III. Fluid Forces on PARTICLES

(Bagchi & Balachandar, JFM, 2003)

momentum coupling as a result of mass transfer (X)
(uniform efflux from the interface) and fluid forces (drag, lift, ...).

· Eq. of motion for small (solid) particle in non-uniform unsteady flows at low Re.
(Maxey & Riley, PoF, 1983)

undisturbed flow (cont. phase)

Stokes drag

Faxen force

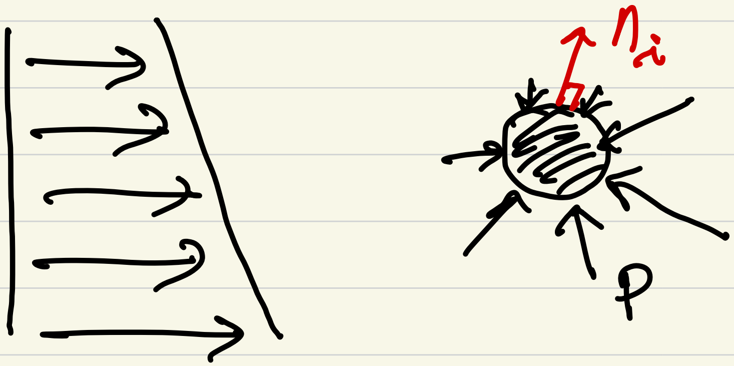
$$m\frac{dv_i}{dt} = mg_i + V_d\left(-\frac{\partial P}{\partial x_i} + \cancel{\frac{\partial \tau_{ij}}{\partial x_j}}\right) + \underbrace{3\pi\mu_c D\left[(u_i - v_i) + \frac{D^2}{24}\nabla^2 u_i\right]}_{\text{steady state drag}}$$

$$+ \underbrace{\frac{1}{2}\rho_c V_d \frac{d}{dt}\left[(u_i - v_i) + \frac{D^2}{40}\nabla^2 u_i\right]}_{\text{added (virtual) mass force}}$$

$$+\frac{3}{2}\pi\mu_c D^2\int_0^t\left[\frac{d/d\tau\left(u_i-v_i+D^2/24\,\nabla^2 u_i\right)}{\pi\nu_c(t-\tau)^{1/2}}\right]d\tau.$$

→ Basset history force.

① Undisturbed (free-stream) flow force.



· force from pressure grad.

$$F_{p,i}=\int_S(-p)\,n_i\,dS.$$

↓ divergence theorem.

$$\oint_S(F\cdot\bar{n})\,dS=\int_V(\nabla\cdot F)\,dV.$$

$$=-\int_V\frac{\partial p}{\partial x_i}\,dV\simeq-V_d\frac{\partial p}{\partial x_i}.$$

volume of particle.

(if the pressure distribution is hydrostatic.

$$\frac{\partial p}{\partial x_i} = -\rho_c g_i \Rightarrow F_p = \rho_c g V_d$$: buoyancy.)

- force due to shear stress.

$$F_{\tau,i} = \int_S \tau_{ik} \cdot n_k \cdot dS = \int_V \frac{\partial \tau_{ik}}{\partial x_k} dV \simeq \underline{V_d} \frac{\partial \tau_{ik}}{\partial x_k}.$$

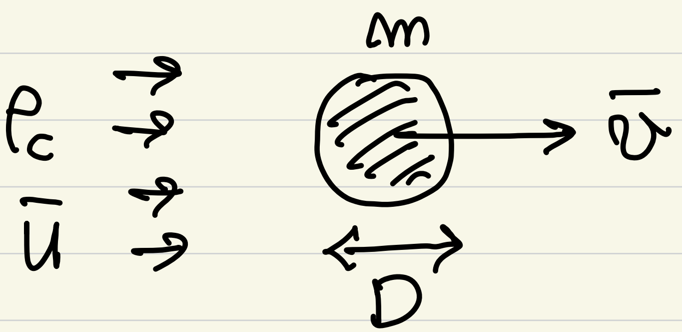
$$\therefore F_{uD} = F_p + F_\tau$$

$$= V_d \left(-\frac{\partial p}{\partial x_i} + \frac{\partial \tau_{ik}}{\partial x_k} \right) = V_d \left(-\nabla p + \nabla \tau \right).$$

Not significant for gas-solid flow (particle-laden flow)

② Steady-state drag.

→ No acceleration of the relative velocity.



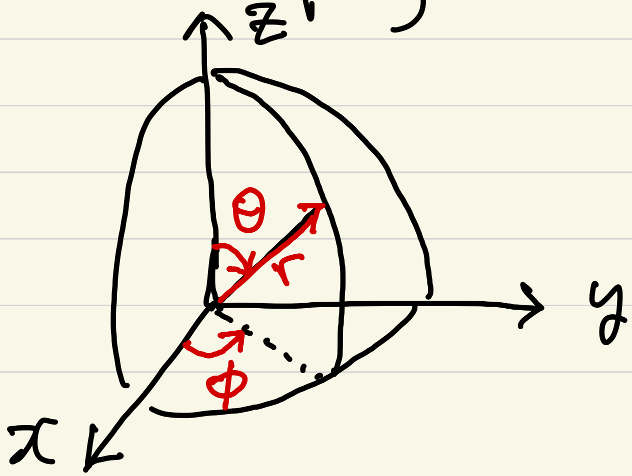
$$F_D = C_D \cdot \frac{1}{2} \rho_c (\bar{u}-\bar{v}) |\bar{u}-\bar{v}| \cdot \frac{\pi D^2}{4}$$

for $Re_r = D|\bar{u}-\bar{v}|/\nu_c \ll 1$.

$$F_D = 3\pi\mu_c D (u-v), \quad C_D = \frac{24}{Re_r}.$$

Stokes Drag.

· In creeping flow, ($Re \ll 1$)



· cont. : $\nabla \cdot \bar{v} = 0$

· mtm : $\rho \left[\cancel{\frac{\partial \bar{v}}{\partial t}} + \cancel{(\bar{v} \cdot \nabla)\bar{v}} \right] = -\nabla P + \mu \nabla^2 \bar{v} + \cancel{\rho g}$

for $Re \ll 1$, $\nabla P = \mu \nabla^2 \bar{v}$.

$\nabla \times (\nabla P = \mu \nabla^2 \bar{v}) \rightarrow \nabla^2 \bar{\omega} = 0$

vorticity

$\nabla \cdot (\nabla P = \mu \nabla^2 \bar{v}) \rightarrow \nabla^2 P = 0$.

By introducing Stoke's stream function for axisymmetric flow,

$$\begin{cases} v_r = \dfrac{1}{r^2 \sin\theta} \cdot \dfrac{\partial \psi}{\partial \theta}, \\ v_\theta = -\dfrac{1}{r \sin\theta} \cdot \dfrac{\partial \psi}{\partial r} \end{cases} \rightarrow \omega_\phi = \frac{1}{r}\left[\frac{\partial}{\partial r}(r u_\theta) - \frac{\partial u_r}{\partial \theta}\right].$$

$$\Rightarrow E^4 \psi = E^2(E^2 \psi) = 0.$$

$$E^2 = \frac{\partial^2}{\partial r^2} + \frac{\sin\theta}{r^2}\cdot\frac{\partial}{\partial\theta}\left(\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\right)$$: axisymmetric Laplacian in spherical coord.

w/ BC's

sphere radius

ⓐ $r = R$, $v_r = v_\theta = 0$

ⓐ $r \to \infty$, free-stream (U_∞) $\to$ $v_r = U_\infty \cos\theta$, $v_\theta = U_\infty \sin\theta$.

$$\psi_\infty = \frac{1}{2} U_\infty r^2 \sin^2\theta.$$

We assume that the solution is in the form of

$$\psi = f(r)\, U_\infty \sin^2\theta. \rightarrow E^4\psi = 0$$

$$\therefore\ r^2\frac{d^4 f}{dr^2} - 4r^2\frac{d^2 f}{dr^2} + r\frac{df}{dr} - 8f = 0.$$

$$\left\{ f(r) = r^{\alpha} \right. \Rightarrow \alpha = -1, 1, 2, 4.$$

$$\therefore \psi = U_\infty \sin^2\theta \left(\frac{A}{r} + Br + Cr^2 + Dr^4 \right)$$

w/ BC's : $\psi(r,\theta) = U_\infty \sin^2\theta \left(\frac{R^3}{4r} - \frac{3R}{4} r + \frac{1}{2} r^2 \right).$

σ_{ij}

$$v_r = 2U_\infty \left(\frac{R^3}{4r^3} - \frac{3R}{4r} + \frac{1}{2} \right) \cos\theta$$

$$v_\theta = U_\infty \left(1 - \frac{3R}{4r} - \frac{R^3}{4r^3} \right) \sin\theta.$$

then, the drag force per unit area on the solid sphere is :

$$\underline{f} = \sigma_{rr} + \sigma_{\theta r} = -p + 2\mu \frac{\partial v_r}{\partial r}.$$

from $\int \nabla p = \int \mu \nabla^2 \underline{u}$

$|$

$$\rightarrow p = p_0 - \frac{3R\mu U_\infty \cos\theta}{2r^2}.$$

$$\therefore \bar{D} = \int_0^{2\pi}\int_0^{\pi} \bar{f}\, dA.$$

$$= \underbrace{\pi \mu_c D U_\infty}_{\text{pressure}} + \underbrace{2\pi \mu_c D U_\infty}_{\text{shear}} = 3\pi \mu_c D U_\infty.$$

$$3\pi\mu_c D(\underbrace{u - v}_{\text{rel. vel.}})$$

$$C_D = \bar{D} / \left(\tfrac{1}{2}\rho U_\infty^2 \cdot \pi R^2\right)$$

$$= \frac{12\mu_c}{\rho U_\infty R} = \frac{24}{Re}.$$

Oseen's approximation (considering the non-negligible inertia term)

$$: C_D = \frac{24}{Re}\left(1 + \frac{3}{16} Re\right)$$

- for a spherical fluid particle (Batchelor, 1967) pp. 235–238.
  - shear stress on the surface induces an internal motion. (Hadamard, 1911)

→ At the interface, the tangential stress by the other fluid is balanced by that from the inner circulation. → "BC"

$$C_D = \boxed{\frac{24}{Re_r}} \left(\frac{1 + \frac{3}{2}\bar{\mu}}{1 + \bar{\mu}} \right)$$: Hadamard–Rybczynski drag law.

$$\bar{\mu} = \mu_c / \mu_d .$$

for a rigid sphere. $\bar{\mu} \to 0$: $C_D = 24/Re$ [Stokes drag]

" droplet in air, $\bar{\mu} \to 0$: "

" bubble in liq. $\bar{\mu} \to \infty$: $C_D = \frac{16}{Re}$

- Faxen force : effect of non-uniform flow.

$$\bar{F}_D = 3\pi\mu_c D(\bar{u} - \bar{v}) \quad : \text{uniform flow.}$$

$$\bar{F}_D = \underbrace{3\pi\mu_c D(\bar{u} - \bar{v})} + \underbrace{\mu_c \pi \frac{D^3}{8} \nabla^2 \bar{u}}.$$

$\nabla^2 \bar{u}$ → flow curvature effect.

(Happel & Brenner, 1973)

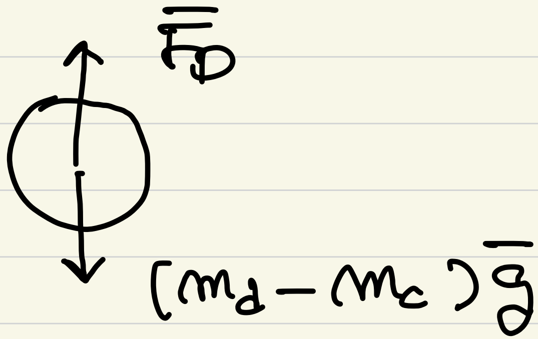
part. size

$$\frac{F_{Faxen}}{F_{Stokes}} \simeq \left(\frac{D}{l}\right)^2$$

l → characteristic length associated w/ the carrier flow field velocity distribution such as

the radius of curvature of
the vel. distribution.

- Terminal (settling) velocity of a particle
in a stationary fluid.



$$m_d \frac{d\bar{v}}{dt} = (m_d - m_c)\bar{g} - \bar{F}_D = 0$$

$$\frac{\pi}{6} D^3 (\rho_d - \rho_c) g = C_D \cdot \frac{1}{2} \rho_c V_{st}^2 \cdot \frac{\pi}{4} D^2$$

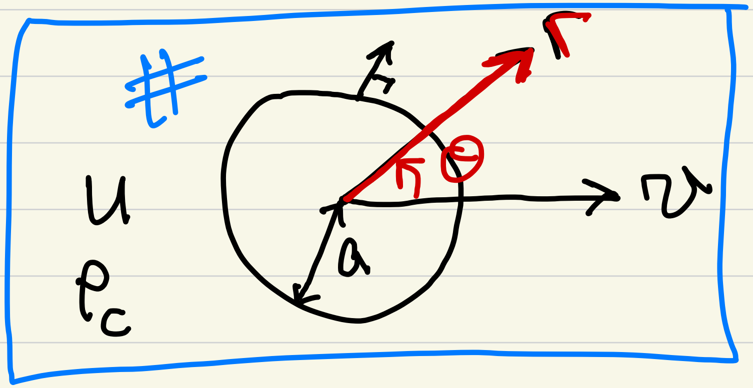
$$\therefore V_{st}^2 = \frac{4}{3} \cdot \frac{D(\rho_d - \rho_c)}{C_D \rho_c} g$$

for stokes drag, $V_{st} = \frac{1}{18} \cdot \frac{D^2 g}{\nu_c} \left(\frac{\rho_d}{\rho_c} - 1 \right)$

" Oseen's approximation,

$$V_{st} = \frac{24\mu_c}{9D}\left(\sqrt{1+\frac{[(\rho_d/\rho_c-1)gD^3}{24\nu_c^2}}-1\right).$$

③ Virtual (apparent, added) mass force.



$\Downarrow$

acceleration of a surrounding fluid at the expense of work done by the moving body.

· kinetic energy of the fluid surrounding the sphere.

$$KE = \frac{1}{2}\rho_c\int_V u^2\, dV = \frac{1}{2}\rho_c\int_V \frac{\partial\phi}{\partial x_i}\frac{\partial\phi}{\partial x_i}\, dV.$$

$$\nabla\cdot\bar{u} = \nabla\cdot\nabla\phi = 0$$
$$= \nabla^2\phi$$

$$= \frac{1}{2}\rho_c \int_V \left[\frac{\partial \phi}{\partial x_i}\frac{\partial \phi}{\partial x_i} + \underline{\phi \frac{\partial^2 \phi}{\partial x_i \partial x_i}} \right] dV.$$

divergence theorem

$$= \frac{1}{2}\rho_c \int_V \frac{\partial}{\partial x_i}\left(\phi \frac{\partial \phi}{\partial x_i} \right) dV.$$

$$= \frac{1}{2}\rho_c \int_S \underline{\phi} \frac{\partial \phi}{\partial x_i} \underline{n_i}\, dA$$

normal vector.

· velocity potential for a sphere moving w/ velocity of u. : $\phi = -\frac{ua^3}{2r^2}\cos\theta.$

$$u_r = \frac{\partial \phi}{\partial r} = \frac{ua^3}{r^3}\cos\theta.$$

$$\therefore \text{KE} = \frac{\pi \rho_c a^3 u^2}{2}\int_0^{\pi} \cos^2\theta \cdot \sin\theta \cdot d\theta$$

$$= \frac{\pi}{3} \rho_c a^3 u^2 .$$

· Work rate required to change the kinetic energy

$$u \cdot F_{VM} = \frac{dKE}{dt} \rightarrow \cancel{u} \cdot F_{VM} = \frac{2\pi \rho_c a^3}{3} \cancel{u} \frac{du}{dt}.$$

$$\Rightarrow \therefore F_{VM} = \frac{M_f}{2} \cdot \frac{du}{dt}. \qquad \boxed{M_f = \frac{4}{3} \rho_c \pi a^3}$$

$\rho_c \cdot V_d$

mass of fluid displaced by the sphere (moving body)

· relative acceleration of the fluid with respect to the particle acceleration

$$F_{VM} = \frac{\rho_c V_d}{2} \left(\frac{d\bar{u}}{dt} - \frac{d\bar{v}}{dt} \right) \quad (\text{Auton et al, 1988})$$

- for non-uniform flow,

$$F_{VM} = \frac{\rho_c V_d}{2}\left(\frac{d\bar{u}}{dt} - \frac{d\bar{v}}{dt} - \underbrace{\frac{D^2}{40}\frac{d}{dt}\nabla^2\bar{u}}_{\text{flow curvature effect}} \right)$$