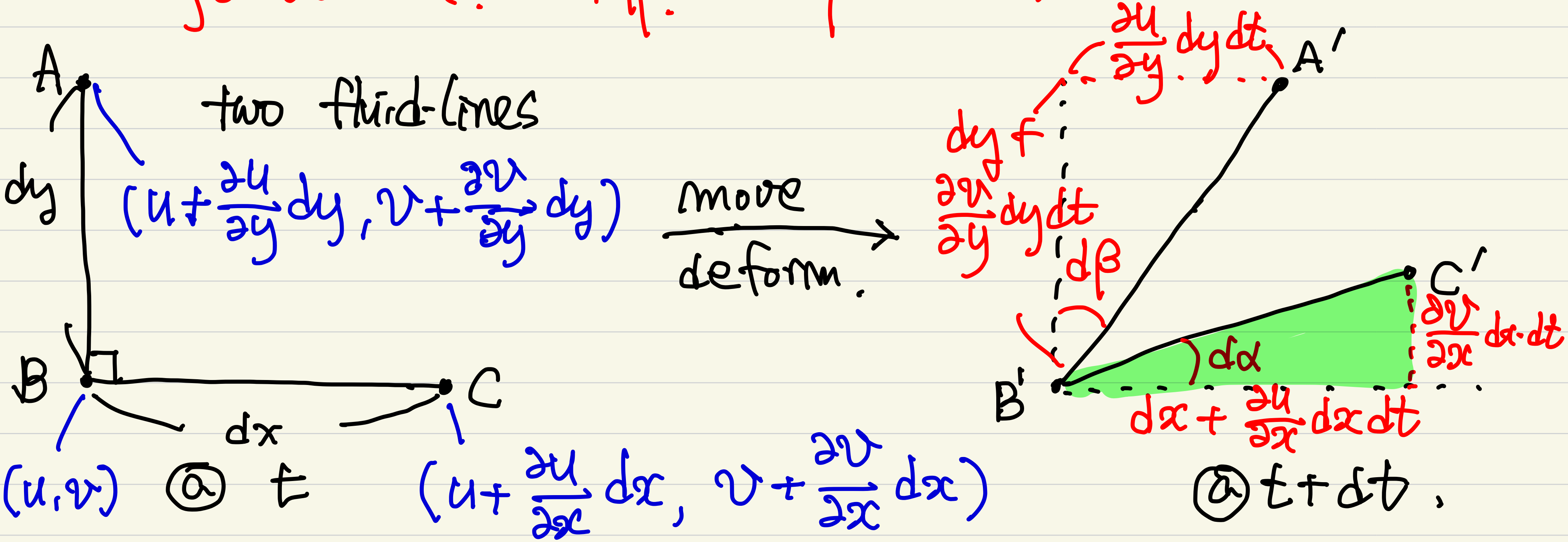
4.8 Vorticity and Irrotationality.

(와도) (비회전성, zero vorticity).

cf) vortex. (와류. eddy, swirl, ...)



· angular velocity $\omega_z \equiv$ average rate of counter-clockwise turning of two lines.

$$\therefore \omega_z = \frac{1}{2}\left(\frac{d\alpha}{dt} - \frac{d\beta}{dt}\right) = \boxed{\frac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)}.$$

$$d\alpha = \tan^{-1}\left(\frac{\frac{\partial v}{\partial x}\cancel{dx}\,dt}{\underbrace{\cancel{dx}}_{1} + \frac{\partial u}{\partial x}\cancel{dx}\,dt}\right) \simeq \frac{\partial v}{\partial x}dt$$

as $dt \to 0$

$$d\beta \simeq \frac{\partial u}{\partial y}dt.$$

also, $\omega_x = \boxed{\frac{1}{2}\left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right)}$, $\omega_y = \boxed{\frac{1}{2}\left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right)}$.

$$\Rightarrow \underline{\omega} = \frac{1}{2}(\nabla \times \underline{V}) \quad : \text{angular velocity}.$$

↳ Vorticity : $\underline{\xi} = 2\underline{\omega} \equiv \nabla \times \underline{V}$.

↓

tendency of a local fluid to rotate.

$$\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ u & v & w \end{vmatrix}$$

$$= \left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right)\hat{i} - \left(\frac{\partial w}{\partial x} - \frac{\partial u}{\partial z}\right)\hat{j} + \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)\hat{k}.$$

- irrotational flow : $\nabla \times \underline{V} = \underline{0}$

$$\zeta_z = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = \underbrace{\frac{\partial}{\partial x}\left(-\frac{\partial \psi}{\partial x}\right) - \frac{\partial}{\partial y}\left(\frac{\partial \psi}{\partial y}\right)} = 0.$$

$$= -\left(\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2}\right) = -\nabla^2 \psi$$

$$\boxed{\nabla^2 \psi = 0}$$

4.9 Frictionless irrotational flows.

(inviscid)

N-S eq. becomes $\rho \frac{\partial \underline{V}}{\partial t} + \rho (\underline{V} \cdot \nabla) \underline{V} = \rho \underline{g} - \nabla p$ ~~$+ \mu \nabla^2 \underline{V}$~~

(Euler eq.).

using a vector identity.

$$(\underline{V} \cdot \nabla) \underline{V} \equiv \nabla \left(\frac{1}{2} V^2 \right) + \underbrace{(\nabla \times \underline{V})}_{= \underline{\zeta} \text{ (vorticity)}} \times \underline{V}.$$

$$\therefore \left[\frac{\partial \underline{V}}{\partial t} + \nabla \left(\frac{1}{2} V^2 \right) + \underline{\zeta} \times \underline{V} + \frac{1}{\rho} \nabla p - \underline{g} \right] \cdot d\underline{r} = 0 \quad (*)$$

for arbitrary displacement vector

$$d\underline{r} = dx\,\hat{i} + dy\,\hat{j} + dz\,\hat{k}.$$

we want $(\underline{\xi} \times \underline{V}) \cdot d\underline{r} = 0$. when?

① $\underline{V} \equiv 0$ (no flow, trivial solution).

② $\underline{\xi} = \nabla \times \underline{V} = 0$: irrotational flow.

③ $d\underline{r} \perp (\underline{\xi} \times \underline{V})$: rare condition (Beltrami flow)

④ $d\underline{r} \parallel \underline{V}$: $d\underline{r}$ is a streamline

⑤ $d\underline{r} \parallel \underline{\xi}$: $d\underline{r}$ is a vortex line

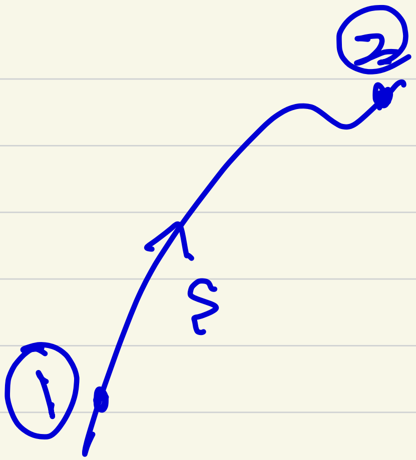
$(d\underline{r} = dx\,\hat{i} + dy\,\hat{j} + dz\,\hat{k})$

④ : $\underline{g} = -g\hat{k}$, ⊛ becomes

$$\frac{\partial \underline{V}}{\partial t} \cdot d\underline{r} + \nabla\left(\frac{1}{2}V^2\right) \cdot d\underline{r} + \frac{1}{\rho}\nabla p \cdot d\underline{r} + g\hat{k} \cdot d\underline{r} = 0$$

$$\nabla \cdot d\underline{r} = dx\frac{\partial}{\partial x} + dy\frac{\partial}{\partial y} + dz\frac{\partial}{\partial z} = d.$$

$$\nabla = \frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}$$



$$\frac{\partial V}{\partial t} ds + d\left(\frac{1}{2}V^2\right) + \frac{1}{\rho} dp + g\, dz = 0.$$

$$\int_1^2 \frac{\partial V}{\partial t} ds + \frac{1}{2}\left(V_2^2 - V_1^2\right) + \int_1^2 \frac{1}{\rho} dp + g(z_2 - z_1) = 0.$$

along the streamline.

if steady, $\frac{p}{\rho} + \frac{1}{2}V^2 + gz = \text{constant}$

(along the streamline)

Bernoulli eq.

② $\underline{\xi} = \underbrace{\nabla \times \underline{V}}$ $= 0$ (irrotational flow).

$\to$ $\underline{V} = \nabla \phi$ ← scalar function

(velocity potential)

$$u = \frac{\partial \phi}{\partial x}, \quad v = \frac{\partial \phi}{\partial y}, \quad w = \frac{\partial \phi}{\partial z}.$$

and lines of constant ϕ = potential lines.

(*) $$\frac{\partial \underline{V}}{\partial t} \cdot d\underline{r} = \frac{\partial}{\partial t}(\nabla \phi) \cdot d\underline{r} = \underline{\nabla}\left(\frac{\partial \phi}{\partial t}\right) \cdot \underline{d\underline{r}} = d\left(\frac{\partial \phi}{\partial t}\right)$$

$$\nabla\left(\frac{1}{2}V^2\right) \cdot d\underline{r} = d\left(\frac{1}{2}V^2\right), \quad \frac{1}{\rho}\nabla p \cdot d\underline{r} = \frac{1}{\rho}dp.$$

$$\Downarrow$$

$$d\left(\frac{\partial \phi}{\partial t} + \int \frac{1}{\rho}dp + \frac{1}{2}|\nabla \phi|^2 + gz\right) = 0.$$

$$\Rightarrow \frac{\partial \phi}{\partial t} + \int \frac{1}{\rho}dp + \frac{1}{2}|\nabla \phi|^2 + gz = \text{constant}.$$

(unsteady irrotational Bernoulli eq.)
(inviscid)

valid all over the flow field!

* Orthogonality of ψ and ϕ (in 2D).

can be defined in 3D.

$$\boxed{\begin{aligned} u &= \frac{\partial \psi}{\partial y} = \frac{\partial \phi}{\partial x} \\ v &= -\frac{\partial \psi}{\partial x} = \frac{\partial \phi}{\partial y}. \end{aligned}}$$

$\nabla \times \underline{V} = 0$ (irrotational flow)

$\rightarrow \underline{V} = \nabla \phi.$

continuity : $\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0. \longrightarrow \boxed{\nabla^2 \phi = 0}$

irrotational flow : $\xi_z = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = 0 \longrightarrow \boxed{\nabla^2 \psi = 0}$

"inviscid & irrotational flow".

→ potential flow.

Solve for ϕ and ψ (Ch.8)

→ u and v → P (Bernoulli eq.)

· lines of constant $\phi = \phi(x, y) = C$

· $d\phi = \underbrace{\frac{\partial \phi}{\partial x}}_{= u} dx + \underbrace{\frac{\partial \phi}{\partial y}}_{= v} dy = 0.$

$u\,dx + v\,dy = 0. \rightarrow \boxed{\left.\frac{dy}{dx}\right)_{\text{const } \phi} = -\frac{u}{v}.}$

· lines of constant $\psi = \psi(x, y) = C$

· $d\psi = \underbrace{\frac{\partial \psi}{\partial x}}_{= -v} dx + \underbrace{\frac{\partial \psi}{\partial y}}_{= u} dy = 0.$

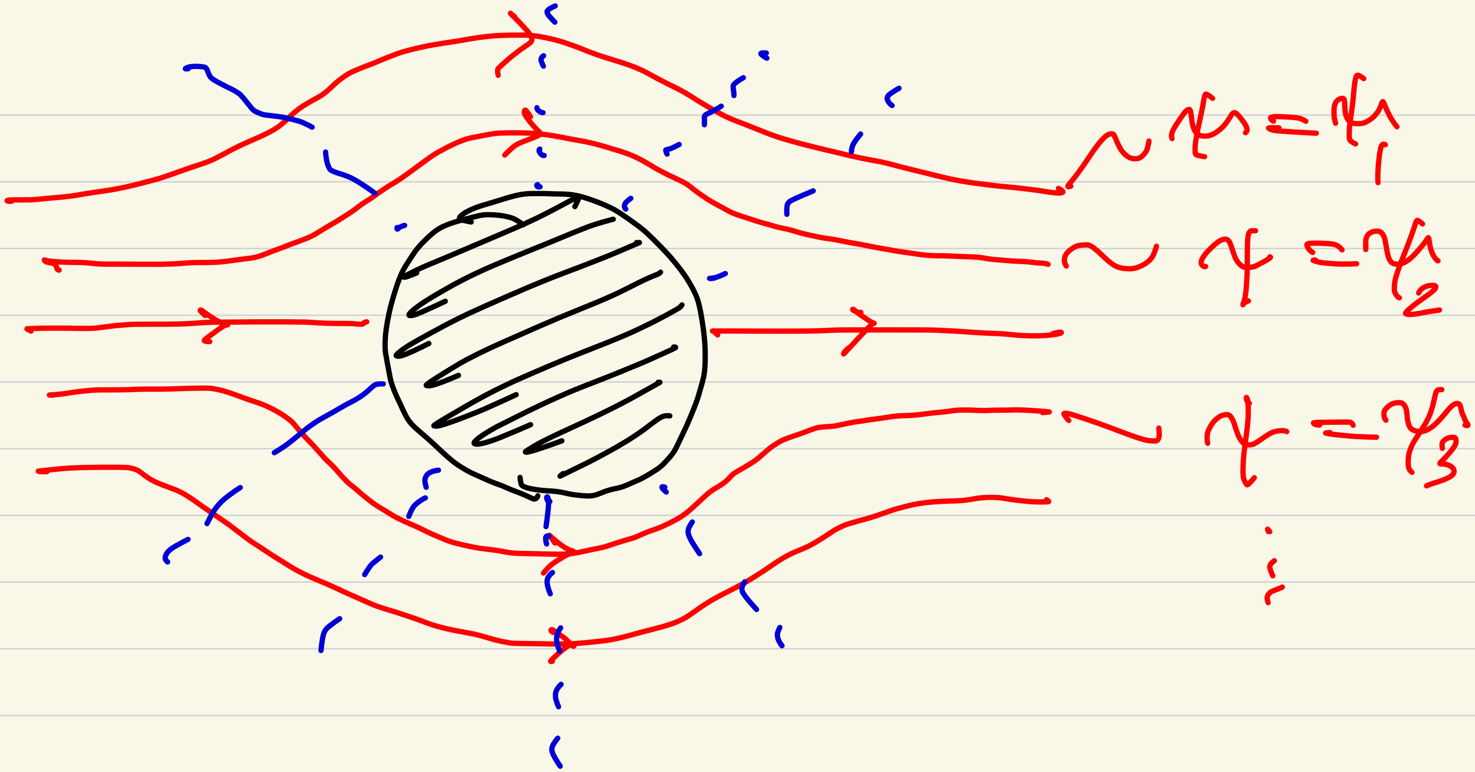
$\Rightarrow \boxed{\left.\frac{dy}{dx}\right)_{\text{const } \psi} = \frac{v}{u}.}$

$\Downarrow$

$\underbrace{\left.\frac{dy}{dx}\right)_{\psi = \text{const}}}_{\text{streamline}} \times \underbrace{\left.\frac{dy}{dx}\right)_{\phi = \text{const}}}_{\text{potential line}} = -1$

$\Downarrow$

"orthogonal"



* Generation of rotationality.

$$\text{irrotational} \xrightarrow{\text{when?}} \text{rotational}$$

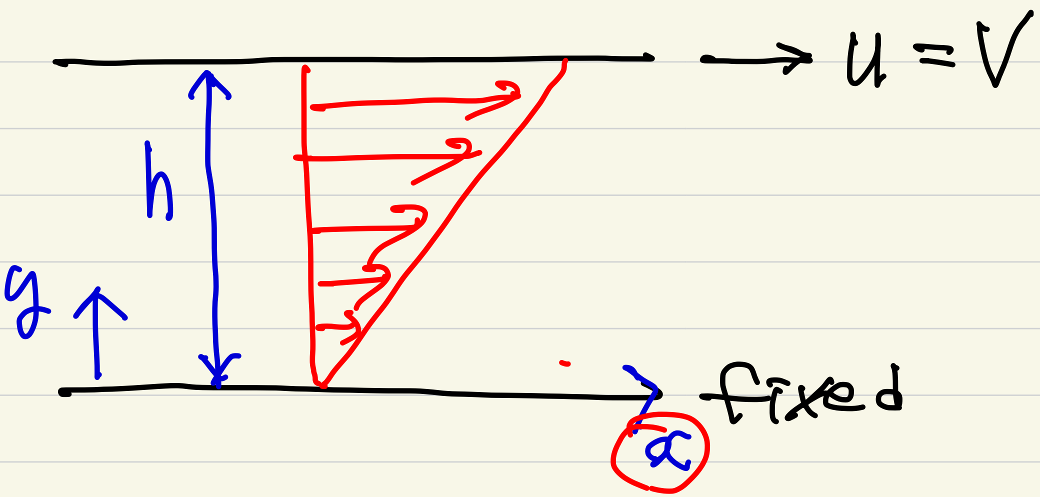
① viscous force dominant.
② entropy gradient (shock wave ...)
③ density 〃 (stratification)

④ non-inertial effect (Coriolis effect)

Bernoulli eq is still valid if the flow is frictionless.

4.10. Some illustrative incompressible viscous flow.

① Couette flow. → relative motion induces the flow.



· steady, 2D.

· $\Delta p = 0$

· $\boxed{\frac{\partial}{\partial x}(\,\cdot\,) = 0}$ ← fully-developed.

· continuity : $\cancel{\frac{\partial u}{\partial x}}^{\,0} + \frac{\partial v}{\partial y} = 0$. $\Rightarrow$ $\frac{\partial v}{\partial y} = 0$. $v = \text{const.}$ $= 0$.

$$u = u(x, y). \Rightarrow u = u(y).$$

· N-S eq

$$x\text{-dir}: \cancel{\rho\frac{\partial u}{\partial t}} + \cancel{\rho u\frac{\partial u}{\partial x}} + \cancel{\rho v\frac{\partial u}{\partial y}} = \cancel{-\frac{\partial p}{\partial x}} + \mu\left(\cancel{\frac{\partial^2 u}{\partial x^2}} + \frac{\partial^2 u}{\partial y^2}\right).$$

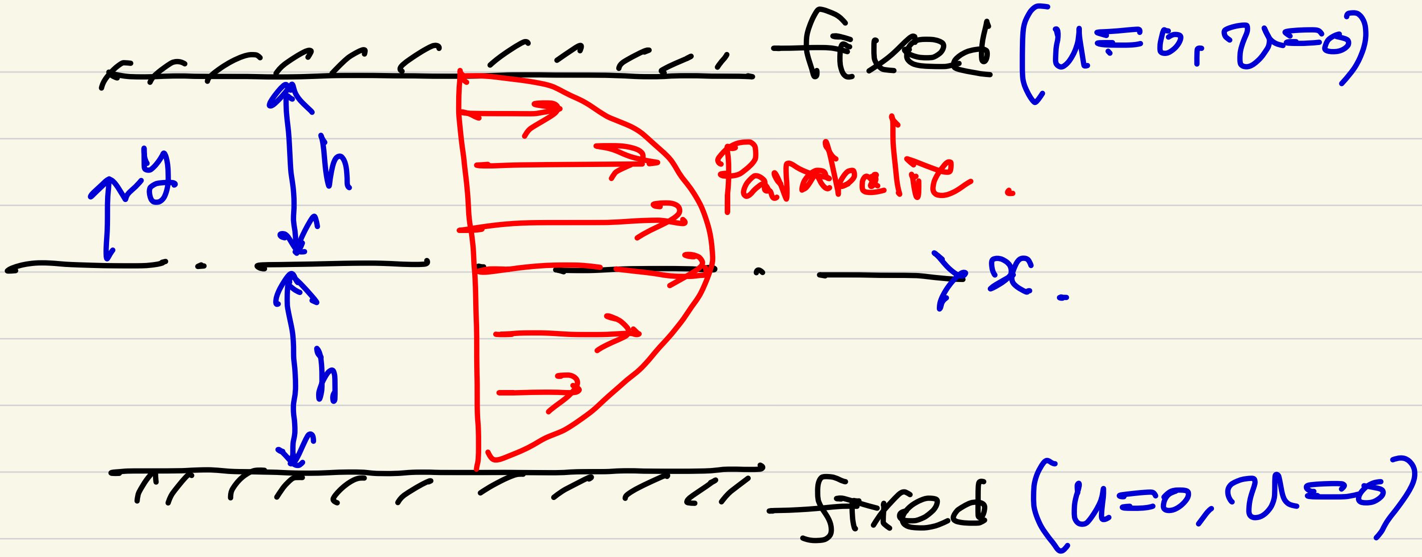
$$\therefore \frac{d^2 u}{dy^2} = 0. \rightarrow u(y) = C_1 y + C_2.$$

BC's : $u = V$ @ $y = h$

$u = 0$ @ $y = 0$.

$$\therefore u(y) = \frac{y}{h} V.$$

② Flow due to pressure gradient between two ~~fixed~~ plates. $(\underbrace{\text{plane}}_{(2D)}$ Poiseuille flow).



· steady ($\partial/\partial t = 0$)

· fully-developed ($\partial/\partial x = 0$)

· $\partial p/\partial x \neq 0$ (<0)

continuity : $\dfrac{\partial u}{\partial x} + \dfrac{\partial v}{\partial y} = 0 \Rightarrow v = 0.$

N-S eq :

$$\rho u \cancel{\frac{\partial u}{\partial x}} + \rho \cancel{v} \frac{\partial u}{\partial y}$$

$$\cancel{\rho \frac{\partial u}{\partial t}} + \cancel{\rho (\underline{V} \cdot \nabla) u} = - \frac{dp}{dx} + \mu \left(\cancel{\frac{\partial^2 u}{\partial x^2}} + \frac{\partial^2 u}{\partial y^2} \right)$$

$$\rightarrow \rho \frac{\partial \cancel{v}}{\partial t} + \rho (\underline{V} \cdot \nabla) \cancel{v} = - \frac{\partial p}{\partial y} + \mu \left(\frac{\partial^2 \cancel{v}}{\partial x^2} + \frac{\partial^2 \cancel{v}}{\partial y^2} \right).$$

$$\rightarrow \frac{\partial p}{\partial y} = 0. \Rightarrow p = p(x).$$

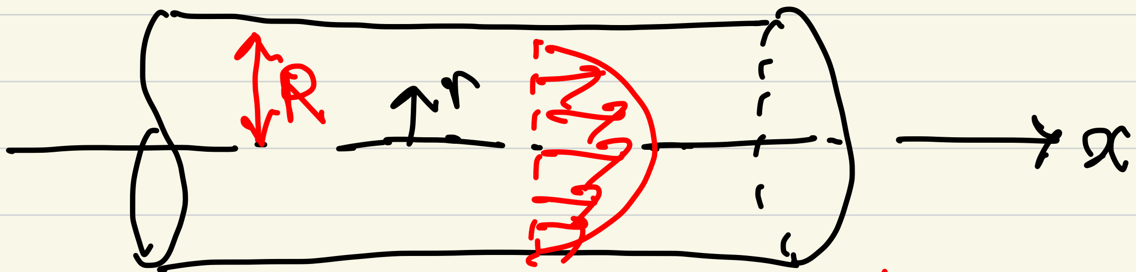
$$\therefore \mu \frac{\partial^2 u}{\partial y^2} = \frac{dp}{dx} = \text{constant.}$$

$$\therefore u(y) = \frac{1}{\mu}\frac{dp}{dx}\cdot\frac{1}{2}y^2 + C_1 y + C_2 . \quad \left(u = 0 \text{ @ } y = \pm h\right)$$

$$\rightarrow \boxed{u(y) = -\frac{dp}{dx}\cdot\frac{h^2}{2\mu}\left(1 - \frac{y^2}{h^2}\right).}$$ Parabolic.

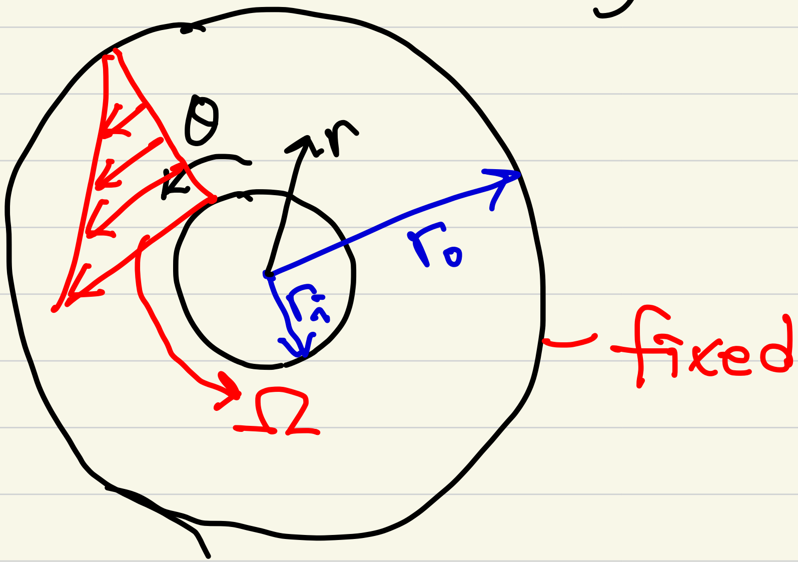
③ Fully-developed laminar pipe flow. (Hagen-Poiseuille flow)

$\frac{\partial p}{\partial x} \neq 0.$



$$u(r) = -\frac{dp}{dx}\cdot\frac{1}{4\mu}\left(R^2 - r^2\right) \Rightarrow \underline{\text{Ch. 6}}$$

④ Flow between long concentric cylinders.



(r, θ, z) coordinate.

- no axial motion

$\therefore v_z = 0, \ \frac{\partial}{\partial z}(\cdot) = 0.$

- no variation in θ $\therefore \frac{\partial}{\partial \theta}(\cdot) = 0.$

- cont. : $\frac{1}{r}\frac{\partial}{\partial r}(r v_r) + \cancel{\frac{1}{r}\frac{\partial v_\theta}{\partial \theta}} = 0.$ $\rightarrow r v_r = \text{constant}.$

ⓐ $r = r_i, \ v_r = 0.$ $\Rightarrow v_r = 0.$

- N-S eq (θ-dir).

$$\rho \cancel{(\underline{v} \cdot \nabla) v_\theta} + \cancel{\frac{\rho}{r} v_r v_\theta} = -\cancel{\frac{1}{r}\frac{\partial p}{\partial \theta}} + \mu\left[\frac{1}{r}\frac{d}{dr}\left(r\frac{dv_\theta}{dr}\right) - \frac{v_\theta}{r^2}\right]$$

$$\therefore \frac{1}{r}\frac{d}{dr}\left(r\frac{dv_\theta}{dr}\right) = \frac{v_\theta}{r^2}. \rightarrow v_\theta(r) = C_1 r + \frac{C_2}{r}.$$

BC's: $v_\theta = 0$ @ $r = r_o$

$v_\theta = r_i\Omega$ @ $r = r_i$.

$$\therefore v_\theta = \Omega r_i \frac{r_o/r - r/r_o}{r_o/r_i - r_i/r_o}$$