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457.646 Topics in Structural Reliability
In-Class Material: Class 15

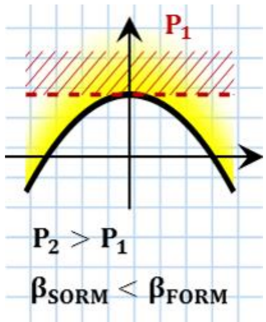
※ FERUM Example (SORM)

$$g(\mathbf{x}) = 1 - \frac{m_1}{s_1 y} - \frac{m_2}{s_2 y} - \left(\frac{P}{Ay}\right)^2 \le 0$$

$$\beta_{FORM} = 2.4661$$

(Curvature fitting)

$$\kappa_i \begin{cases} -1.548\times10^{-1} \\ -3.997\times10^{-2} \\ 8.903\times10^{-7} \end{cases}$$



$$\beta_{SORM} = 2.3506(T),\ 2.3596(B),\ 2.341(iB)$$

(Point fitting)

$$a_i \begin{cases} & + & - \\ & -6.2969\times10^{-2} & -4.0358\times10^{-2} \\ & -1.1986\times10^{-2} & -9.7461\times10^{-3} \\ & -1.3778\times10^{-1} & -1.1050\times10^{-1} \end{cases}$$

$$\beta_{SORM} = 2.3599(T),\ 2.3693(B),\ 2.3537(iB)$$

See supplement, "Importance and Sensitivity Vectors" (by A. Der Kiureghian)

➔ Main reference: Bjerager & Krenk (1989)

◎ FORM importance vector $\hat{\boldsymbol{\alpha}}$

FORM approximation of the limit-state function

$$G(\mathbf{u}) \cong G(\mathbf{u}^*) + \nabla G(\mathbf{u}^*)(\mathbf{u} - \mathbf{u}^*)$$

$$=$$

$$= \qquad (\beta - \hat{\boldsymbol{\alpha}}\mathbf{u})$$

$$G'(\mathbf{u}) = \frac{G(\mathbf{u})^{FO}}{\|\nabla G(\mathbf{u}^*)\|} =$$

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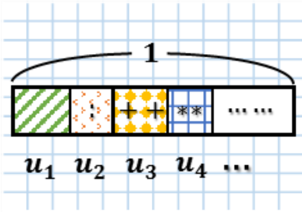

Note $\sigma_{G'}^2 = (\quad)\mathbf{\Sigma}_{\mathbf{uu}}(\quad)$

$= \quad \hat{\boldsymbol{\alpha}} \quad \hat{\boldsymbol{\alpha}}^T \quad = \hat{\boldsymbol{\alpha}}\hat{\boldsymbol{\alpha}}^T = \boxed{\qquad\qquad} =$

Contribution (percentage) of u_i

to the total (variability)

of the limit-state function $G'(\mathbf{u})$

① Magnitude of α_i^2 $\Rightarrow$ measure of relative importance (contribution to the uncertainty) of u_i's

② Sign of α_i $\Rightarrow$ nature of u_i's e.g., $g(\mathbf{X}) = R - S$

$G'(\mathbf{u}) = \beta - \hat{\boldsymbol{\alpha}}\mathbf{u} = \beta -$

α_i positive $\Rightarrow$ u_i capacity or demand

α_i negative $\Rightarrow$ u_i capacity or demand

Question) Importance of u_i $\overset{?}{=}$ Importance of X_i

i) Independent : $u_i = \Phi^{-1}\left[F_{X_i}(x_i)\right]$ OK

ii) Dependent: e.g., Nataf NOT OK

$$\mathbf{u} = \mathbf{L}_0^{-1}\mathbf{z} = \mathbf{L}_0^{-1}\begin{Bmatrix} \Phi^{-1}\left[F_{X_1}(x_1)\right] \\ \vdots \\ \Phi^{-1}\left[F_{X_n}(x_n)\right] \end{Bmatrix}$$

$\therefore$ $\hat{\alpha}_i$ does NOT (Measure importance / Indicate the nature) of x_i's

when X_i's are .

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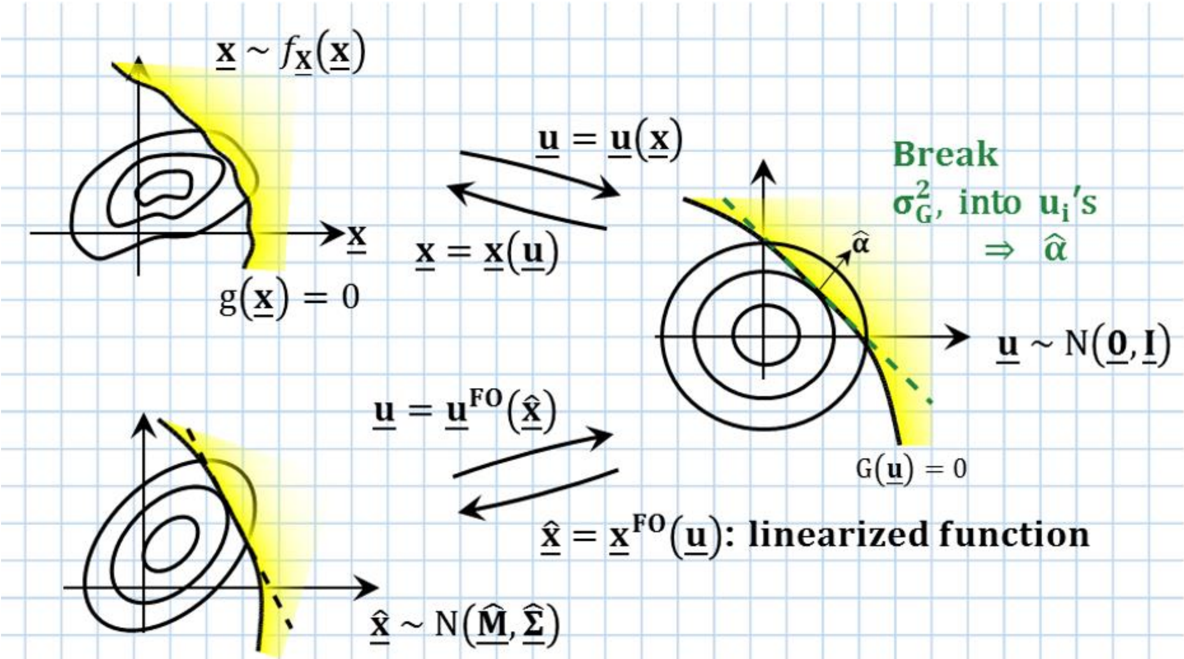
◎ **Form importance vector** $\hat{\gamma}$ **(Question: contribution/nature of** x_i **? Not** u_i**'s)**



Transform to "normal equivalent" of $\mathbf{x}$

Why? Want to keep ( ) distribution

Want to recover ( )

$\mathbf{u}^{FO}(\mathbf{x})$?

$$\begin{cases} \mathbf{u} = & \mathbf{u}(\mathbf{x}^*) + J_{\mathbf{u},\mathbf{x}}(\hat{\mathbf{x}} - \mathbf{x}^*) \\ \hat{\mathbf{x}} = & \mathbf{x}^* + J_{\mathbf{u},\mathbf{x}}^{-1}(\mathbf{u} - \mathbf{u}^*) \end{cases} \quad (*)$$

Note: Jacobians evaluated at $\mathbf{x} =$

$\hat{\mathbf{X}} \sim N(\hat{\mathbf{M}}, \hat{\mathbf{\Sigma}})$

$$\begin{cases} \hat{\mathbf{M}} = \\ \hat{\mathbf{\Sigma}} = \end{cases}$$

Substituting (*) into $G'(\mathbf{u}) = \beta - \hat{\boldsymbol{\alpha}}\mathbf{u}$,

$$G'(\mathbf{u}) = G''(\hat{\mathbf{x}}) = \beta - \hat{\boldsymbol{\alpha}}[\mathbf{u}^* + J_{\mathbf{u},\mathbf{x}}(\hat{\mathbf{x}} - \mathbf{x}^*)]$$

$$= \beta - \hat{\boldsymbol{\alpha}}\mathbf{u}^* - \hat{\boldsymbol{\alpha}}J_{\mathbf{u},\mathbf{x}}(\hat{\mathbf{x}} - \mathbf{x}^*)$$

$$= -\hat{\boldsymbol{\alpha}}J_{\mathbf{u},\mathbf{x}}(\hat{\mathbf{x}} - \mathbf{x}^*)$$

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$$\sigma_{G''}{}^2 = (-\hat{\boldsymbol{\alpha}} J_{\mathbf{u},\mathbf{x}})\hat{\Sigma}(-J_{\mathbf{u},\mathbf{x}}{}^{\mathrm{T}}\hat{\boldsymbol{\alpha}}^{\mathrm{T}})$$

$$= \hat{\boldsymbol{\alpha}} J_{\mathbf{u},\mathbf{x}} J_{\mathbf{u},\mathbf{x}}{}^{-1}(J_{\mathbf{u},\mathbf{x}}{}^{-1})^{\mathrm{T}} J_{\mathbf{u},\mathbf{x}}{}^{\mathrm{T}}\hat{\boldsymbol{\alpha}}^{\mathrm{T}}$$

$$= \qquad = \| \qquad \|^2 \quad =$$ Contribution of each $\hat{x}_i$?

$$\hat{\boldsymbol{\Sigma}} = \hat{\mathbf{D}}\hat{\mathbf{D}} + (\hat{\boldsymbol{\Sigma}} - \hat{\mathbf{D}}\hat{\mathbf{D}})$$

diagonal off-diagonal

$$\sigma^2_{G''} = \quad \hat{\boldsymbol{\alpha}} J_{\mathbf{u},\mathbf{x}}(\hat{\mathbf{D}}\hat{\mathbf{D}}) J_{\mathbf{u},\mathbf{x}}{}^T \hat{\boldsymbol{\alpha}}^T + \quad \hat{\boldsymbol{\alpha}} J_{\mathbf{u},\mathbf{x}}(\hat{\boldsymbol{\Sigma}} - \hat{\mathbf{D}}\hat{\mathbf{D}}) J_{\mathbf{u},\mathbf{x}}{}^T \hat{\boldsymbol{\alpha}}^T = 1$$

Contribution from variances $\sigma^2_{\hat{x}_i}$ Contribution from covariances $COV[\hat{x}_i, \hat{x}_j]$

Then, how about using $\hat{\boldsymbol{\alpha}} J_{\mathbf{u},\mathbf{x}}\hat{\mathbf{D}}$ instead of $\hat{\boldsymbol{\alpha}}$?

But not normalized yet.

$$\therefore \quad \hat{\boldsymbol{\gamma}} = \frac{\qquad}{\qquad}$$

i) Magnitude of $\hat{\gamma}_i^2$ $\rightarrow$ contribution (importance) of $\hat{x}_i$ or x_i

ii) Sign of $\hat{\gamma}_i$ $\rightarrow$ nature of $\hat{x}_i$ or x_i

Note : $G''(\hat{\mathbf{x}}) = -\hat{\boldsymbol{\alpha}} J_{\mathbf{u},\mathbf{x}}(\hat{\mathbf{x}} - \mathbf{x}^*)$

$\hat{\gamma}_i$ positive $\rightarrow$ ______________ type r.v x_i

$\hat{\gamma}_i$ negative $\rightarrow$ ______________ type r.v x_i

Note : when $\mathbf{x}$ are independent, $\hat{\boldsymbol{\alpha}} = \hat{\boldsymbol{\gamma}}$?

$$\hat{\boldsymbol{\Sigma}} = (J_{\mathbf{u},\mathbf{x}}{}^{-1})(J_{\mathbf{u},\mathbf{x}}{}^{-1})^T = \hat{\mathbf{D}}\hat{\mathbf{D}} + (\hat{\boldsymbol{\Sigma}} - \hat{\mathbf{D}}\hat{\mathbf{D}})$$

$$\hat{\mathbf{D}} =$$

$$\hat{\boldsymbol{\gamma}} = \frac{\hat{\boldsymbol{\alpha}} J_{\mathbf{u},\mathbf{x}}\hat{\mathbf{D}}}{\left\| \hat{\boldsymbol{\alpha}} J_{\mathbf{u},\mathbf{x}}\hat{\mathbf{D}} \right\|} =$$

※ **FERUM Example** ($\hat{\boldsymbol{\alpha}}$ and $\hat{\boldsymbol{\gamma}}$)

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457.646 Topics in Structural Reliability

In-Class Material: Class 16

◎ FORM importance vectors: $\hat{\boldsymbol{\alpha}}, \ \hat{\boldsymbol{\gamma}}$

◎ Generalized Reliability Importance Measure (GRIM; Kim and Song, 2018)

See supplement "Generalized Reliability Importance Measure"

◎ FORM parameter sensitivities of β **:** $\dfrac{\partial \beta}{\partial \theta}$ (Bjerager & Krenk, 1989; See Supp)

θ

- $\theta \in \boldsymbol{\theta}_g$: parameters in , $g(\mathbf{x};\boldsymbol{\theta}_g)$

  e.g. $g(\mathbf{x};\boldsymbol{\theta}_g) = 1 - \dfrac{M}{M_u} - \left(\dfrac{P}{P_u}\right)^2 \le 0 \qquad \boldsymbol{\theta}_g = \{M_u \ P_u$

- $\theta \in \boldsymbol{\theta}_f$: ________________ parameters in $f_{\mathbf{x}}(\mathbf{x};\boldsymbol{\theta}_f)$

  e.g. $\sigma, \ \mu, \ \rho, \ \lambda, \ \xi, \ b$

① Case $\theta \in \boldsymbol{\theta}_f$ (distribution) ※ Derivations → see Supplement

$$\frac{d\beta}{d\theta} = \hat{\boldsymbol{\alpha}} \frac{\partial \mathbf{u}(\mathbf{x}^*, \theta)}{\partial \theta}$$

Obtain $\hat{\boldsymbol{\alpha}}$ by FORM analysis

Derive $\dfrac{\partial \mathbf{u}(\mathbf{x}, \theta)}{\partial \theta}$ from $\mathbf{u}(\mathbf{x}, \theta)$ and evaluate it at $\mathbf{x} = \mathbf{x}^*$

⇒ Vector version $\nabla_{\boldsymbol{\theta}_f} \beta = \hat{\boldsymbol{\alpha}} J_{\mathbf{u},\boldsymbol{\theta}_f}(\mathbf{x}^*, \boldsymbol{\theta}_f)$

e.g. $\mathbf{x} \sim$ s.i. Normal

$$\mathbf{u} = \mathbf{L}^{-1}\mathbf{D}^{-1}(\mathbf{X} - \mathbf{M})$$
$$= \mathbf{D}^{-1}(\mathbf{X} - \mathbf{M})$$

$u_1 =$, $u_2 =$ $\cdots$

$\dfrac{\partial u_1}{\partial \sigma_1} =$ $\qquad \therefore \dfrac{\partial u_1}{\partial \sigma_1}(\mathbf{x}^*) =$

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② Case $\theta \in \boldsymbol{\theta}_g$ (limit-state function)

$$\frac{d\beta}{d\theta} = \frac{1}{\left\| \nabla_{\mathbf{u}} G(\mathbf{u}^*, \theta) \right\|} \frac{\partial g(\mathbf{x}^*, \theta)}{\partial \theta}$$

↖ FORM ↖ derive from $g(\mathbf{x})$

⇒ Vector version

$$\nabla_{\boldsymbol{\theta}_g} \beta = \frac{1}{\left\| \nabla_{\mathbf{u}} G(\mathbf{u}^*, \boldsymbol{\theta}) \right\|} \nabla_{\boldsymbol{\theta}_g} g(\mathbf{x}^*, \boldsymbol{\theta}_g)$$

e.g.

$$g(\mathbf{x}, \boldsymbol{\theta}_g) = 1 - \frac{M}{M_u} - \left(\frac{P}{P_u}\right)^2 \leq 0$$

θ_g

$$\frac{\partial g}{\partial \theta} = \qquad\qquad \therefore \frac{\partial g}{\partial \theta}(\mathbf{x}^*) =$$

◎ Parameter Sensitivities of failure probability P_f : $\frac{\partial P_f}{\partial \theta}$ **?**

Recall $P_f = \Phi(\qquad)$

$$\frac{dP_f}{d\theta} =$$

Vector version:

$$\nabla_{\boldsymbol{\theta}} P_f = -\varphi(-\beta) \nabla_{\boldsymbol{\theta}} \beta$$

◎ Parameter sensitivities w.r.t. alternative parameters

$$\boldsymbol{\theta}_f = \boldsymbol{\theta}_f(\boldsymbol{\theta}_{f'})$$

↑ λ, ξ ↑ μ, σ

$$e.g. \begin{bmatrix} \lambda \\ \xi \end{bmatrix} = \begin{bmatrix} \ln \mu - 0.5 \ln[1 + (\frac{\sigma}{\mu})^2] \\ \sqrt{\ln[1 + (\frac{\sigma}{\mu})^2]} \end{bmatrix}$$

$\boldsymbol{\theta}_f$ $\qquad$ $\boldsymbol{\theta}_f(\boldsymbol{\theta}_{f'})$ ← μ, σ

$$\nabla_{\boldsymbol{\theta}_{f'}} \beta = \nabla_{\boldsymbol{\theta}_f} \beta \cdot$$

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% FERUM Input File for CRC CH14 Example (with Parameter)

```
clear probdata femodel analysisopt gfundata randomfield systems results
output_filename

output_filename = 'output_Ch14_Example_param.txt';


probdata.marg(1,:) =  [ 1   2.5e5   2.5e5*0.3   2.5e5  0 0 0 0 0];
probdata.marg(2,:) =  [ 1  1.25e5  1.25e5*0.3  1.25e5  0 0 0 0 0];
probdata.marg(3,:) =  [15   2.5e6   2.5e6*0.2   2.5e6  0 0 0 0 0];
probdata.marg(4,:) =  [16  4.0e7   4.0e7*0.1   4.0e7  0 0 0 0 0];


probdata.correlation = [1.0 0.5 0.3 0.0;
                    0.5 1.0 0.3 0.0;
                    0.3 0.3 1.0 0.0;
                    0.0 0.0 0.0 1.0];

probdata.parameter = distribution_parameter(probdata.marg);

analysisopt.ig_max    = 100;
analysisopt.il_max    = 5;
analysisopt.e1       = 0.001;
analysisopt.e2       = 0.001;
analysisopt.step_code = 0;
analysisopt.grad_flag = 'DDM';
analysisopt.sim_point = 'dspt';
analysisopt.stdv_sim  = 1;
analysisopt.num_sim   = 100000;
analysisopt.target_cov = 0.05;


gfundata(1).evaluator = 'basic';
gfundata(1).type = 'expression';
gfundata(1).parameter = 'yes'; % "We have a parameter in the limit-state
function"
gfundata(1).thetag = [0.03]; % default value of S1
gfundata(1).expression = '1-x(1)/gfundata(1).thetag(1)/x(4)-
x(2)/0.015/x(4)-(x(3)/0.190/x(4))^2';
gfundata(1).dgdq = { '-1/gfundata(1).thetag(1)/x(4)' ;
                  '-1/0.015/x(4)';
                  '-2*x(3)/0.190^2/x(4)^2';

'x(1)/gfundata(1).thetag(1)/x(4)^2+x(2)/0.015/x(4)^2+2*x(3)^2/0.190^2/x(4
)^3'};
gfundata(1).dgthetag = {'x(1)/x(4)/gfundata(1).thetag(1)^2'}; %
Derivative w.r.t. S1

femodel = 0;
randomfield.mesh = 0;
```

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◎ Importance Vectors Using Parameter Sensitivities

⇒ Use $\nabla_{\mathbf{M}}\beta$ and $\nabla_{\mathbf{D}}\beta$ to quantify importance of random variables?

$\frac{\partial\beta}{\partial\mu_1} \gg \frac{\partial\beta}{\partial\mu_2}$ → more to than

① Importance vector $\boldsymbol{\delta}$

$$\boldsymbol{\delta} = \nabla_{\mathbf{M}}\beta \cdot \mathbf{D}$$

$$= \left[\frac{\partial\beta}{\partial\mu_1}\cdot \quad , \frac{\partial\beta}{\partial\mu_2}\cdot \quad \cdots, \frac{\partial\beta}{\partial\mu_n}\cdot \quad \right]$$

Why?

- X_i's Can have different units & dimensions (therefore μ_i's) ⇒ make it dimensionless
- Assume variations in $\mu_i \propto$
- Change in β when μ_i change by

② Importance vector $\boldsymbol{\eta}$

$$\boldsymbol{\eta} = \nabla_{\mathbf{D}}\beta \cdot \mathbf{D}$$

$$= \left[\frac{\partial\beta}{\partial\sigma_1}\cdot \quad , \frac{\partial\beta}{\partial\sigma_2}\cdot \quad ,\cdots, \frac{\partial\beta}{\partial\sigma_n} \quad \right]$$

Change in β when σ_i change by

③ Upgrade worth $\mathbf{I}_{\boldsymbol{\theta}}$

$$\mathbf{I}_{\boldsymbol{\theta}} = -\nabla_{\boldsymbol{\theta}} P_f \mathbf{D}_{\boldsymbol{\theta}}$$

$$= \left[-\frac{\partial P_f}{\partial\theta_1} \quad \cdots, -\frac{\partial P_f}{\partial\theta_n} \quad \right]$$

$$\mathbf{D}_{\boldsymbol{\theta}} = \begin{bmatrix} \nwarrow & & \\ & \Delta\theta_i & \\ & & \searrow \end{bmatrix}$$

Change in θ_i that can be achieved by unit _______

- Der Kiureghian, Ditlevsen & Song (2007)
- Song & Kang (2009)

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◎ Use of sensitivity / Importance Vectors

$(\nabla_{\boldsymbol{\theta}}\beta)$ $(\hat{\boldsymbol{\alpha}}, \hat{\boldsymbol{\gamma}}\ \boldsymbol{\delta}\ \boldsymbol{\eta})$

① To identify important rv's

② To update β for small increment

$$\beta_{new} \cong \beta_{old} + \sum_i \frac{\partial\beta}{\partial\theta_i} \cdot \Delta\theta_i$$

③ Reliability Based Design Optimization

$\Rightarrow \frac{\partial\beta}{\partial\theta}$ needed to facilitate the use of ( )-based optimizers

④ To compute PDF of a function $y(\mathbf{x})$

$$F_Y(\theta) = P(Y(\mathbf{x}) \le \theta)$$

$= P(Y(\mathbf{x}) - \theta \le 0)$ here consider $Y(\mathbf{x})$-θ as the limit state function $g(\mathbf{x},\theta)$

$$\simeq \Phi(-\beta(\theta))$$

$$f_Y(\theta) = \frac{dF_y(\theta)}{d\theta} = -\varphi(-\beta(\theta))\frac{d\beta}{d\theta}$$

⑤ To help gain insight of the reliability problem

IV. System Reliability

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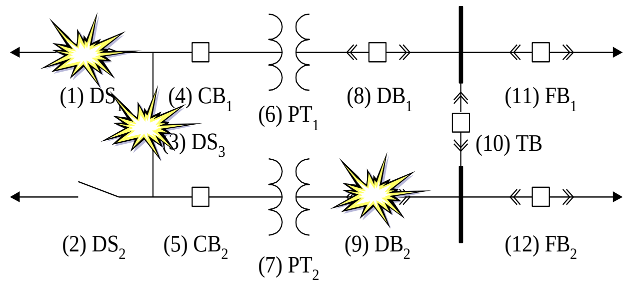
System reliability?

Failure event	E_{sys}
Abnormal flight (engine)	$E_1 \cup E_2$
Emergency	$E_1 E_2$
Landing at nearby airport	$E_1 \bar{E}_2 \cup \bar{E}_1 E_2$

$P(E_{sys})$?

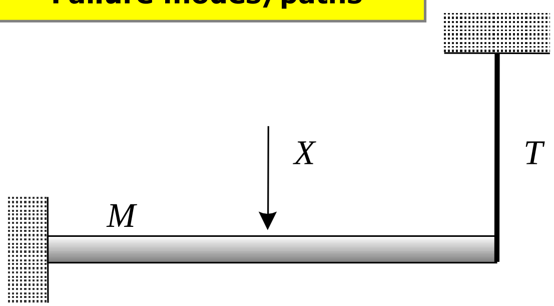
System reliability in structural engineering

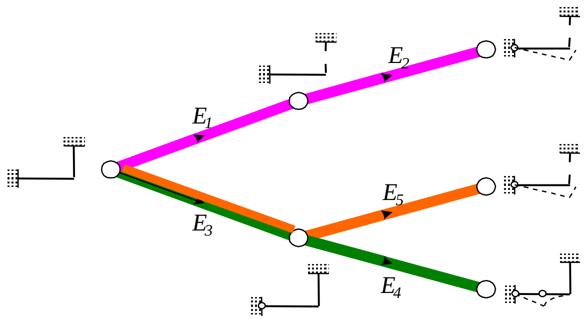
Lifeline networks



$$\begin{aligned}E_{system} = {} & E_1E_2 \cup E_4E_5 \cup E_4E_7 \cup E_4E_9 \cup E_5E_6 \cup \\ & E_6E_7 \cup E_6E_9 \cup E_5E_8 \cup E_7E_8 \cup E_8E_9 \cup \\ & E_{11}E_{12} \cup E_1E_3E_5 \cup E_1E_3E_7 \cup \boxed{E_1E_3E_9} \cup \\ & E_2E_3E_4 \cup E_2E_3E_6 \cup E_2E_3E_8 \cup E_4E_{10}E_{12} \cup \\ & E_6E_{10}E_{12} \cup E_8E_{10}E_{12} \cup E_5E_{10}E_{11} \cup E_7E_{10}E_{11} \cup \\ & E_9E_{10}E_{11} \cup E_1E_3E_{10}E_{12} \cup E_2E_3E_{10}E_{11}\end{aligned}$$

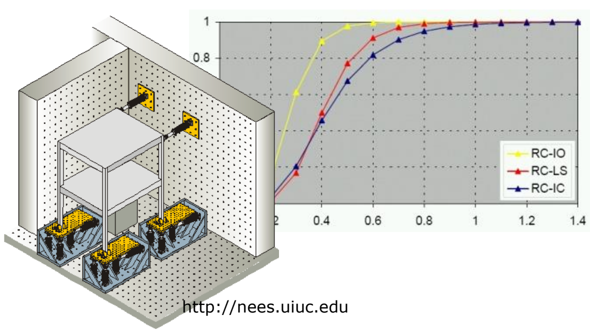
Failure modes/paths





$$E_{system} = (E_1 \cap E_2) \cup (E_3 \cap E_4) \cup (E_3 \cap E_5)$$

Damage/loss aggregation



$$L_{system} = f(\mathbf{D}, \mathbf{\Theta})$$

$$\mathrm{E}[L_{system}] \cong f(\mathrm{E}[\mathbf{D}], \mathrm{E}[\mathbf{\Theta}])$$

$$\mathrm{Var}[L_{system}] \cong \nabla f^{\mathrm{T}} \mathbf{\Sigma} \nabla f$$

$$P(L_{system} \geq c) \cong 1 - \Phi\left(\frac{c - \mathrm{E}[L_{system}]}{\sqrt{\mathrm{Var}[L_{system}]}}\right)$$

Outline

I. System reliability: definitions, existing methods and challenges

II. Bounds of system reliability by linear programming ('LP bounds')

III. Matrix-based system reliability (MSR) method

I. System Reliability:

- definitions, existing methods and challenges

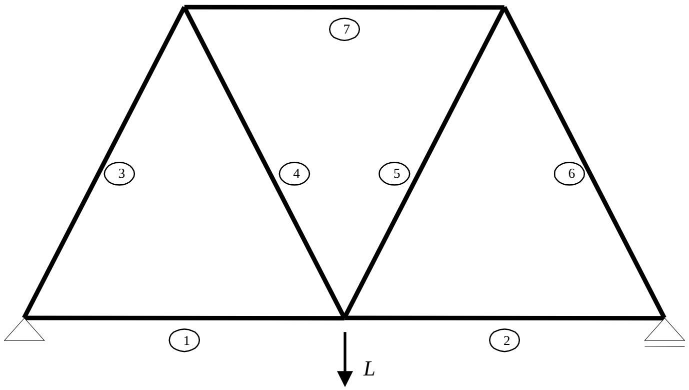
Definition of system: (1) series system

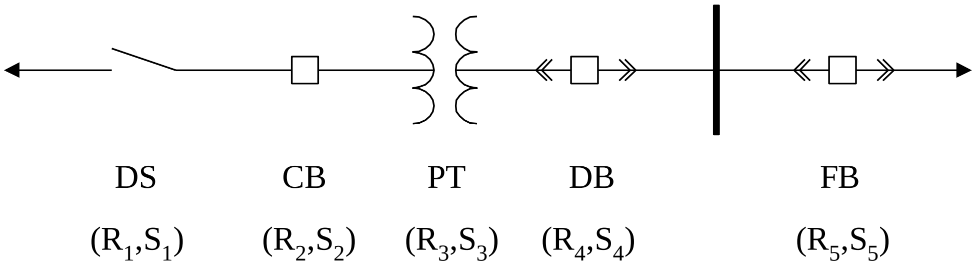
- System fails ***if any*** of its component events occur

$$E_{\text{system}} = \bigcup_{i=1}^{n} E_i$$

- Systems with no redundancy
- Examples: 1) statically determinate structure

  2) electrical substation with single-transmission-line





Song, J., and A. Der Kiureghian (2003, JEM ASCE)

Song, J., and A. Der Kiureghian (2003, ICASP9)

Definition of system: (2) parallel system

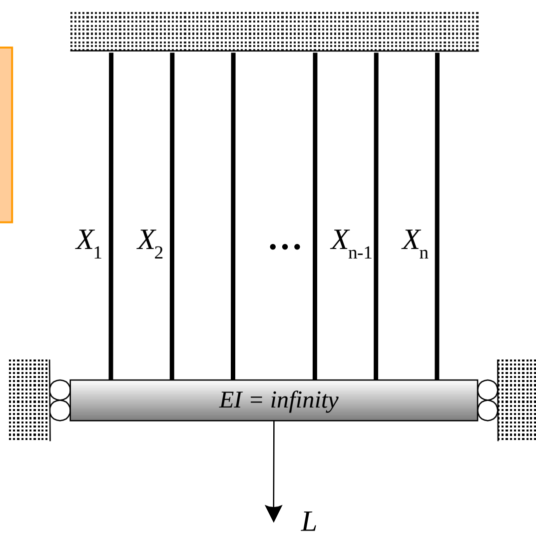
- System fails ***only if every*** component event occurs

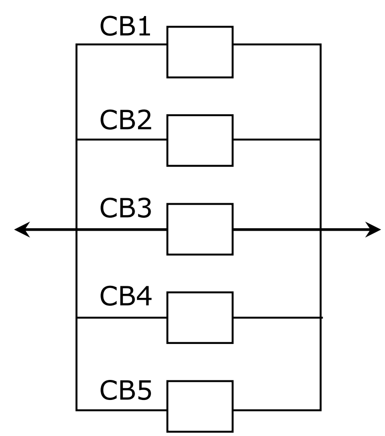
$$E_{\text{system}} = \bigcap_{i=1}^{n} E_i$$

- Systems with maximum redundancy
- Examples: 1) a bunch of wires or cables.

  2) electrical substation with equipment items in parallel.

Song, J., and A. Der Kiureghian (2003, JEM ASCE)





Definition of system: (3) general system

- System that is ***neither series or parallel*** system

  1) Cut-set system:

  - a series system of sub-parallel systems

$$E_{\text{system}} = \bigcup_{k=1}^{K} C_k = \bigcup_{k=1}^{K} \bigcap_{i \in C_k} E_i$$

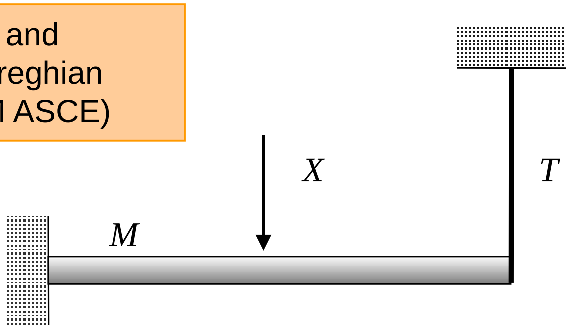
  2) Link-set system:

  - a parallel system of sub-series systems

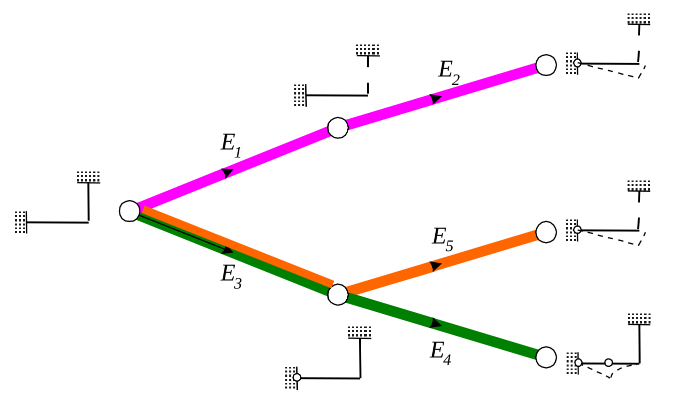
$$E_{\text{system}} = \bigcap_{l=1}^{L} L_l = \bigcap_{l=1}^{L} \bigcup_{i \in L_l} E_i$$

- Example: a structure with multiple failure paths (scenarios) ~ a cut-set system

Song, J., and
A. Der Kiureghian
(2003, JEM ASCE)



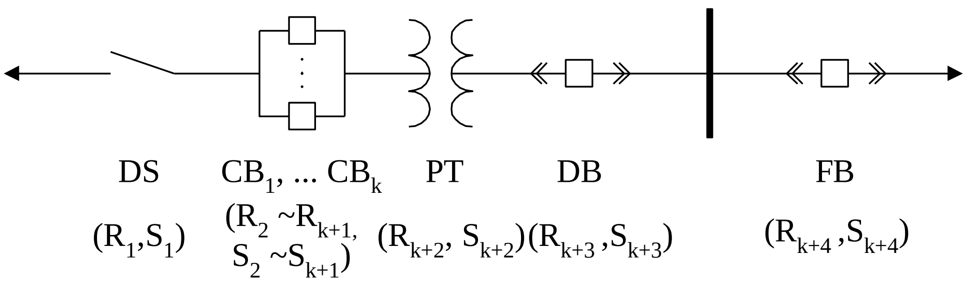
$$E_{\text{system}} = \underbrace{(E_1 \cap E_2)}_{\text{Scenario 1}} \cup \underbrace{(E_3 \cap E_4)}_{\text{Scenario 2}} \cup \underbrace{(E_3 \cap E_5)}_{\text{Scenario 3}}$$



* Component failure events and failure paths

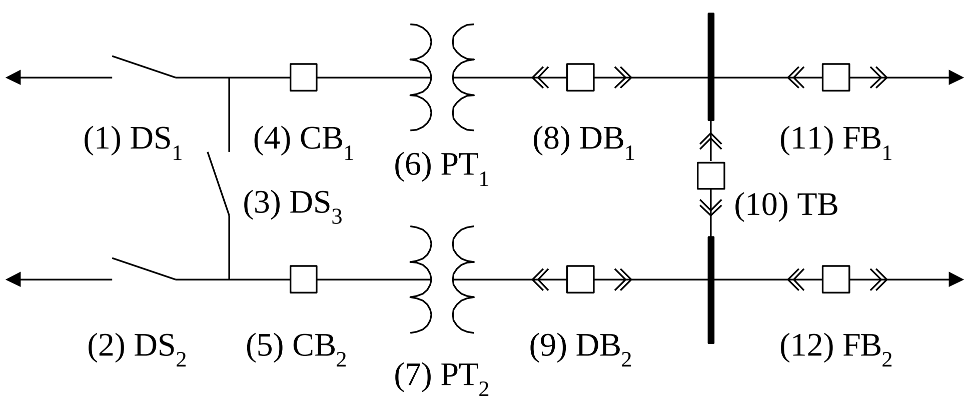
(3) General system (contd.)

- Example: electrical substations (cut-set systems)



$$P(E_{system}) =$$
$$P[E_1 \cup (E_2 E_3 \cdots E_{k+1}) \cup E_{k+2} \cup E_{k+3} \cup E_{k+4}]$$

* 5 cut sets, k+4 components



$$P(E_{system}) =$$
$$P(E_1E_2 \cup E_4E_5 \cup E_4E_7 \cup E_4E_9 \cup E_5E_6 \cup E_6E_7 \cup$$
$$E_6E_9 \cup E_5E_8 \cup E_7E_8 \cup E_8E_9 \cup E_{11}E_{12} \cup$$
$$E_1E_3E_5 \cup E_1E_3E_7 \cup E_1E_3E_9 \cup E_2E_3E_4 \cup$$
$$E_2E_3E_6 \cup E_2E_3E_8 \cup E_4E_{10}E_{12} \cup E_6E_{10}E_{12} \cup$$
$$E_8E_{10}E_{12} \cup E_5E_{10}E_{11} \cup E_7E_{10}E_{11} \cup E_9E_{10}E_{11} \cup$$
$$E_1E_3E_{10}E_{12} \cup E_2E_3E_{10}E_{11})$$

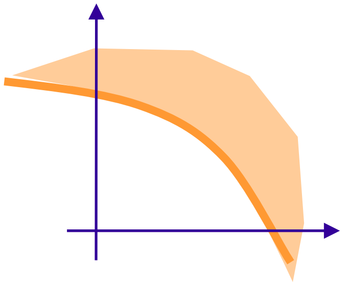
* 25 cut sets, 12 components

Song, J., and A. Der Kiureghian (2003, ICASP9)

“component” reliability vs “system” reliability

- Component reliability analysis: $P(E_i) = P\left(g_i(\mathbf{X}) \le 0\right) = \int_{g(\mathbf{X})\le 0} f_{\mathbf{X}}(\mathbf{x})d\mathbf{x}$

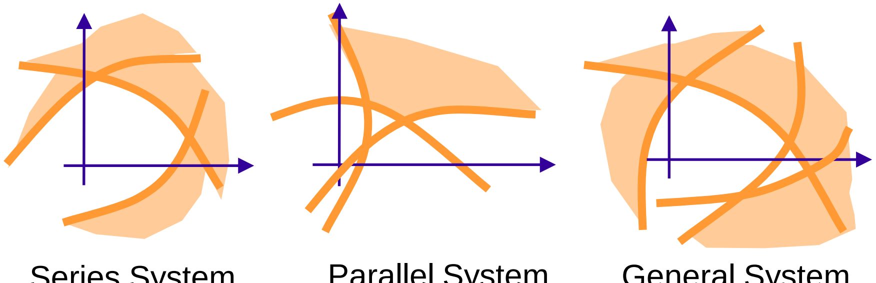
  1) FORM/SORM
  2) Response surface method
  3) Monte Carlo simulations
  4) Importance samplings



- System reliability analysis: $P(E_{\text{system}}) = P\left(\bigcup\bigcap g_i(\mathbf{x}) \le 0\right) = \int_{D_{system}} f_{\mathbf{X}}(\mathbf{x})d\mathbf{x}$

  1) Complexity
  2) Dependence between component events
  3) Lack of information

  ~ synthesize components reliabilities or perform simulations



Existing methods: (1) inclusion-exclusion formula

* Series system

$$P\left(\bigcup_{i=1}^{n} E_i\right) = \sum_{i=1}^{n} P(E_i) - \sum_{i=1}^{n-1}\sum_{j=i+1}^{n} P(E_i E_j) + \cdots + (-1)^{n-1} P(E_1 E_2 \cdots E_n)$$

* Parallel system

$$P\left(\bigcap_{i=1}^{n} E_i\right) = 1 - P\left(\bigcup_{i=1}^{n} \overline{E}_i\right) = 1 - \sum_{i=1}^{n} P(\overline{E}_i) + \cdots$$

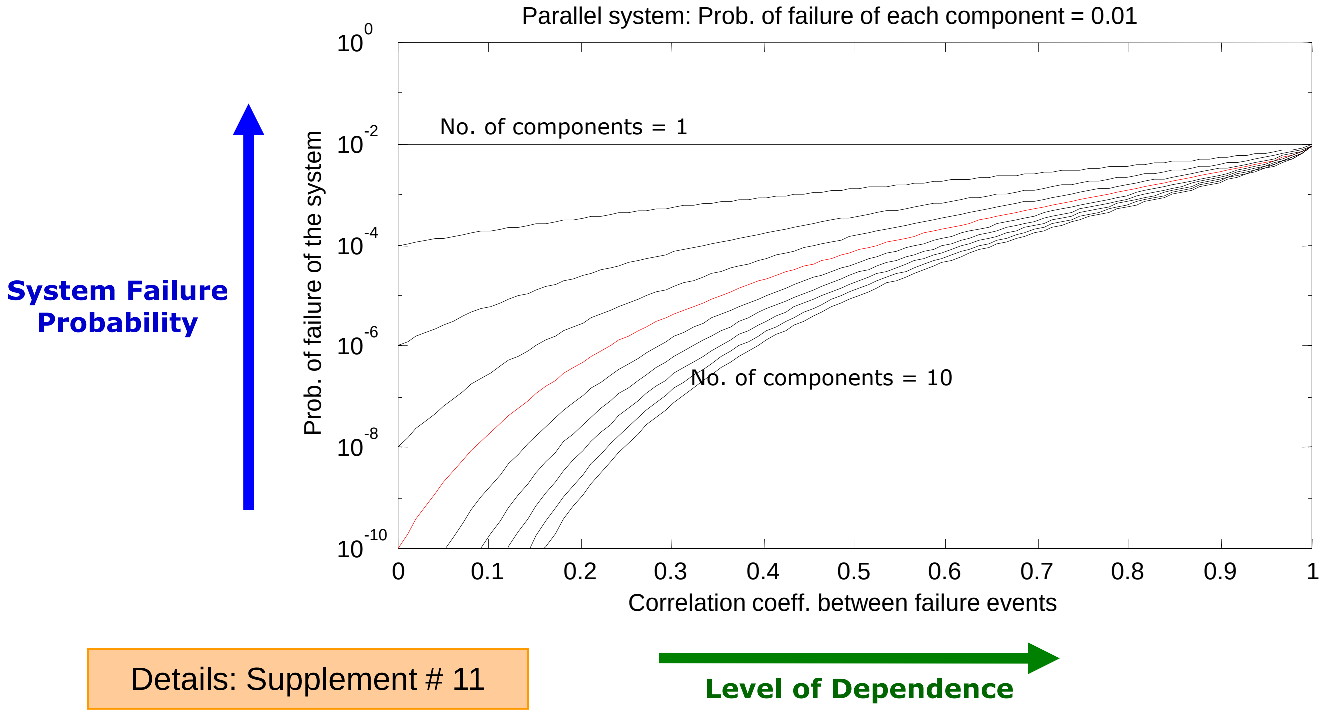
* Cut-set system

$$P\left(\bigcup_{i=1}^{n} C_i\right) = \sum_{i=1}^{n} P(C_i) - \sum_{i=1}^{n-1}\sum_{j=i+1}^{n} P(C_i C_j) + \cdots + (-1)^{n-1} P(C_1 C_2 \cdots C_n)$$

- the number of terms increase exponentially; $2^n - 1$
- requires all the joint probabilities: $P(E_i)$, $P(E_i E_j)$, $P(E_i E_j E_k)$, …
- useful only if component events are statistically independent: $P(E_i E_j) = P(E_i)P(E_j)$
  ~ need marginal probabilities only

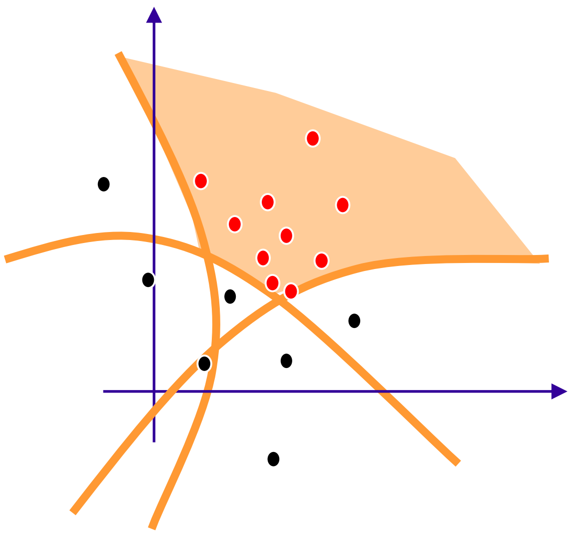
** Dependence and system reliability

- A parallel system with 1~10 components with $P(E_i) = 0.01$
  - ~ e.g. n=5: 10^{-10} (independent) ~10^{-2} (perfectly dependent)



Details: Supplement # 11

Existing methods: (2) simulations



$$P(E_{\text{system}}) = \int_D f_{\mathbf{X}}(\mathbf{x}) d\mathbf{x}$$

$$\cong \frac{\#(\mathbf{x} \in D)}{\#(\mathbf{x})}$$

~ Count the number of samples in the system failure domain and estimate the ratio.

- Monte Carlo simulations, importance sampling, directional sampling, etc.
- Independent random variables: easily generated.
- Dependent random variables: need joint probability density function
  ~ not available in many cases.
- Independence assumption will lead to errors in estimating system reliability

Existing methods: (3) bounding formulas

It is desirable to derive **bounds** on system probability which involve **low-order component probabilities:**

- ✓ Uni-component probabilities: $P(E_i) = P_i$
- ✓ Bi-component probabilities: $P(E_i E_j) = P_{ij}$
- ✓ Tri-component probabilities: $P(E_i E_j E_k) = P_{ijk}$

➢ **Series System**

1) Uni-component bounds (Boole 1854; Fréchet 1953) $\max_i P_i \le P\left(\bigcup_{i=1}^{n} E_i\right) \le \min\left(1, \sum_{i=1}^{n} P_i\right)$

2) Bi-component bounds (Kounias 1968; Hunter 1976; Ditlevsen 1979)

$$P_1 + \sum_{i=2}^{n} \max\left(0, P_i - \sum_{j=1}^{i-1} P_{ij}\right) \le P\left(\bigcup_{i=1}^{n} E_i\right) \le P_1 + \sum_{i=2}^{n} \left(P_i - \max_{j<i} P_{ij}\right)$$

3) Tri-component bounds (Hohenbichler & Rackwitz 1983; Zhang 1993)

$$P_1 + P_2 - P_{12} + \sum_{i=3}^{n} \max\left(0, P_i - \sum_{j=1}^{i-1} P_{ij} + \max_{k\in\{1,2,\ldots,i-1\}} \sum_{\substack{j=1 \\ j\neq k}}^{i-1} P_{ijk}\right) \le P\left(\bigcup_{k=1}^{n} E_k\right) \le P_1 + P_2 - P_{12} + \sum_{i=3}^{n} \left[P_i - \max_{\substack{k\in\{2,3,\ldots,i-1\} \\ j<k}} \left(P_{ik} + P_{ij} - P_{ijk}\right)\right]$$

Existing method: (3) bounding formulas (contd.)

➢ Parallel System

- Uni-component bounds (Boole 1854; Fréchet 1953)

$$\max\left(0, \sum_{i=1}^{n} P_i - (n-1)\right) \leq P\left(\bigcap_{i=1}^{n} E_i\right) \leq \min_i P_i$$

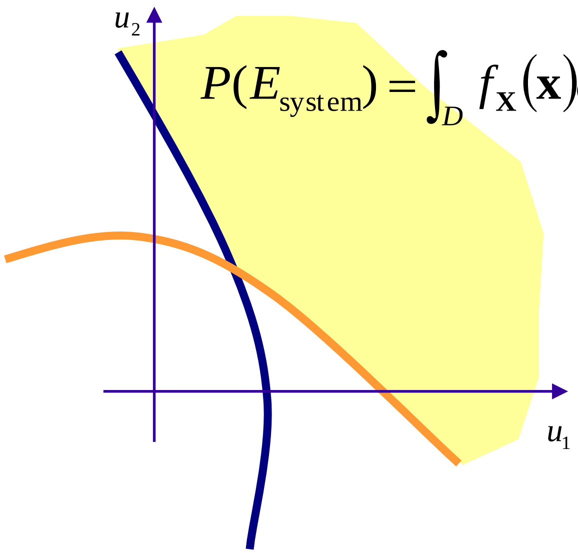
- No higher-order bounds available.

Note: De Morgan’s rule can be used to convert a parallel system to a series system, allowing use of bi- and tri-component bounding formulas for series systems.

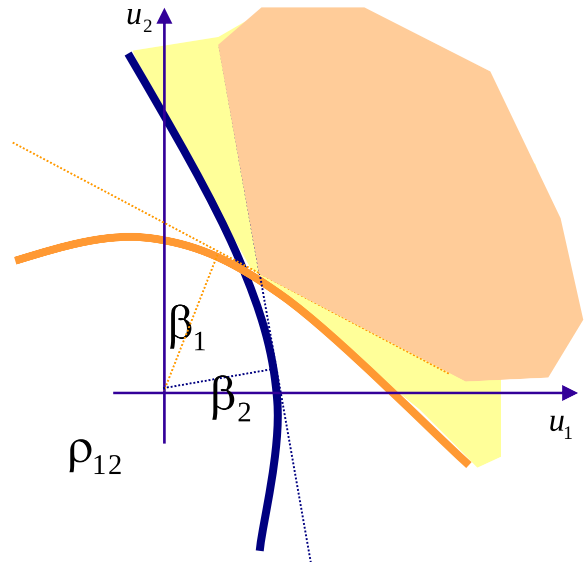
➢ General System

- No bounding formulas exist.

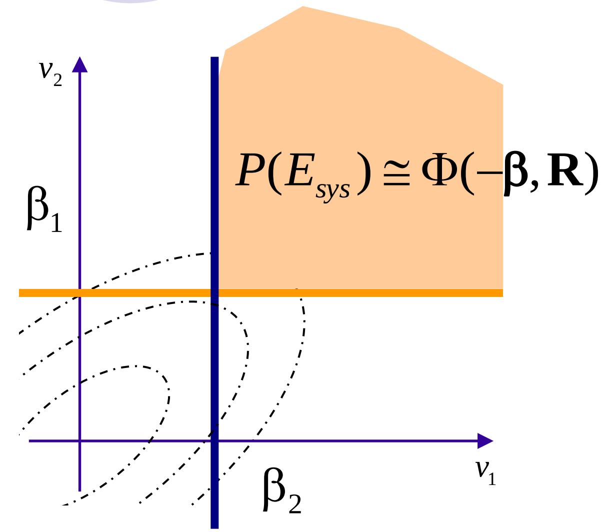
Existing methods: (4) FORM approximation



Original system reliability problem



FORM analysis for each component



Integration in standard normal space

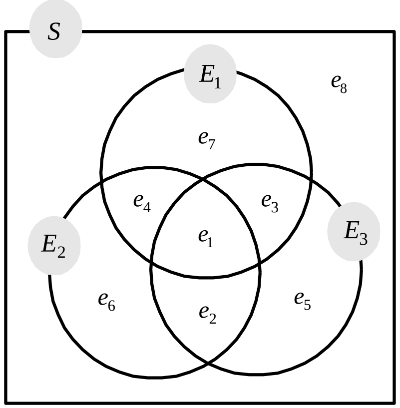
- For parallel and series system
- Find the corresponding volume in standard normal space based on FORM analyses of component events
- Errors depend on the level of nonlinearity and complexity of domain.

System reliability: challenges

- **Complexity** of system problems
  - large number of components, component states, cut sets, link sets, etc.
  - difficulty in identifying cut sets or link sets
  - computational challenges (speed and memory)
- **Dependence** between component states
  - “environmental dependence” or “common source effect”
  - members and materials by the same manufacturer or supplier
  - analysis as “independent components” is simple, but may be misleading.
- **Diversity/Lack** of available information on components
  - missing information
  - various types of information
  - should be flexible in obtaining information

II. Bounds on System Reliability by Linear Programming (‘LP Bounds’)

Bounds by linear programming (LP)



Probabilities of basic MECE events: $p_i \equiv P(e_i), \quad i = 1, 2, \ldots, 2^n$

1. The system failure probability $$P(E_{\text{system}}) = \sum_{r: e_r \subseteq E_{\text{system}}} p_r = \mathbf{c}^{\mathrm{T}}\mathbf{p}$$

2. Axioms of probability: $$\sum_{i=1}^{2^n} p_i = 1 \quad \text{and} \quad p_i \geq 0, \ \forall i$$

3. Available information on component probabilities $$P(E_i) = \sum_{r: e_r \subseteq E_i} p_r = P_i \ \ (\geq P_i, \ \leq P_i)$$ $$P(E_i E_j) = \sum_{r: e_r \subseteq E_i E_j} p_r = P_{ij} \ \ (\geq P_{ij}, \ \leq P_{ij}) \ \cdots$$

$$\begin{aligned} &\text{minimize (maximize)} \ \mathbf{c}^{\mathrm{T}}\mathbf{p} \\ &\text{subject to} \quad \mathbf{a}_1\mathbf{p} = \mathbf{b}_1 \\ &\qquad\qquad\quad\ \mathbf{a}_2\mathbf{p} \geq \mathbf{b}_2 \end{aligned}$$

Linear Programming Problem

* Song, J., and A. Der Kiureghian (2003). Bounds on system reliability by linear programming. *Journal of Engineering Mechanics*, ASCE, 129(**6**): 627-636.

Merits of LP approach

✓ Bounds for general systems.

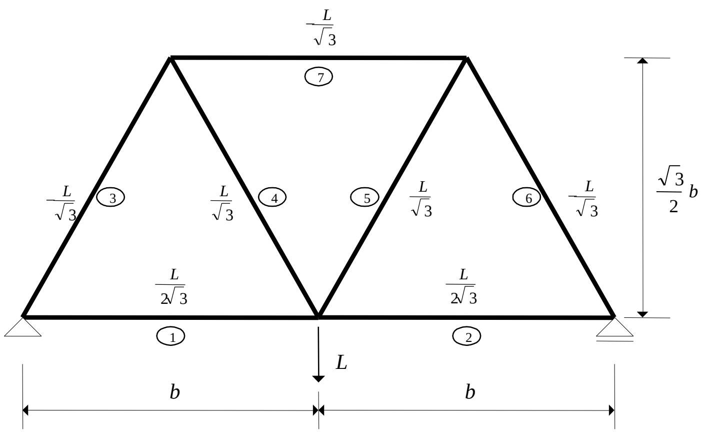
✓ Any type of information on component probabilities can be used.
- Equality: $P_{ij} = 0.02$
- Inequality: $P_{ij} \leq 0.01,\ \ 0.05 \leq P_i \leq 0.07, P_3 \leq P_2$
- Partial: $P_1 = 0.01$, $\boldsymbol{P_2 = ?}$, $P_3 = 0.03$

✓ Finds the *narrowest* possible bounds for the given information. (This is not guaranteed for existing formulas for series systems involving bi- or higher-order component probabilities.)

✓ Can be used to compute importance and sensitivity measures, and updated system reliability.

Application to structural system reliability

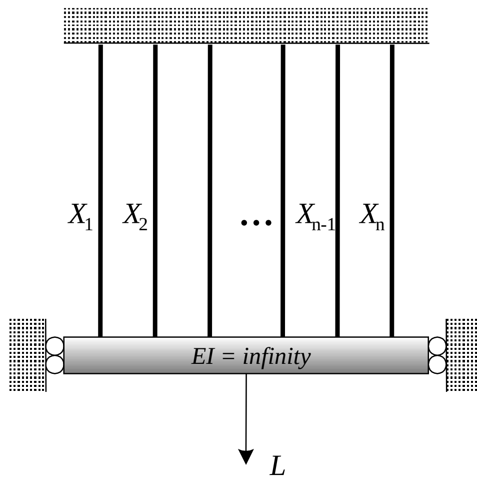
Statically determinate truss (series system)



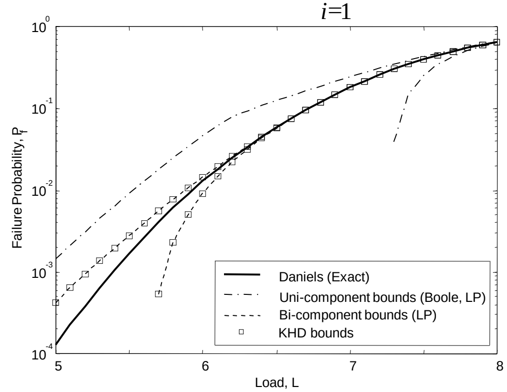
$$E_{\text{system}} = \bigcup_{i=1}^{n} E_i$$

1. Narrowest bounds
2. Incomplete set of probabilities
3. Inequality-type information

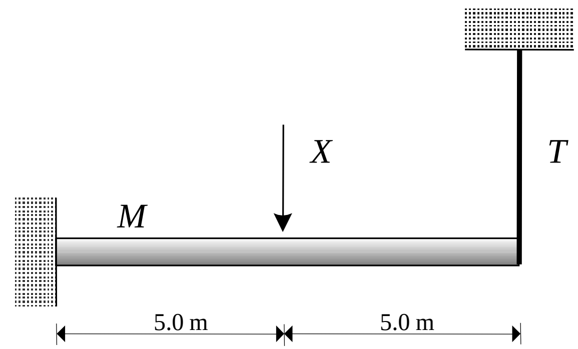
Daniels' parallel system



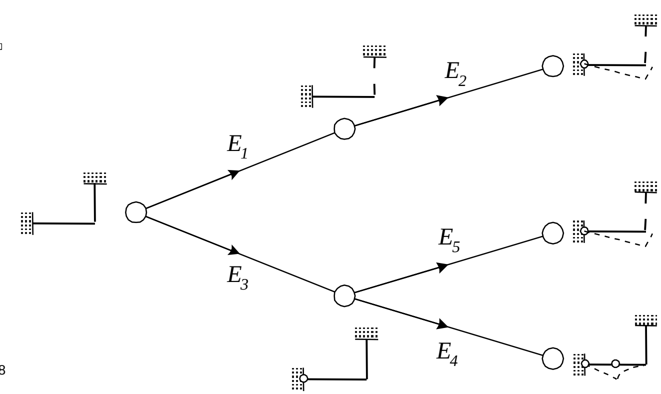
$$E_{\text{system}} = \bigcap_{i=1}^{n} E_i$$



Cantilever beam – bar (general system)

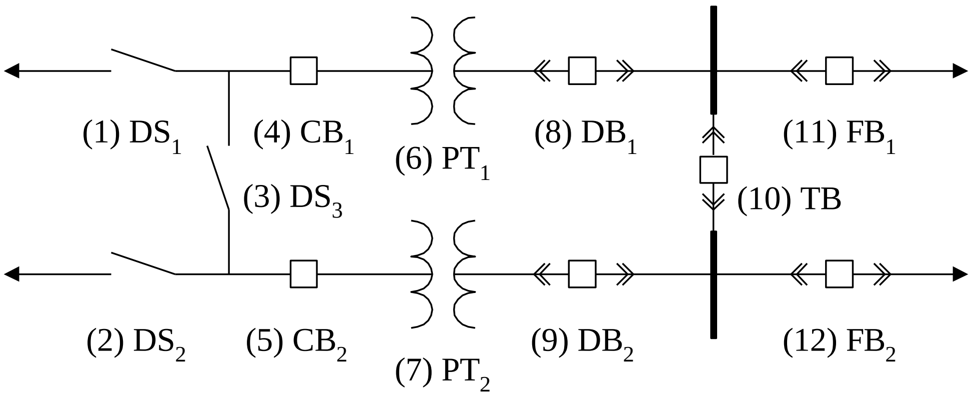


$$E_{\text{system}} = E_1E_2 \cup E_3E_4 \cup E_3E_5$$



* Song, J., and A. Der Kiureghian (2003). Bounds on system reliability by linear programming. *Journal of Engineering Mechanics*, ASCE, 129(**6**): 627-636.

Application to electrical substation systems



Two-transmission-line substations

- Component failure event, E_i

$$E_i = \{\ln R_i - \ln A - \ln S_i \le 0\},\ i = 1, \ldots, n$$

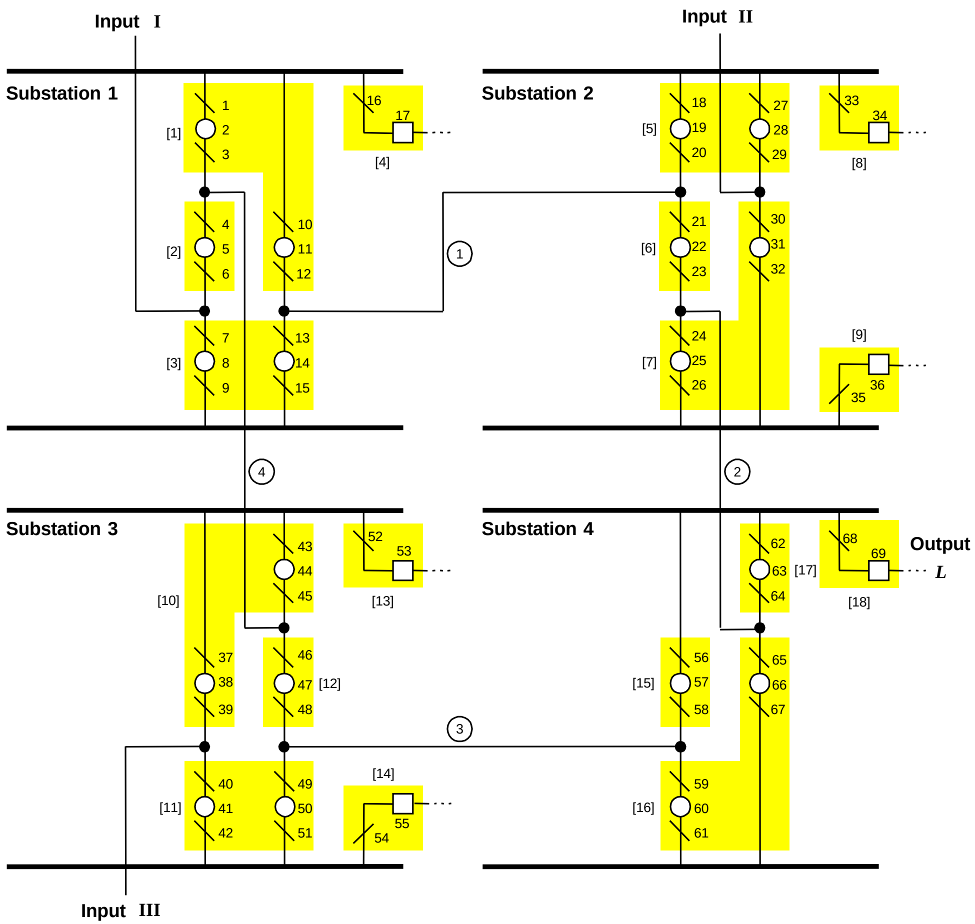
A = LN(mean=0.15, c.o.v.=0.5) PGA
S_i = LN(mean=1, c.o.v.=0.2) local site effect
R_i = LN(mean,c.o.v.,corr.) equipment capacity

DS: Disconnect Switch (0.4, 0.3, 0.3)
CB: Circuit Breaker (0.3, 0.3, 0.3)
PT: Power Transformer (0.5, 0.5, 0.5)
DB: Drawout Breaker (0.4, 0.3, 0.3)
TB: Tie Breaker (1.0, 0.3, 0.3)
FB: Feeder Breaker (1.0, 0.3, 0.3)

Case	Uni-comp.	Bi-comp.	Tri-comp.	M.C. δ=0.01
As shown in figure	1.13×10^{-12}~0.202	0.0436~0.146	0.0616~0.0942	0.0752
No information available on TB (E_{10})	1.82×10^{-11}~0.202	0.0436~0.146	0.0615~0.0943	N/A
No information available on CB_1 (E_4)	1.26×10^{-9}~0.202	0.0267~0.147	0.0395~0.1360	N/A
Upper bound available on CB_1, $P_4 \le 0.01$	5.19×10^{-9}~0.120	0.0267~0.0995	0.0395~0.0701	N/A

* Song, J., and A. Der Kiureghian (2003). Bounds on system reliability by linear programming and applications to electrical substations. *Proc. of ICASP9*, San Francisco, USA, July 6-9.

Multi-scale system reliability analysis



System of four electrical substations

(n = 59: 5.76x10^{17} design variables)

- System decomposition

  - consider a subset of the components of a system as “super-components”
  - bounds on marginal and joint probabilities of the super-components are computed by LP approach
  - the computed bounds are used as constraints in solving the LP problem for the entire system
  - reduced to 35 LP problems, the largest of which has 2^{15} = 32,768 variables

- multi-scale system modeling

  - helps the analyst see the “big picture,” while not disregarding system details
  - particularly effective when many similar subsystems exist
  - allows different teams of analysts to work on different subsystems (parallel computing)

System reliability updating

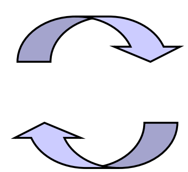
- In the analysis of system reliability, it is often of interest to compute the **conditional probability** of a system or subsystem event, given that another system or subsystem event is known or presumed to have occurred.
  - Examples: $P(E_i \mid E_{system}),\ P(E_i \mid \overline{E}_{system})$, etc.

$$P(B \mid A) = \frac{P(AB)}{P(A)} = \frac{\sum_{r \in AB} p_r}{\sum_{r \in A} p_r}$$ ~ **Nonlinear** function of **p**'s

- The bounds on the conditional probabilities can be obtained after a few iterations of a parameterized LP problem (Dinkelbach 1967).

$$\begin{aligned} \text{minimize/maximize} \quad & (\mathbf{c}_{AB}^{\mathrm{T}} - \lambda \mathbf{c}_{A}^{\mathrm{T}})\mathbf{p} \\ \text{subject to} \quad & \mathbf{a}_1\mathbf{p} = \mathbf{b}_1 \\ & \mathbf{a}_2\mathbf{p} \geq \mathbf{b}_2 \end{aligned}$$

optimal solution, $\hat{\mathbf{p}}$

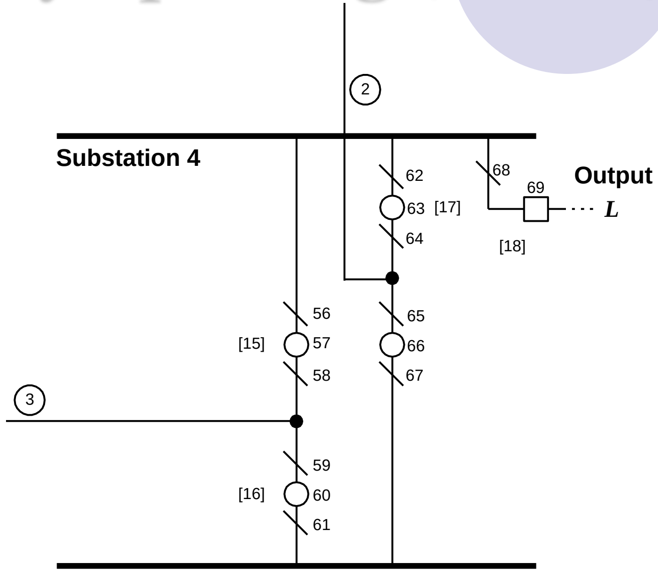


parameter, λ

$$\lambda = \frac{\mathbf{c}_{AB}^{\mathrm{T}}\hat{\mathbf{p}}}{\mathbf{c}_{A}^{\mathrm{T}}\hat{\mathbf{p}}}$$

* Der Kiureghian, A. and J. Song (2008). Multi-scale reliability analysis and updating of complex systems by use of linear programming. *Journal of Reliability Engineering & System Safety*, 93(2): 288-297.

System reliability updating (contd.)



Updated failure probabilities of equipment items in Substation 4

Type	Equipment No.	$P(E_i)$	$P(E_i \mid E_{sys})$	$P(E_i \mid \overline{E}_{sys})$
DS	56, 58, 62, 64	0.00371	0.243 ~ 0.375	0.000431 ~ 0.00125
	59, 61, 65, 67	0.00371	0.175 ~ 0.372	0.000431 ~ 0.00182
	68	0.00371	0.331 ~ 0.468	0
CB	57, 63	0.00953	0.506 ~ 0.660	0.00345 ~ 0.00458
	60, 66	0.00953	0.338 ~ 0.623	0.00357 ~ 0.00613
PT	69	0.00232	0.206 ~ 0.292	0

Identification of critical components and cut sets

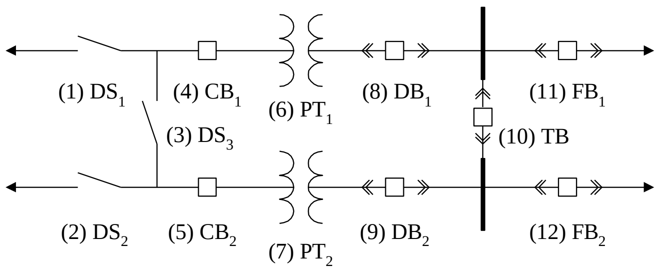
- LP approach can identify components and cut sets which make **significant contributions** to the system failure probability by iteratively solving parameterized LP's.

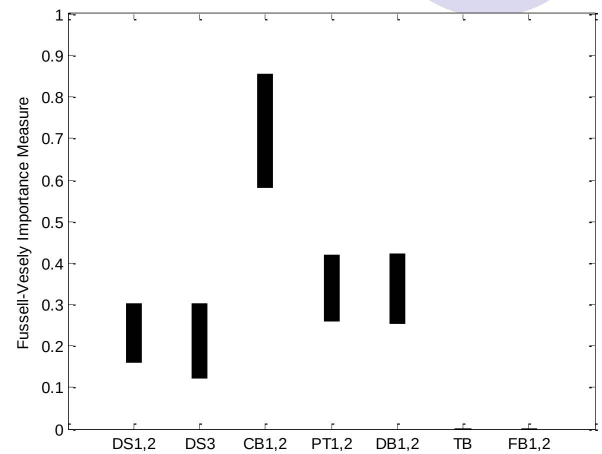
- **Importance Measures (IM)**

  quantifies **participation** in system failure probability

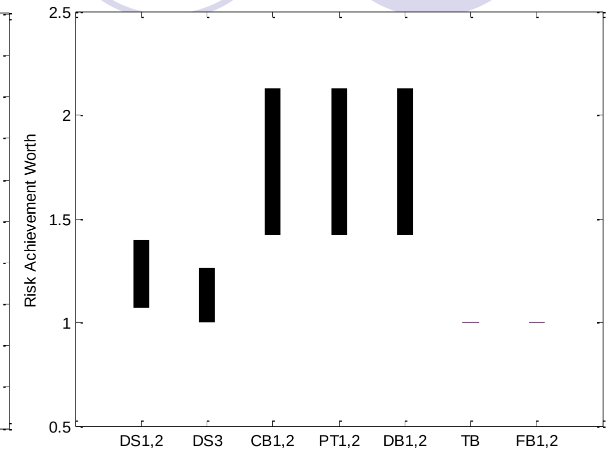
  - Fussell-Vesely: $FV_i = P(\bigcup_{k:E_i \subseteq C_k} C_k) / P(E_{system})$
  - Risk Achievement Worth: $RAW_i = P(E_{system}^{(i)}) / P(E_{system})$
  - Risk Reduction Worth: $RRW_i = P(E_{system}) / P(E_{system}^{\overline{(i)}})$
  - Boundary Probability: $BP_i = P(E_{system}^{(i)}) - P(E_{system}^{\overline{(i)}})$
  - Fussell-Vesely Cutset IM: $FVC_k = P(C_k) / P(E_{\text{system}})$

Identification of critical components and cut sets (contd.)

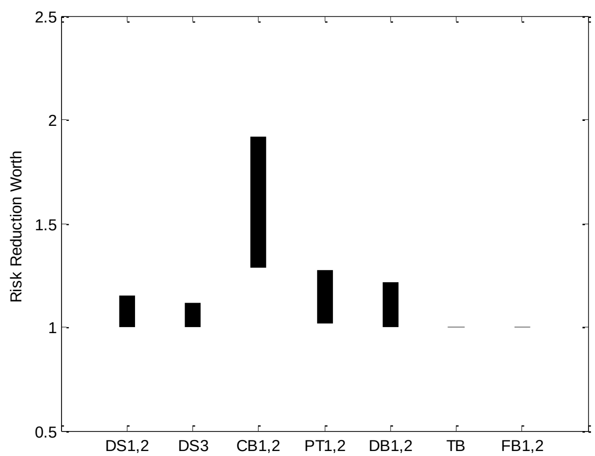




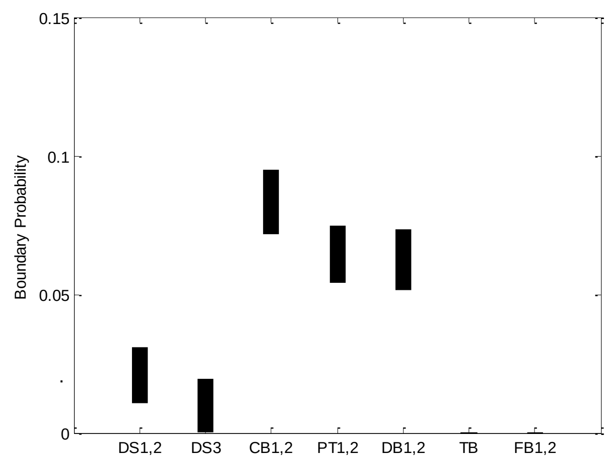
FV IM



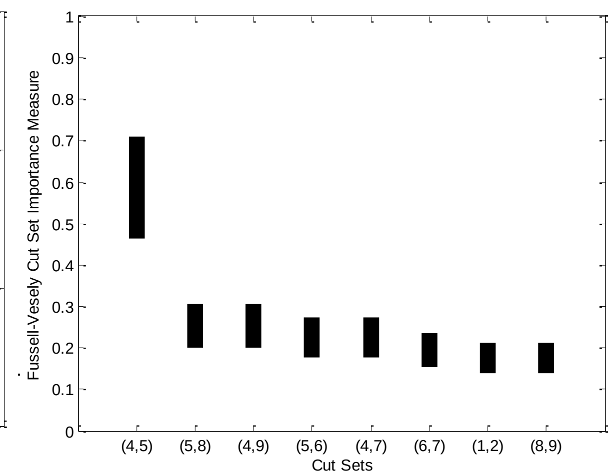
RAW



RRW



BP



FVC IM

* Song, J. and A. Der Kiureghian. Component importance measures by linear programming bounds on system reliability. *Proc. of ICOSSAR9*, Rome, Italy, June 19-23.

Sensitivity and optimal upgrade

- General-purpose LP algorithms provide the sensitivity of an optimal solution with respect to the values in the right-hand side vector, **b**.

$$\begin{aligned}&\text{minimize (maximize) } \mathbf{c}^{\mathrm{T}}\mathbf{p}\\ &\text{subject to} \quad \mathbf{a}_1\mathbf{p} = \mathbf{b}_1\\ &\qquad\qquad\quad\;\; \mathbf{a}_2\mathbf{p} \geq \mathbf{b}_2\end{aligned}$$

LP problem for system reliability

LP algorithm →

$$\frac{\partial P_{sys}}{\partial b_i}$$

Sensitivity w.r.t. failure probabilities

$$\frac{\partial b_i}{\partial \theta_j}$$

Chain rule →

$$\frac{\partial P_{sys}}{\partial \theta_j}$$

Sensitivity w.r.t. design parameters

- **Optimal upgrade** of system reliability within the limit of upgrade cost (*in progress*)

$$\begin{aligned}&\min_{\mathbf{x}} \max_{\mathbf{p}} \quad \mathbf{c}^{\mathrm{T}}\mathbf{p}(\mathbf{x})\\ &\text{subject to} \quad \mathbf{a}_1\mathbf{p} = \mathbf{b}_1(\mathbf{x}), \quad \mathbf{a}_2\mathbf{p} \geq \mathbf{b}_2(\mathbf{x})\\ &\qquad\qquad\quad\;\; \mathbf{Qx} \leq \mathbf{q}, \;\; \mathbf{m}^{\mathrm{T}}\mathbf{x} \leq m_c\\ &\qquad\qquad\quad\;\; \mathbf{x}: \text{binary integers}\end{aligned}$$

~ minimize the upper bound of P_{sys}

~ component failure probabilities: f(actions)

~ constraints on the actions (workability, cost)

~ indicators for upgrade actions (1: yes, 0: no)

LP Bounds approach and decision-making

$$\begin{aligned} &\text{minimize (maximize)}\ \mathbf{c}^{T}\mathbf{p} \\ &\text{subject to}\quad \mathbf{a}_1\mathbf{p} = \mathbf{b}_1 \\ &\qquad\qquad\quad\ \mathbf{a}_2\mathbf{p} \geq \mathbf{b}_2 \end{aligned}$$

LP Bounds Approach

- **System Reliability** — consequence-based engrg. Life-cycle cost analysis
- **Identification of Critical Components and Cut sets** — Priority in upgrade project (cost limit not considered)
- **System Reliability Updating** — Strategy for post-hazard inspection/ recovery
- **Sensitivity of System Reliability** — Plan for optimal upgrade (cost limit considered)

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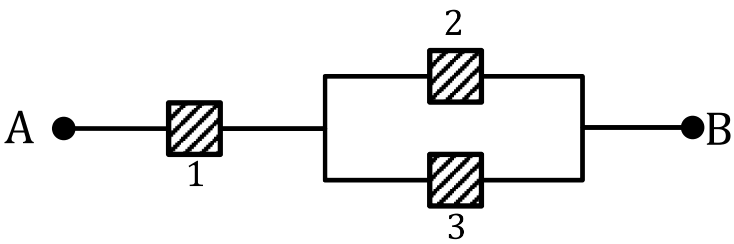
457.646 Topics in Structural Reliability

In-Class Material: Class 16

◎ General system by cut set formulation



E_{sys} : cannot travel from A to B

① Cut set: a subset of components whose joint _________ constitutes the _______ of the system

$C = \{ \qquad \}$

$E_{sys} =$

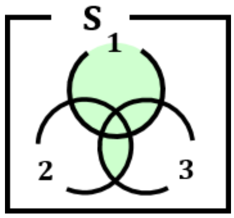
② "Minimum" cut sets ~ cut sets with no r_________ components

$C = \{ \qquad \}$

$E_{sys} =$

$\Rightarrow$cut sets which cease to be a cut set if any of the components is ________

③ "Disjoint" cut sets $P(E_{sys}) = P(\cup C_k) = \Sigma P(C_k)$



$C_{disj} = \{ \qquad \}$

$\quad = \{ \qquad \}$

$E_{sys} = E_1 \cup \bar{E}_1 E_2 E_3$

★ $E_{sys} = \bigcup_{k=1}^{Ncut} C_k$

$= \bigcup_{k=1}^{Ncut} \bigcap_{i \in C_k} E_i$

1	2	3	
O	O	O	$\overline{E_1} \cdot \overline{E_2} \cdot \overline{E_3}$
O	O	X	$\overline{E_1} \cdot \overline{E_2} \cdot E_3$
O	X	O	$\overline{E_1} \cdot E_2 \cdot \overline{E_3}$
X	O	O	$E_1 \cdot \overline{E_2} \cdot \overline{E_3}$
	⋮		⋮
	⋮		⋮

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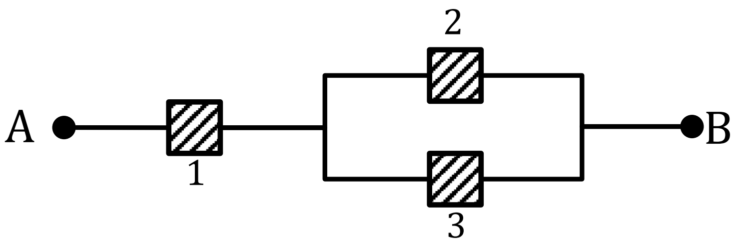
◎ General system by link set formulation



① Link set: a subset of components whose joint (   ) assures (   ) of the system

$L = \{ \qquad\qquad\qquad\qquad\qquad \}$

② “Minimum” link sets ~ link sets with no r__________ component

$L_{\min} = \{ \qquad\qquad\qquad\qquad \}$

③ “Disjoint” Link set

$L_{disj} = \{ \qquad\qquad\qquad\qquad \}$

★ $\bar{E}_{sys} = \bigcup_{k=1}^{Nlink} L_k \qquad = \bigcup_{k=1}^{Nlink} \bigcap_{i=L_k} \bar{E}_i$

De morgan’s law

$\therefore E_{sys} = \bigcap_{k=1}^{Nlink} \left(\bigcup_{i=L_k} \qquad \right)$