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457.646 Topics in Structural Reliability
In-Class Material: Class 09

III. Structural Reliability (Component) - continued

◎ Joint Probability Distribution Models

⑤ Joint distribution models with marginal & corr. coeff (contd.)

a) Morgenstern: $F_{X_i}(x_i),\ i=1,...,n\ \&\ \alpha_{ij}$ but $|\rho_{ij}|<0.30$

b) Nataf model (Nataf, 1962)

★ Joint PDF by Nataf model

$$f_{\mathbf{X}}(\mathbf{x}) = f_{\mathbf{Z}}(\mathbf{z})\cdot|\det J_{\mathbf{Z},\mathbf{X}}|$$

$$= \varphi_n(\mathbf{z};\mathbf{R}')\text{―――――}$$

$$J_{\mathbf{Z},\mathbf{X}} = \left[\qquad\qquad\right]$$

$$= \left[\prod_{i=1}^{n} f_{X_i}(x_i)\right]\cdot\text{―――――}$$

Note:

$$F_{X_i}(x_i) = \Phi(z_i)$$
$$f_{X_i}(x_i)dx_i = \varphi(z_i)dz_i$$

★ ρ'_{ij} (corr. coeff. b/w Z_i and Z_j)?

$$\rho_{ij} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left(\text{―――}\right)\left(\text{―――}\right) f_{X_iX_j}(x_i,x_j)dx_idx_j$$

$$\therefore \rho_{ij} = \int\int\left(\text{―――}\right)\left(\text{―――}\right)\varphi_2(z_i,z_j;\rho'_{ij})\text{――――― ――― ―――}dz_idz_j$$

In general, $|\rho'_{ij}|$ $\quad$ $|\rho_{ij}|$

$\therefore |\rho_{ij}| \le A < 1$ may not cover the whole range of ρ_{ij}

$\rho'_{ij} \cong F\ \cdot \rho_{ij}$ Liu & ADK (Table 4~6) for pairs of selected distribution types

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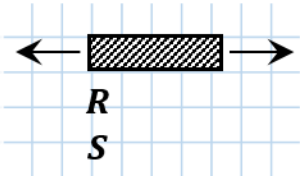
Table 9: Range of ρ_{ij} ~ wider (than Morgenstern)

Later used for transformation of dependent RVs into $\mathbf{U} \sim N(\mathbf{0}, \mathbf{I})$

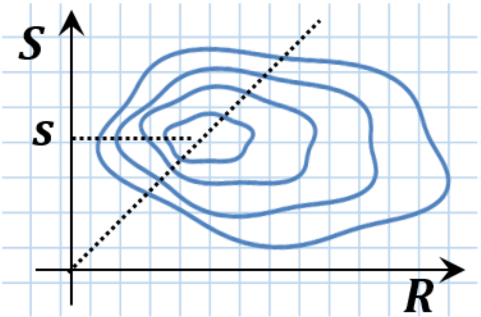
$\mathbf{X}$ $\mathbf{Z}$ $\mathbf{U}$

◎ Elementary Structural Reliability Problem

Describe the failure event in terms of _________ & __________



① **Failure :** $g(\mathbf{x}) = g(\quad,\quad) = \qquad \leq 0$



② **Failure probability :** $P_f = P(\qquad \leq 0)$

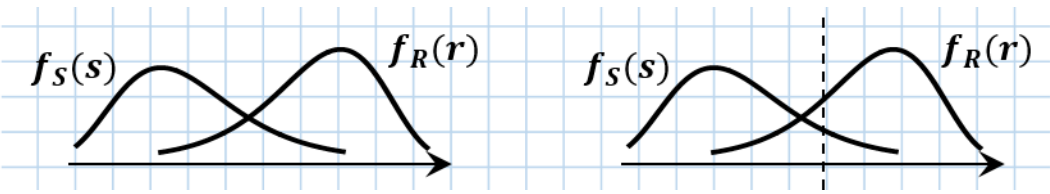
$$P_f = \iint f_{R,S}(r,s)drds$$

$$= \iint f_{R|S}(r|s) \cdot f_s(s)drds$$

$$= \iint f_{R|S}(r|s)\ dr\ f_s(s)ds$$

$$= \int \qquad f_s(s)ds$$

If R&S are s.i $P_f = \int \qquad ds$



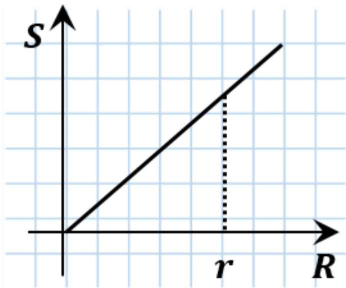
OR

$$P_f = \iint_{r \le s} f_{S|R}(s|r)\, f_R(r)\, ds\, dr$$

$$= \iint f_{S|R}(s|r)\, ds f_R(r)\, dr$$

$$= \int [\qquad\qquad] \; f_R(r)\, dr$$

$$\text{if s.i} = \int \qquad\qquad\qquad dr$$



③ **Reliability Index by "Safety Margin,"** β_{SM}

$$\boxed{\quad M = \qquad\qquad\qquad\qquad}$$

: Safety Margin

Failure : $\{R - S \le 0\} \quad \Leftrightarrow \{\qquad \le 0\}$

$$\Leftrightarrow \{U_M \le \qquad\qquad \}$$

※ Standardization

$$U_M = \frac{M}{\qquad\qquad} \qquad \begin{cases} E[U_M] = \\ Var[U_M] = \end{cases}$$

For n RVs, $\quad \mathbf{U} = \mathbf{L}^{-1}\mathbf{D}^{-1}(\mathbf{X} - \mathbf{M})$

$$\therefore P_f = P(U_M \le \qquad) = F_{U_M}(\qquad)$$

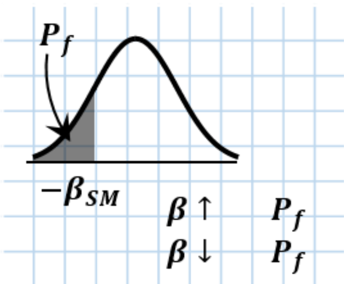
$$= F_{U_M}(\qquad)$$

β_{SM} : reliability index by safety margin

$$= \frac{\qquad}{\qquad\qquad}$$

$$= \frac{\qquad}{\qquad\qquad\qquad}$$

$$= \frac{r - 1}{\sqrt{r^2\delta_R^2 + \delta_s^2 - 2r\delta_R\delta_S\rho_{RS}}}, \quad r = \mu_R / \mu_S$$



F_{U_M} : depends on distribution of R and S

e.g. special case ~ R and S are jointly normal

Then $U_M \sim$

Therefore $P_f = F_{U_M}(-\beta_{SM}) =$

※ A. Cornell (1968. ACI codes)

Assumed R&S are jointly normal & used β_{SM} to compute P_f

④ **Reliability Index by "Safety Factor"**

$F = \ln \frac{\quad}{\quad} = \ln \quad - \ln$

Failure $: \{ \qquad \leq 0\}$ (※ used for LRFD $\phi R_n \geq \sum \gamma_k Q_k$)

$\Leftrightarrow \{ \qquad \leq 0\}$

$\Leftrightarrow \{u_F \leq - \frac{\quad}{\quad}\}$

$\therefore \beta_{SF} = \frac{\quad}{\quad}$

$\mu_F =$

$\sigma_F^2 =$

$P_f = F_{u_F}(- \quad)$

⇒ special case: R & S are jointly lognormal

$U_F \sim$

$\therefore P_f = \Phi(\quad)$

$\mu_F^{(LN)} =$

$\sigma_F^{(LN)} =$

$$\beta_{SF}^{(LN)} = \frac{\ln\left(r \cdot \sqrt{\frac{1+\delta_S^2}{1+\delta_R^2}}\right)}{\sqrt{\ln(1+\delta_R^2) - 2\ln(1+\rho_{RS}\delta_R\delta_S) + \ln(1+\delta_S^2)}}, \quad r = \frac{\mu_R}{\mu_s}$$

Safety factor-based reliability-index when R & S are jointly lognormal

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In-Class Material: Class 10

◎ Second moment reliability index β_{MVFOSM}

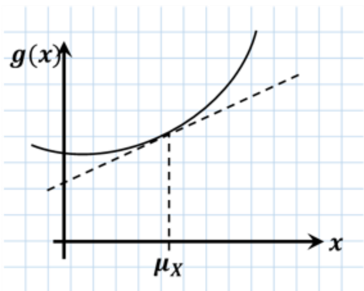
M V F O S M

- Failure : $g(\mathbf{x}) \le 0$ (NOT "elementary")
- Use ( ) & ( ) only. Therefore, can't compute P_f (index, not method)

- Ang & Cornell (1974) ASCE Journal of Structural Engineering

Use ( ) order approximation to estimate & of $g(\mathbf{x})$

$P_f = P(g \le 0) = P(u_g \le \quad)$



μ_g^{FO} & σ_g^{FO} ?

$g(\mathbf{x}) \simeq g(\quad) + \sum_{i=1}^{n} \qquad (x_i - \mu_{X_i})$

$$\begin{cases} \mu_g^{FO} = \\ \sigma_g^{2FO} = \sum \qquad\qquad + 2\sum_{i=1}^{n-1}\sum_{j=i+1}^{n} \end{cases}$$

$=$

$$\boxed{\therefore \ \beta_{MVFOSM} = \frac{\mu_g^{FO}}{\sigma_g^{FO}} = \underline{\qquad\qquad\qquad}}$$

If we assume $u_g \sim N(0, 1^2)$

$P_f \cong \Phi(\quad)$

$\Rightarrow$ Popular for a while

$\Rightarrow$ But problem

i.e. equivalent limit-state functions could give different β_{MVFORM}

$$\left.\begin{aligned} g_1(x) &= X_1^2 + 3X_2 < 0 \\ g_2(x) &= g_1(X) \big/ X_1^2 = 1 + 3\frac{X_2}{X_1^2} < 0 \end{aligned}\right\}$$ equivalent $\Rightarrow$ the same β_{MVFORM} ?

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Example: lack of invariance of MVFOSM

Consider a structural reliability problem with two random variables X_1 and X_2.
The mean vector and the covariance matrix of X_1 and X_2 are

$$\mathbf{M}_{\mathrm{X}} = \begin{bmatrix} 5 \\ 10 \end{bmatrix}, \quad \mathbf{\Sigma}_{\mathrm{XX}} = \begin{bmatrix} 4 & 5 \\ 5 & 25 \end{bmatrix}$$

<u>Case 1:</u> $g(X_1, X_2) = X_1^2 + 3X_2$

Gradient $\nabla g = [2X_1 \quad 3]$. At the mean point $\mathbf{X} = \mathbf{M}_{\mathrm{X}}$, $\nabla g = [10 \quad 3]$.

First order approximation on μ_g and σ_g^2:

$\mu_g \cong 5^2 + 3 \times 10 = 55$

$\sigma_g^2 \cong \nabla g \mathbf{\Sigma}_{\mathrm{XX}} \nabla g^{\mathrm{T}} = 925$

$$\beta_{MVFOSM} = \frac{\mu_g}{\sigma_g} = \frac{55}{\sqrt{925}} = 1.81$$

$P_f = \Phi(-1.81) = 0.0351$

<u>Case 2:</u> $g(X_1, X_2) = 1 + \dfrac{3X_2}{X_1^2}$

$\nabla g = [-6X_2X_1^{-3} \quad 3X_1^{-2}]$.

At the mean point $\mathbf{X} = \mathbf{M}_{\mathrm{X}}$, $\nabla g = [-0.48 \quad 0.12]$.

$\mu_g \cong 1 + 3 \times 10 / 25 = 2.20$

$\sigma_g^2 \cong \nabla g \mathbf{\Sigma}_{\mathrm{XX}} \nabla g^{\mathrm{T}} = 0.706$

$$\beta_{MVFOSM} = \frac{\mu_g}{\sigma_g} = \frac{2.20}{\sqrt{0.706}} = 2.62$$

$P_f = \Phi(-2.62) = 0.00440$

Although the two limit-state functions are equivalent ones with the same failure domains, the second order reliability method yields different reliability indices and failure probability estimates.

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Summary:

$$\beta_{SM} = \frac{\mu_M}{\sigma_M} = \frac{\mu_R - \mu_S}{\sqrt{\sigma_R^2 + \sigma_S^2 - 2\sigma_R\sigma_S\rho_{RS}}}$$

$$\beta_{SF} = \frac{\mu_F}{\sigma_F}, \text{ for LN } \beta_{SF} = \frac{\lambda_R - \lambda_S}{\sqrt{\zeta_R^2 + \zeta_S^2 - 2\zeta_R\zeta_S\rho_{\ln R \ln S}}}$$

$$\beta_{MVFOSM} = \frac{\mu_g^{FO}}{\sigma_g^{FO}} = \frac{g(\mathbf{M}_{\mathrm{X}})}{\nabla g(\mathbf{M}_{\mathrm{X}})\mathbf{\Sigma}_{\mathrm{XX}}\nabla g(\mathbf{M}_{\mathrm{X}})^{\mathrm{T}}} \quad \text{(Oct1974)}$$

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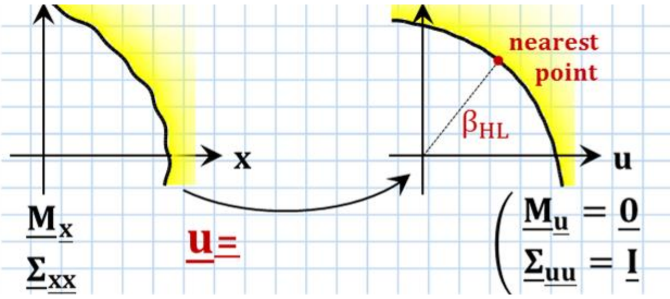
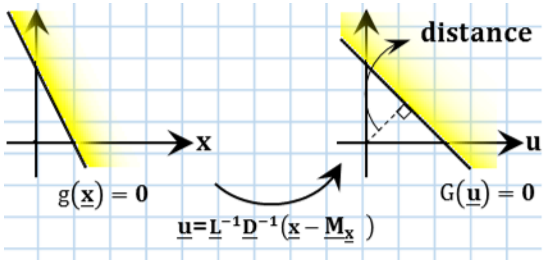
◎ Hasofer-Lind Reliability Index, β_{HL} (JEM, May 1974)

(or β_{AFOSM} "Advanced" FOSM)



Linear Limit-State Function

$$g(\mathbf{x}) = a_0 + \mathbf{a}^{\mathrm{T}}\mathbf{x}$$
$$= a_0 + \mathbf{a}^{\mathrm{T}}(\qquad\qquad\qquad)$$
$$= a_0 + \mathbf{a}^{\mathrm{T}}\mathbf{M} + \mathbf{a}^{\mathrm{T}}\mathbf{DLu}$$
$$= b_0 + \mathbf{b}^{\mathrm{T}}\mathbf{u} = G(\mathbf{u})$$

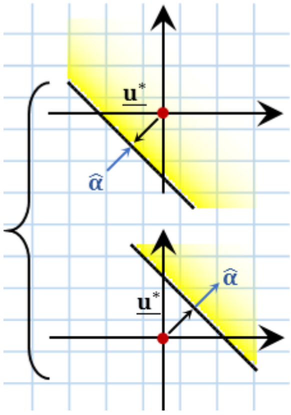


$$\boxed{\beta = \frac{\mu_G}{\sigma_G} = \frac{b_0}{\|\mathbf{b}\|}} \qquad \text{VS} \qquad \text{distance} = \frac{|b_0|}{\|\mathbf{b}\|}$$

Can have +/− sign

always positive

For $G(\mathbf{u}) = b_0 + \mathbf{b}^{\mathrm{T}}\mathbf{u}$



$b_o = G(\mathbf{0}) <$ (

(in failure domain)

$\beta < 0$

$b_o = G(\mathbf{0}) > 0$

(in safe domain)

$\beta > 0$

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i. $\boldsymbol{\alpha} = -\dfrac{\nabla G}{\|\nabla G\|}$: "Negative normalized gradient vector"

: Unit row vector pointing toward the _______ domain

e.g. linear function : $\boldsymbol{\alpha} = -\dfrac{\mathbf{b}^{\mathrm{T}}}{\|\mathbf{b}\|}$

ii. $\mathbf{u}^*$: "Design point"

"Most probable failure point (MPP)"

"Beta point"

e.g. linear function : $\mathbf{u}^* \equiv -b_0 \dfrac{\mathbf{b}}{\|\mathbf{b}\|^2}$

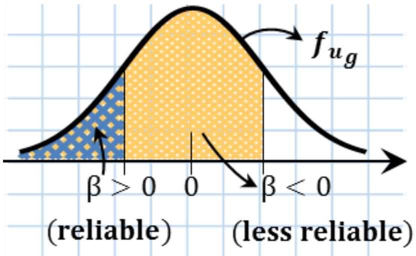
iii.

$$\boxed{\beta_{HL} \equiv \boldsymbol{\alpha}\mathbf{u}^*}$$

Hasofer-Lind Reliability Index

$$\begin{cases} |\beta_{HL}| : \text{distance between origin and } \mathbf{u}^* \\ \text{sign : directions of } \boldsymbol{\alpha} \text{ and } \mathbf{u}^* \end{cases}$$

e.g. linear function : $\beta_{HL} = \dfrac{b_0}{\|\mathbf{b}\|} \left(= \dfrac{\mu_G}{\sigma_G} \right)$



$P_f = F_{u_g}(-\beta_{HL})$

What if $\mathbf{X} \sim N(\mathbf{M}_{\mathrm{X}}, \Sigma_{\mathrm{XX}})$ and $g(\mathbf{x})$ linear?

$\Rightarrow G, g \sim N$

$P_f = \Phi(-\beta_{HL})$