\mathbf{0},\quad) \qquad \varphi_n(\mathbf{u},\quad) = \frac{1}{(2\pi)^{n/2}\sqrt{\det\qquad}} \cdot \exp\left[-\frac{1}{2}\qquad\right]$$



$$= \prod_{i=1}^{n}$$

U used for FORM/SORM

For normal,

$$\begin{cases} \mathbf{x} = \mathbf{DLu} + \mathbf{M} \\ \mathbf{u} = \mathbf{L}^{-1}\mathbf{D}^{-1}(\mathbf{x} - \mathbf{M}) \end{cases}$$

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III. Structural Reliability (Component)

◎ Joint Probability Distribution Models

② Joint Lognormal

$X_1, \cdots, X_n$ are jointly lognormal if $\ln X_1, \cdots, \ln X_n$ are jointly __________

a) Parameters

$\lambda_i = E[\qquad] = \ln \mu_i - 0.5 \ln(1+\delta_i^2)$

$\xi_i^2 = Var[\qquad] = \ln(1+\delta_i^2)$ $(\cong \delta_i^2$ for $\delta \quad 1)$

$\rho_{\ln X_i, \ln X_j} = \dfrac{1}{\xi_i \xi_j} \ln(1+\rho_{ij}\delta_i\delta_j)$

b) Properties

- Completely defined in terms of (   ) & (   )
- All lower order distribution are jointly
- Conditional distribution are jointly
- Uncorrelated ⇄ S.I.
- Product / Quotient of jointly lognormal r.v.'s follows
- $\rho_{X_i, \ln X_j} = \dfrac{1}{\xi_i} \delta_j \rho_{ij}$

③ General Joint Distribution Forms

e.g. Johnson & Kotz (1976)

⇒ on multivariate prob. distribution models

④ Joint Distribution by conditioning (e.g. Bayesian Networks)

$f(x_1, \cdots, x_n) = f(x_n | x_1, \cdots, x_{n-1}) \times$

⑤ Joint Distribution model with : Prescribed marginals: $\qquad , i = 1, \cdots, n$ **and**

correlation coefficient matrix :

- Read CRC Ch.14
- **See Liu & Der Kiureghian (1986)**   a) Morgenstern

  b) Nataf

※ "Copula": a class of functions satisfying certain conditions that can be used to construct joint probability functions using marginal probability functions

$$F(x_1, \dots, x_n) = C(F_1(x_1), \dots, F_n(x_n))$$

Nataf transformation/model employs "normal copula" to describe the dependence structure

See Lebrun and Dutfoy (2009a,b,c) for details about copulas

a) Morgenstern distribution

$$F_{\mathbf{X}}(\mathbf{x}) = \prod_{i=1}^{n} F_{X_i}(x_i) \cdot \left\{1 + \sum_{i<j} \alpha_{ij}[1 - F_{X_i}(x_i)][1 - F_{X_j}(x_j)]\right\}$$

Q) Can we derive $F_{X_i}(x_i)$ from $F_{\mathbf{X}}(\mathbf{x})$?

i.e. $x_2, x_3, \cdots, x_n \rightarrow$ then $F_{\mathbf{X}}(\mathbf{x}) =$?

Q) Can we describe dependence using α_{ij} ?

$F_{X_iX_j}(x_i, x_j) =$

$f_{X_iX_j}(x_i, x_j) = \overline{\qquad\qquad}$

$$= f_{X_i}(x_i) \cdot f_{X_j}(x_j) \cdot \left\{1 + \alpha_{ij}[1 - 2F_{X_i}(x_i)][1 - 2F_{X_j}(x_j)]\right\}$$

$\Rightarrow \qquad \leq \alpha_{ij} \leq$

$$\begin{cases} \alpha_{ij} = 0 \\ \alpha_{ij} \neq 0 \end{cases}$$

Therefore, α_{ij} is a parameter that represents (corr coeff.)

But α_{ij} ρ_{ij}

Liu & Der Kiureghian (1986) showed

$$\rho_{ij} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left(\frac{x_i-\mu_i}{\sigma_i}\right)\left(\frac{x_j-\mu_j}{\sigma_j}\right) f_{X_iX_j}(x_i,x_j)dx_idx_j$$

$$= 4\alpha_{ij}Q_iQ_j \qquad \Rightarrow \qquad |\rho_{ij}| \le 0.30$$

Where $Q_i = \int_{-\infty}^{\infty}\left[\left(\frac{x_i-\mu_i}{\sigma_i}\right)F_{X_i}(x_i)\right]f_{X_i}(x_i)dx_i \approx 0.28$

Table 1: Distributions considered

Table 2: Q_i

Table 3 : maximum $|\rho_{ij}|$

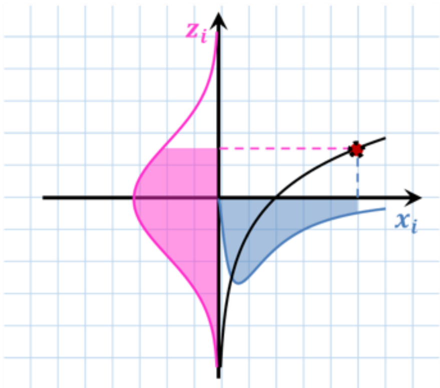
$\Rightarrow$ In summary, using Morgenstern's model, you cannot describe X_i, X_j whose $|\rho_{ij}| > 0.30$

b) Nataf model (Nataf, 1962) ("Gaussian Copula")

$\mathbf{X} \sim F_{X_i}(x_i),\ i=1,\cdots,n$ $\mathbf{R} = [\rho_{ij}]$	$\xrightarrow{(1)}$ $\xleftarrow[\text{Nataf}]{(2)}$	$\mathbf{Z} \sim N(\mathbf{0},\mathbf{R}')$ $\mathbf{R}' = [\rho'_{ij}]$

Transformation to $\mathbf{Z}$

$Z_i =$



Why?

$$f_{Z_i}(z_i) = f_{X_i}(x_i)\cdot\left|\frac{dx_i}{dz_i}\right|$$

$$f_{Z_i}(z_i)\cdot \qquad = f_{X_i}(x_i)\cdot$$

$$\Phi(\quad) = F_{X_i}(\quad)$$