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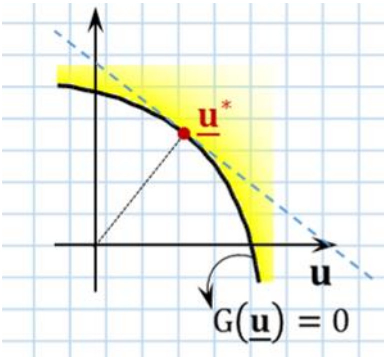
457.646 Topics in Structural Reliability
In-Class Material: Class 11

◎ Hasofer-Lind Reliability Index, β_{HL} (contd.)

② Nonlinear Limit-State Function



Transform $g(\quad)$ to $G(\quad)$ by

$\mathbf{X} =$
$\mathbf{u} =$

- suppose one can find $\mathbf{u}^*$
- Linearize $G(\mathbf{u})$ at $\mathbf{u} =$

→ $G(u)^{FO} G(\quad) + \quad (\quad)$

$= \quad =0$

Reliability index

Try $\frac{\mu_G}{\sigma_G} \simeq \frac{\mu_G^{FO}}{\sigma_G^{FO}}$?

$\mu_G^{FO} =$

$\sigma_G^{2FO} = \quad = \| \quad \|^2$

$\therefore \frac{\mu_G^{FO}}{\sigma_G^{FO}} = \quad = \quad =$

In summary, the "distance" between the origin and the design point $\mathbf{u}^*$ in $\mathbf{u}$-space gives reliability index based on first-order approximation

★ **Note!**

$$\begin{cases} MVFOSM & \frac{\mu^{FO}}{\sigma^{FO}} \text{ at } \mathbf{x} = \\ HL & \frac{\mu^{FO}}{\sigma^{FO}} \text{ at } \mathbf{u} = \end{cases}$$

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※ Procedure : i) Transform $g(\mathbf{x})$ to $G(\mathbf{u})$ using $\mathbf{x} =$

ii) Find

iii) Find at

iv) $\beta_{HL} =$

※ Description of β_{HL} in $\mathbf{x}$ space?

$$\nabla G(\mathbf{u}^*)(\mathbf{u}-\mathbf{u}^*) = 0$$

$\mathbf{u}=$ ⇄ $\mathbf{x}=$

$$\nabla g(\mathbf{x}^*)(\mathbf{x}-\mathbf{x}^*) = 0$$

Approx. Limit state space in $\mathbf{u}$

Proof :

$$\nabla_{\mathbf{x}} g(\mathbf{x}^*) = \nabla_{\mathbf{u}} G(\mathbf{u}^*) \times$$

$$=$$

$\mathbf{x}^* =$

$\mathbf{x} =$

$$\therefore g^{FO} = \nabla g(\mathbf{x}^*)(\mathbf{x}-\mathbf{x}^*)$$

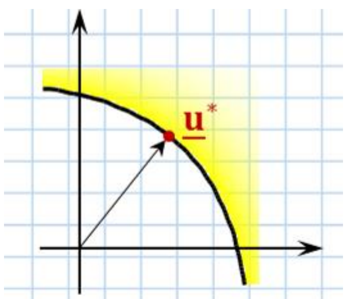
$$\beta_{HL} = \frac{\mu_g^{FO}}{\sigma_g^{FO}} = \frac{}{\sqrt{}}$$

FO at $\underline{\mathbf{x}} =$

cf. $$\beta_{MVFOSM} = \frac{\mu_g^{FO}}{\sigma_g^{FO}} = \frac{g(\mathbf{M}_{\mathbf{x}})}{\sqrt{\nabla g(\mathbf{M}_{\mathbf{x}}) \Sigma_{\mathbf{xx}} \nabla g(\mathbf{M}_{\mathbf{x}})^{\mathrm{T}}}}$$

FO at $\underline{\mathbf{x}} =$

③ Finding the design point $\mathbf{u}^*$



$\mathbf{u}^* = \text{argmin}\{ \quad | \quad \}$

Then evaluate $\boldsymbol{\alpha} =$ at

And compute $\beta_{HL} = \boldsymbol{\alpha}\mathbf{u}^*$

⇒ constrained nonlinear optimization problem

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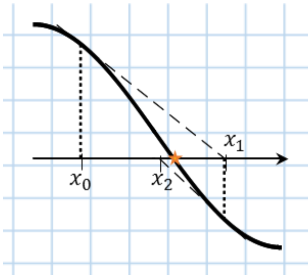
Reviews on optimization algorithm of finding $\mathbf{u}^*$

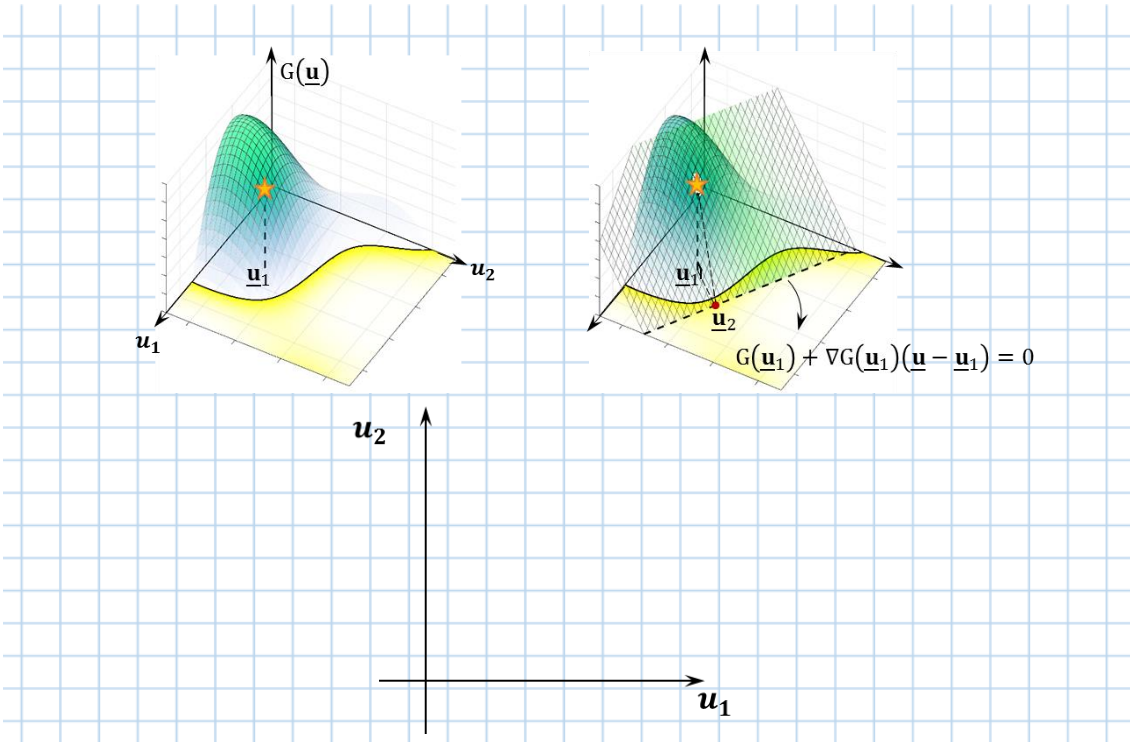
- Liu & ADK (1990)
- Papaioannou et al. (2010)

HL-RF, SQP, GP, DFO



a) HL-RF algorithm (Rackwitz & Fissler 1978)

"Newton-Raphson-like algorithm" solve $f(x)=0$ for $x=x^*$?



$\mathbf{u}_1$: initial point (e.g $\mathbf{u}_1$ = $\mathbf{M}_{\mathbf{u}}=\mathbf{0}$)

$\mathbf{u}_2 = (\qquad\qquad) \times (\qquad\qquad)$

$=$

$=$

$\mathbf{u}_{i+1} =$

To update $\mathbf{u}_i$ to , $\mathbf{u}_{i+1}$, one needs

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$G(\mathbf{u}_i) =$

$\nabla_{\mathbf{u}} G(\mathbf{u}_i) =$

Iterate until 1)

2)

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In-Class Material: Class 12

HL-RF Algorithm for HL Reliability Index and FORM/SORM

Initialization

- $i = 1$ (step index)
- Set convergence tolerances $\varepsilon_1, \varepsilon_2$ (e.g. 10^{-3})
- Pick a starting point $\mathbf{x}_1$ (e.g. $\mathbf{x}_1 = \mathbf{M}_X$)
- $\mathbf{u}_1 = \mathbf{u}(\mathbf{x}_1)$
- Scale parameter G_0 (e.g. $G_0 = g(\mathbf{M}_X)$)

Computation

- $\mathbf{x}_i = \mathbf{x}(\mathbf{u}_i)$ (skip if $i = 1$)
- $\mathbf{J}_{\mathbf{x},\mathbf{u}}$ $(= \mathbf{J}_{\mathbf{u},\mathbf{x}}^{-1})$
- $G(\mathbf{u}_i) = g(\mathbf{x}_i)$
- $\nabla G(\mathbf{u}_i) = \nabla g(\mathbf{x}_i)\mathbf{J}_{\mathbf{x},\mathbf{u}}$
- $\hat{\alpha}_i = -\nabla G(\mathbf{u}_i) / \|\nabla G(\mathbf{u}_i)\|$

Convergence Check

- $|G(\mathbf{u}_i)/G_0| \le \varepsilon_1$?
- $\|\mathbf{u}_i - \hat{\alpha}_i \mathbf{u}_i \hat{\alpha}_i^{\mathrm{T}}\| \le \varepsilon_2$?

No → Update

Update

- $\mathbf{u}_{i+1} = \left[\hat{\alpha}_i \mathbf{u}_i + \dfrac{G(\mathbf{u}_i)}{\|\nabla G(\mathbf{u}_i)\|}\right]\hat{\alpha}_i^{\mathrm{T}}$
- Modified and improved algorithms available for better convergence.

$i \rightarrow i+1$ (return to Computation)

Yes → Post-process

Post-process

- $\mathbf{u}^* = \mathbf{u}_i$
- $\hat{\alpha} = \hat{\alpha}_i$
- $\beta = \hat{\alpha}\mathbf{u}^*$
- $\mathbf{x}^* = \mathbf{x}(\mathbf{u}^*)$
- $P_f \cong \Phi(-\beta)$ (FORM)
- $P_f \cong \Phi(-\beta)\displaystyle\prod_{i=1}^{n-1}\frac{1}{\sqrt{1+\psi(\beta)\kappa_i}}$ (SORM)

Note

(1) β_{FOSM}:

$$\mathbf{x}_i = \mathbf{D}\mathbf{L}\mathbf{u}_i + \mathbf{M}_X$$

$$\mathbf{J}_{\mathbf{x},\mathbf{u}} = \mathbf{D}\mathbf{L}$$

(2) FORM/SORM:

$\mathbf{x}_i = \mathbf{x}(\mathbf{u}_i)$ and $\mathbf{J}_{\mathbf{x},\mathbf{u}}$ are the transformation and the Jacobian between the joint distribution of $\mathbf{x}$ and the uncorrelated standard normal distribution of $\mathbf{u} \sim N(\mathbf{0},\mathbf{I})$.

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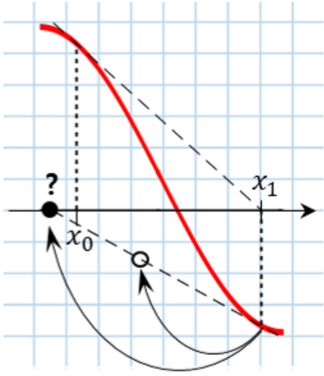
Example (PPT animations): $g(X_1, X_2) = 0.5X_1^2 - X_2 + 3\sin(2X_1)$

☆ **Convergence Issue**

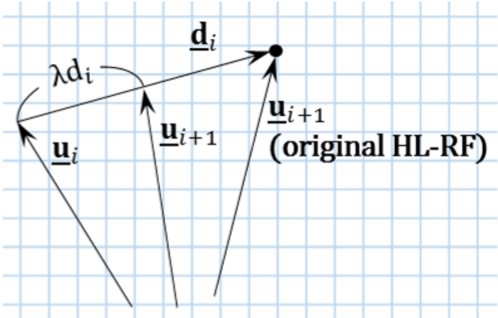


Solution: Does not go full step, i.e. "step size" control

- Modified HL-RF (Liu & ADK 1990)

- Improved HL-RF (Zhang & ADK 1995)

$$\begin{cases} \mathbf{u}_{i+1} = \mathbf{u}_i + \lambda \mathbf{d}_i \quad (\lambda, \text{ stepsize}<1) \\ \mathbf{d}_i = \left(\hat{\boldsymbol{\alpha}}_i \mathbf{u}_i + \dfrac{G(\mathbf{u}_i)}{\|\nabla G(\mathbf{u}_i)\|} \right) \boldsymbol{\alpha}_i^{\mathrm{T}} - \mathbf{u}_i \end{cases}$$



How? "Merit" function $m(\mathbf{u})$ is defined such that $m(\mathbf{u})$ is minimum at $\mathbf{u} =$

Then, select λ at each step such that $m(\mathbf{u})$ d____________

e.g. 1) Modified HL-RF: $m(\mathbf{u}) = \dfrac{1}{2}\left\|\mathbf{u} - \boldsymbol{\alpha}\mathbf{u}\boldsymbol{\alpha}^{\mathrm{T}}\right\|^2 + \dfrac{1}{2}c \cdot G(\mathbf{u})^2$

($m(\mathbf{u})$ can have minima that are not solution)

2) Improved HL-RF: $m(\mathbf{u}) = \dfrac{1}{2}\|\mathbf{u}\|^2 + c|G(\mathbf{u})|$

Select λ such that $m(\mathbf{u}_{i+1})$ $m(\mathbf{u}_i)$ because the direction vector is a descent direction in terms of merit function

as long as $c > \dfrac{\|\mathbf{u}_{i+1}\|}{\|\nabla G(\mathbf{u}_{i+1})\|}$

※ Zhang & ADK(1995) proved this based on so-called "Armijo's rule" and provided detailed updating rule for *c* (but FERUM uses a simple rule)

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Example: β_{HL} by improved HL-RF algorithm

Limit-state function $g(X_1, X_2) = 0.5X_1^2 - X_2 + 3\sin(2X_1)$

Mean vector and covariance matrix of X_1 and X_2:

$$\mathbf{M} = \begin{bmatrix} 5 \\ 3 \end{bmatrix}, \quad \mathbf{\Sigma} = \begin{bmatrix} 4 & 5 \\ 5 & 25 \end{bmatrix}$$

Gradient $\nabla g = [X_1 + 6\cos(2X_1) \quad -1]$

Preparation:

$$\mathbf{D} = \begin{bmatrix} 2 & 0 \\ 0 & 5 \end{bmatrix}, \qquad \mathbf{R} = \begin{bmatrix} 1 & 0.5 \\ 0.5 & 1 \end{bmatrix}$$

$$\mathbf{R} = \mathbf{L}\mathbf{L}^{\mathrm{T}} \text{ (Cholesky decomposition): } \mathbf{L} = \begin{bmatrix} 1 & 0 \\ 0.5 & 0.87 \end{bmatrix}, \ \mathbf{L}^{-1} = \begin{bmatrix} 1 & 0 \\ -0.58 & 1.15 \end{bmatrix}$$

$$\mathbf{u}(\mathbf{x}) = \mathbf{L}^{-1}\mathbf{D}^{-1}(\mathbf{x} - \mathbf{M}_{\mathbf{x}}); \quad \mathbf{x}(\mathbf{u}) = \mathbf{DLu} + \mathbf{M}$$

$$\mathbf{J}_{\mathbf{u},\mathbf{x}} = \mathbf{L}^{-1}\mathbf{D}^{-1} = \begin{bmatrix} 0.5 & 0 \\ -0.29 & 0.23 \end{bmatrix}; \quad \mathbf{J}_{\mathbf{x},\mathbf{u}} = \mathbf{DL} = \begin{bmatrix} 2 & 0 \\ 2.5 & 4.33 \end{bmatrix} \text{ (constant since linear)}$$

Initialization:

$i = 1; \ \varepsilon_1 = \varepsilon_2 = 10^{-3}$

Starting point: $\mathbf{x}_1 = \mathbf{M} = \begin{bmatrix} 5 \\ 3 \end{bmatrix}; \ \mathbf{u}_1 = \mathbf{u}(\mathbf{x}_1) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

Scale parameter: $G_0 = g(\mathbf{M}) = 0.5 \cdot 5^2 - 3 + 3 \cdot \sin(2 \cdot 5) = 7.87$

Computation (1st step):

$$G(\mathbf{u}_1) = g(\mathbf{x}_1) = 7.8679$$

$$\nabla G(\mathbf{u}_1) = \nabla g(\mathbf{x}_1)\mathbf{J}_{\mathbf{x},\mathbf{u}} = [-0.03 \quad -1]\begin{bmatrix} 2 & 0 \\ 2.5 & 4.33 \end{bmatrix} = [-2.57 \quad -4.33]$$

$$\hat{\alpha}_1 = -\frac{[-2.57 \quad -4.33]}{(2.57^2 + 4.33^2)^{1/2}} = [0.51 \quad 0.86]$$

Convergence check (1st step): Skipped.

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Update (1st→2nd):

$$c_1 \geq \frac{\|\mathbf{u}_1\|}{\|\nabla G(\mathbf{u}_1)\|} = 0\text{; Set } c_1 = 10$$

Current value of the merit function:

$$m(\mathbf{u}_1) = 0.5\|\mathbf{u}_1\|^2 + c_1|G(\mathbf{u}_1)| = 0.5(0)^2 + 10(7.87) = 78.7$$

$$\mathbf{d}_1 = \left[\hat{\alpha}_1\mathbf{u}_1 + \frac{G(\mathbf{u}_1)}{\|\nabla G(\mathbf{u}_1)\|}\right]\hat{\alpha}_1^{\mathrm{T}} - \mathbf{u}_1$$

$$= \left\{\begin{bmatrix}0.51 & 0.86\end{bmatrix}\begin{bmatrix}0\\0\end{bmatrix} + \frac{7.87}{5.03}\right\}\begin{bmatrix}0.51\\0.86\end{bmatrix} - \begin{bmatrix}0\\0\end{bmatrix}$$

$$= \begin{bmatrix}0.80\\1.34\end{bmatrix}$$

Try a step size: $\lambda = 1$ (original HL-RF)

$$\mathbf{u}_2 = \mathbf{u}_1 + \lambda\mathbf{d}_1$$

$$= \begin{bmatrix}0\\0\end{bmatrix} + (1)\begin{bmatrix}0.80\\1.34\end{bmatrix} = \begin{bmatrix}0.80\\1.34\end{bmatrix}$$

Check $m(\mathbf{u}_2) < m(\mathbf{u}_1)$

$$\mathbf{x}_2 = \mathbf{x}(\mathbf{u}_2) = \begin{bmatrix}6.59\\10.81\end{bmatrix}$$

$$G(\mathbf{u}_2) = g(\mathbf{x}_2) = 0.5\cdot 6.59^2 - 10.81 + 3\sin(2\cdot 6.59) = 12.68$$

$m(\mathbf{u}_2) = 0.5(6.59^2 + 10.81^2) + 10(12.68) = 126.82 > 78.7$ **N.G. (reject: $\lambda = 1$)**

Try a step size: $\lambda = 0.5$

$$\mathbf{u}_2 = \mathbf{u}_1 + \lambda\mathbf{d}_1$$

$$= \begin{bmatrix}0\\0\end{bmatrix} + (0.5)\begin{bmatrix}0.80\\1.34\end{bmatrix} = \begin{bmatrix}0.40\\0.67\end{bmatrix}$$

Check $m(\mathbf{u}_2) < m(\mathbf{u}_1)$

$$\mathbf{x}_2 = \mathbf{x}(\mathbf{u}_2) = \begin{bmatrix}5.80\\6.91\end{bmatrix}$$

$$G(\mathbf{u}_2) = g(\mathbf{x}_2) = 0.5\cdot 5.08^2 - 6.91 + 3\sin(2\cdot 5.08) = 7.42$$

$m(\mathbf{u}_2) = 0.5(0.40^2 + 0.67^2) + 10(7.42) = 74.60 < 78.7$ **O.K. (accept: $\lambda = 0.5$)**

Computation (2nd step):

$$\nabla g = [X_1 + 6\cos(2X_1) \quad -1]$$

$$G(\mathbf{u}_2) = 7.4$$

$$\nabla G(\mathbf{u}_2) = \begin{bmatrix}9.18 & -1\end{bmatrix}\begin{bmatrix}2 & 0\\2.5 & 4.33\end{bmatrix} = \begin{bmatrix}15.86 & -4.33\end{bmatrix}$$

$$\hat{\alpha}_2 = \begin{bmatrix}-0.97 & 0.\end{bmatrix}$$

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Convergence check (2nd step):

$$\left|G(\mathbf{u}_2)/G_0\right| = \frac{7.42}{7.67} = 0.94 > \varepsilon_1 \quad \textbf{N.G.}$$

$$\left\|\mathbf{u}_2 - \hat{\alpha}_2 \mathbf{u}_2 \hat{\alpha}_2^{\mathrm{T}}\right\| = 0.75 > \varepsilon_2 \quad \textbf{N.G.}$$

Update (2nd→3rd):

$$c_2 \geq \|\mathbf{u}_2\| / \|\nabla G(\mathbf{u}_2)\| = 0.05\text{; set } c_2 = 10$$

$$\vdots$$

Repeat until the convergence criteria are satisfied.

Note: If $m(\mathbf{u}_{i+1}) \geq m(\mathbf{u}_i)$, reduce the value of λ until you satisfy $m(\mathbf{u}_{i+1}) < m(\mathbf{u}_i)$

☆ **Santos, Matioli & Beck (2012)**

New optimization algorithms for structural reliability Analysis

⇒ provides a good review on HLRF, mHLRF and iHLRF
⇒ proposes nHLRF and two Lagrangian methods
⇒ nHLRF→ as efficient as iHLRF & more robust
⇒ Lagrangian→ Less efficient than HLRF's but more general and probably more suitable than HLRFs for large no. of rvs

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◎ Reliability Indices VS Reliability Methods

(β_{SM}, β_{SF}, β_{MVFOSM}, β_{HL}) (P_f)

Reliability indices

- Use partial & (i.e. ∇)
- Do not provide a framework to consider type of of input r.v's
- P_f could be estimated for special cases only

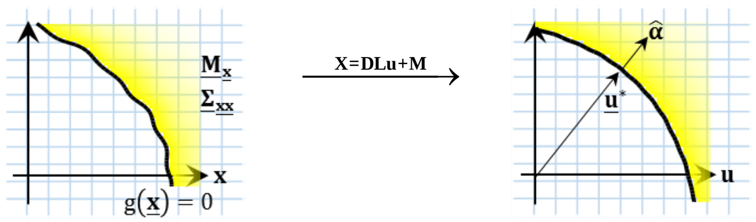
  (e.g., $P_f = \Phi(-\beta_{SM})$ when R, S ~ Normal)

→ Therefore, cannot be considered as reliability ________

cf. FORM/SORM ~ reliability methods

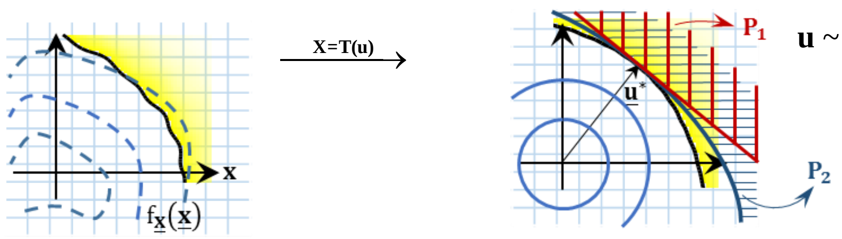
$$= \text{design point concept (e.g. } \beta_{HL}) + \begin{cases} \text{1) transformati achi } \mathbf{u} \sim N(\mathbf{0},\mathbf{I}) \\ \text{2) procedure t} \end{cases}$$

β_{HL} approach $\beta_{HL} = \alpha \mathbf{u}^*$



$$\begin{cases} \mathbf{M_u} = \mathbf{0} \\ \sum_{\mathbf{uu}} = \mathbf{I} \end{cases}$$

FORM/SORM



$\mathbf{u} \sim N(\mathbf{0},\mathbf{I})$

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◎ Probability in the Uncorrelated Standard Normal Space

$\mathbf{u} \sim N(\mathbf{0}, \mathbf{I})$ (cf. $\mathbf{Z} \sim N(\mathbf{0}, \mathbf{R})$)

Joint PDF

$$\varphi(\mathbf{u}) = \frac{1}{(2\pi)^{n/2}} \exp(-\frac{1}{2}\|\mathbf{u}\|^2) = \prod_{i=1}^{n} \varphi(u_i)$$

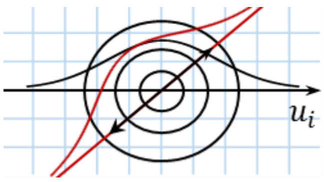
where $\varphi(u_i) = \frac{1}{\sqrt{2\pi}} \exp(-\frac{1}{2} u_i^2)$

① Rotational Symmetry

~the probability density is completely defined by from origin

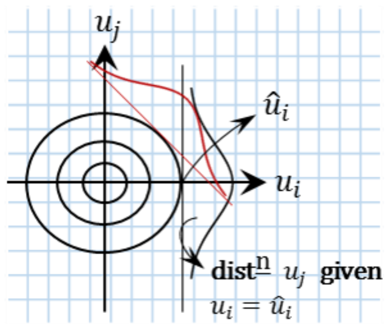
② Exponential Decay of Density

In r direction



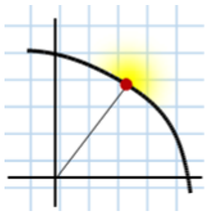
③ Exponential Decay of Density

In t direction



$\mathbf{u}^*$: Richest point in terms of prob. density

Therefore, approximation around $\mathbf{u}^*$ should be good



④ FORM : First Order Reliability Method