V. BUBBLE DYNAMICS.

- Non-dimensional variables (subscript 'l' : liquid
  'g' : gas, bubble).

- Reynolds number, $Re = \rho_l \underbrace{u_b d_b}_{\text{vel/size of bubble}} / \mu_l$.

  $\sim$ inertia/viscous force.

- Weber number, $We = \rho_l u_b^2 d_b / \sigma_l$

  $\sim$ inertia/surf. tension.

- Bond (Eötvös) number,

  $$Bo\ (Eo) = g \Delta\rho d_b^2 / \sigma_l \sim \frac{\text{body force}}{\text{surf. tension}}$$

- Froude number, $Fr = u_b^2 / g d_b \left(u_b / \sqrt{g d_b} \right)$

~ inertia/gravity

↪ wave instability

- Morton number, $Mo \equiv We^3/Fr\cdot Re^4$

  $= \underline{g}\mu_{\underline{l}}^4/\rho_{\underline{l}}\sigma_{\underline{l}}^3$

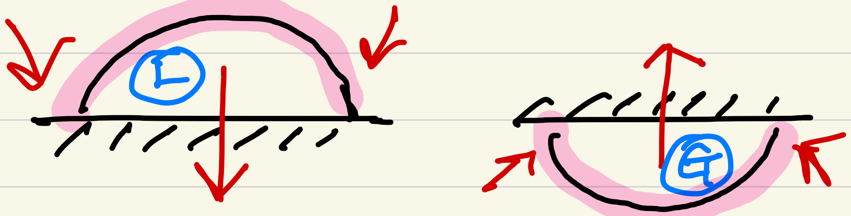
- Galilei number, $Ga = \rho_l g d_b^3/\mu_l$. $\sim \dfrac{\text{gravity}}{\text{viscous force}}$

- Archimedes ", $Ar = g d_b^3 \rho_l \Delta\rho/\mu_l^2$

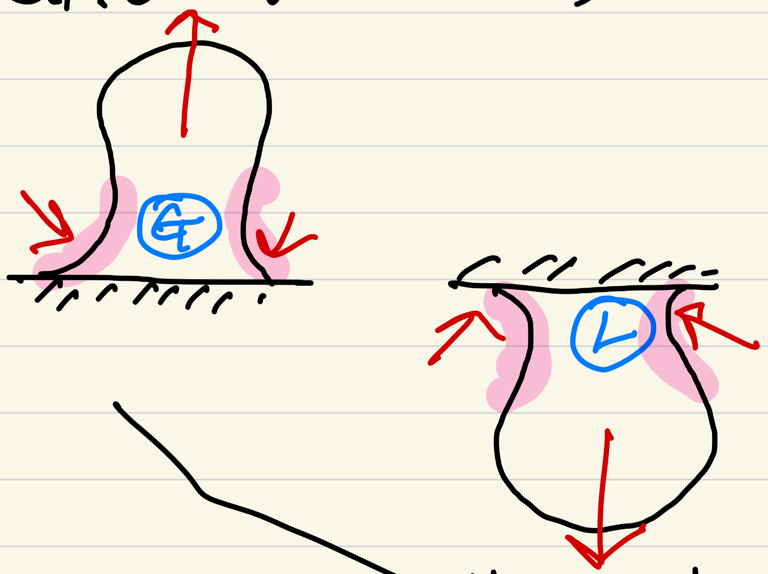
  $\sim \dfrac{\text{density difference}}{\text{viscous force}}$.

· shape of fluid particle : bubbles and drops tend to deform subject to external fluid flows through the balance between stresses at the interface. → smaller range of shapes than the solid particles.

- static bubbles and drops
  - sessile and pendant (Clift et al. 1978)



prevented from moving by a solid plate under the gravity, and surface tension. (adhesion force)



remain attached, with gravity acting to pull away the bubble/drop.

- The profiles of pendant and sessile bubble/drop are used to determine the interfacial tension and contact angle. → hydrostatic pressure gradient is balanced by the normal stress due to the surf. tension at the points on the curved interface.

$$: \underline{\Delta\rho} g \underline{y} = \sigma\left(\frac{2}{\underline{R_0}} - \frac{1}{R_1} - \frac{1}{R_2}\right)$$

elevation

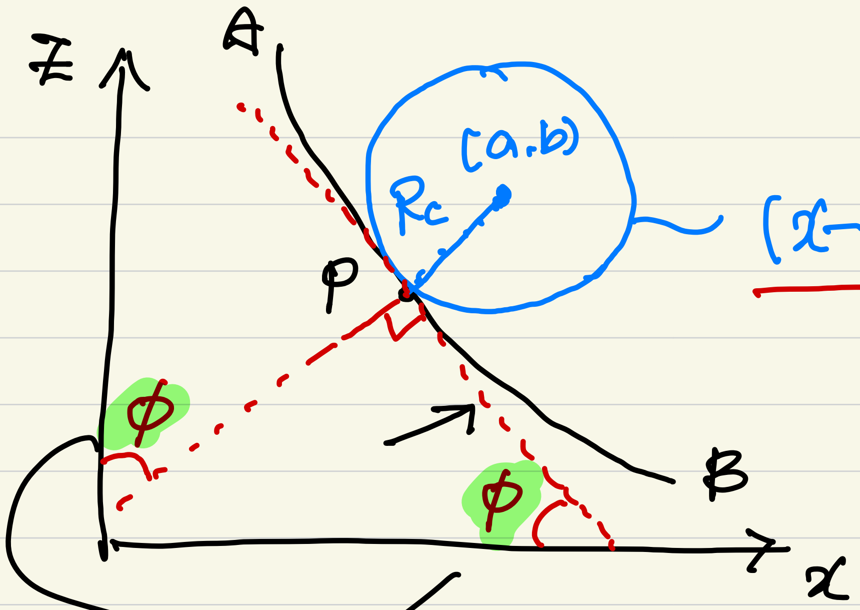
radius of curvature

principal radii of curvature at the point of interest.

※ radius of curvature.

- radius of curvature of $\overset{\frown}{AB}$ at P → R_c

→ curvature, $H = 1/R_c$



$y = f(x)$

$\hookrightarrow dy/dx$

$$(x-a)^2+(z-b)^2=R_c^2$$

$$\frac{dz}{dx} = -\left(\frac{x-a}{z-b}\right).$$

$$\tan\phi = -\frac{dz}{dx} = \frac{x-a}{z-b}.$$

$$\cos\phi = \frac{1}{(1+\tan^2\phi)^{0.5}} = \frac{1}{\left\{1+\left(\frac{x-a}{z-b}\right)^2\right\}^{0.5}} = \frac{z-b}{R_c}.$$

$$\frac{d\cos\phi}{dz} = \frac{1}{R_c}. = H \rightarrow \begin{cases} \frac{d\cos\phi}{dz} < 0 \rightarrow H < 0 \\ \text{: concave outward} \\ \frac{d\cos\phi}{dz} > 0 \rightarrow H > 0 \end{cases}$$

On the other hand.

; convex outward

$$\cos\phi = \frac{1}{(1+\tan^2\phi)^{0.5}} = \frac{1}{\{1+(\frac{dz}{dx})^2\}^{0.5}}$$

↓

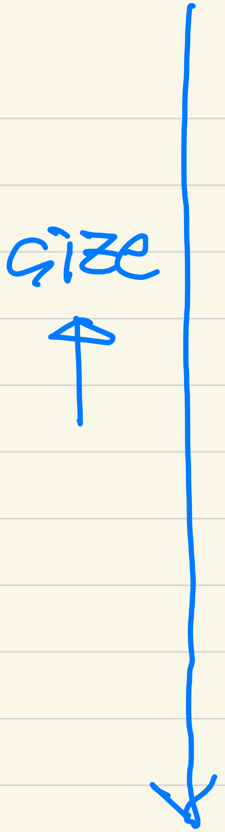
$$H = \frac{d\cos\phi}{dz} = \frac{d^2z/dx^2}{\{1+(dz/dx)^2\}^{0.5}}$$

$$\therefore H\left(= 1/R_c\right) = \pm\frac{d^2z/dx^2}{\{1+(dz/dx)^2\}^{0.5}}$$

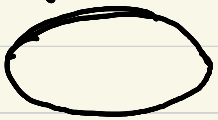
depends on the geometry
(concave or convex)

- Bubbles (and drops) in a dynamic motion.

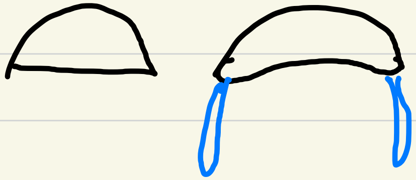
· spherical : in general, bubbles and drops can o be approximated as spheres if interfacial tension and/or viscous forces are dominant.

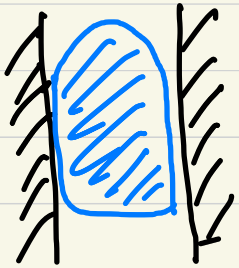


- ellipsoidal : oblate w/ a convex interface (viewed from inside), undergoes periodic dilation or random wobbling.



- cap : large size. flat or indented bottom. may trail thin envelops of dispersed fluid, called as "skirts".



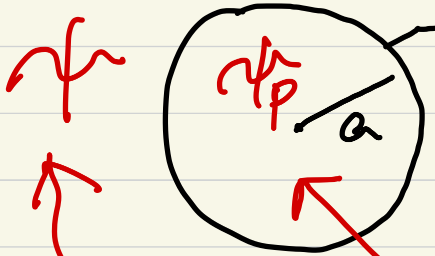


- slug (Taylor bubble) : in a confined condition, if sufficient large, it

fills most of the
cross section.

- Regime of bubble shape is determined, in general, by Re and Eo (Bo).
  - spherical bubble at low Re and Eo.
  - ellipsoidal at high Re and moderate Eo
  - cap bubble at high Re and Eo.

- spherical fluid particle (Hadamard, Rybczynski, 1911).
  - Re ≪ 1. free from surface contamination (perfect slip BC @ the interface)



- for the Stokes flow.

$$\underline{E^4\psi} = \underline{E^4\psi_p} = 0 .$$

$$E^2 \equiv \frac{\partial^2}{\partial r^2} + \frac{\sin\theta}{r}\frac{\partial}{\partial\theta}\left(\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\right)$$ Laplacian in sph. coord. (axisymmetry)

BC) ⓐ $r \to \infty$, $\psi \to \frac{1}{2}U_\infty r^2 \sin^2\theta$ (free stream)

ⓐ $r = a$, $\psi = \psi_p = 0$ (no flow across the interface)

" $$\frac{\partial\psi}{\partial r} = \frac{\partial\psi_p}{\partial r}$$ (continuity of tangential velocity)

" $$\frac{\partial}{\partial r}\left(\frac{1}{r^2}\frac{\partial\psi}{\partial r}\right) = K\frac{\partial}{\partial r}\left(\frac{1}{r^2}\frac{\partial\psi_p}{\partial r}\right),$$

$K = \mu_p/\mu$ (continuity of tangential stress)

" $$p - 2\mu\frac{\partial}{\partial r}\left(\frac{1}{r^2\sin\theta}\cdot\frac{\partial\psi}{\partial\theta}\right) + \frac{2\sigma}{a}$$

$$= p_p - 2\mu_p\frac{\partial}{\partial r}\left(\frac{1}{r^2\sin\theta}\cdot\frac{\partial\psi_p}{\partial\theta}\right).$$

(balance of normal stress).

$$\Rightarrow \psi = \frac{1}{2} U_\infty r^2 \sin^2\theta \left(1 - \frac{a(2+3k)}{2r(1+k)} + \frac{ka^3}{2r^3(1+k)}\right)$$

$$\psi_p = \frac{U_\infty r^2 \sin^2\theta}{4(1+k)} \left(1 - \frac{r^2}{a^2}\right).$$

for creeping flow ($Re \ll 1.0$),

N-S equation reduces to $\underline{\nabla P = \mu \nabla^2 \vec{u}}$.

$$\rightarrow P = \underline{P_o} + \left(\frac{\mu U_\infty a \cos\theta\,(2+3k)}{2r^2(1+k)}\right)$$

$$P_p = \underline{P_{op}} - \left(\frac{5\mu_p U_\infty r \cdot \cos\theta}{a^2(1+k)}\right).$$

($P_{op} - P_o = 2\sigma/a$. for normal stress BC)

- bubbles and drops are spherical when the creeping flow approximation is valid, and only deform when the inertia becomes significant.

- it is not necessary for surf. tension to be dominant for a bubble/drop to be spherical.

  → if Re is sufficiently low, it is spherical irrespective of the surf. tension. $K = \mu_p/\mu$.

- integration of pressure. $\Rightarrow C_{D1} = \frac{8}{3Re}\left(\frac{2+3K}{1+K}\right)$.

- $\hat{\tau}_{ij} = \sigma_{ij} + \underline{P}\delta_{ij} \longrightarrow C_{D2} = \frac{32}{3Re}\left(\frac{1}{1+K}\right)$

  ↳ deviatoric normal stress.

· from shear stress, $C_{D3} = \frac{16}{Re}\left(\frac{K}{1+K}\right)$.

$$\rightarrow C_D = C_{D1} + C_{D2} + C_{D3}$$

$$= \frac{8}{Re}\left(\frac{2+3K}{1+K}\right).$$

(for all K, $C_{D1} \simeq \frac{1}{3} C_{D3}$.

for gas bubble ($K \simeq 0$), $C_{D3} \doteq 0$

· terminal velocity, $U_T = \frac{2}{3} \cdot \frac{g a^2 \Delta\rho}{\mu}\left(\frac{1+K}{2+3K}\right)$.

(drag ~ gravity)

· vorticity at the interface, $\underline{\omega_s = \frac{U_\infty \sin\theta}{2a}\left(\frac{2+3K}{1+K}\right)}$.

⑥ Deformation of bubble/droplet.

. particle in a fluid flow ← surface stress
creates a deformation
(may affect the
particle motion)

bubble/drop : external stress is counteracted
by surf. tension to keep the
particle spherical.

solid particle : elasticity may work.

surface tension force, $F_\sigma \sim \sigma/R$

fluid force, $F_F \sim \rho u_\infty^2$ (for higher Re)

$\sim \mu u_\infty/R$ (for lower Re) ⇐ stokes drag.

↓

ratio of $F_F/F_\sigma = \rho u_\infty^2/(\sigma/R) = \rho u_\infty^2 R/\sigma$ ($\equiv We$).

at lower Re, $(\mu u_\infty/R)/(\sigma/R)$

$$= \underbrace{\frac{\rho u_\infty^2 R}{\sigma}}_{We} \cdot \underbrace{\frac{\mu}{\rho u_\infty R}}_{1/Re} = \frac{We}{\boxed{Re}}.$$

∴ significant deformation will occur

at $We > 1.0$ (when $Re \gg 1$) or

$F_F > F_\sigma$ ⇐ $We/Re > 1.0$ (" $Re \ll 1$).

To solve this phenomena, the relative velocity should be known first. In cases where the gravity works, it depends on the deformation state of the particle.

↳ $F(Re, We, Fr) = 0$, $Fr \equiv \dfrac{u_\infty^2}{gR|1-\rho_p/\rho_f|}$.

competition between gravity wave to the convective flow in the free-surface flow.

Is not suitable to explain the bubble/droplet deformation. → Morton number, Mo.

$$Mo \equiv \frac{g|1-\rho_p/\rho_f|\mu^4}{\rho_f \sigma^3} = \frac{We^3}{Fr^2 Re^3}.$$

→ $F(Re, We, Mo) = 0$

or

$F[Re, Mo, \underline{C_D}) = 0$. $u_\infty = f(Re, C_D)$.

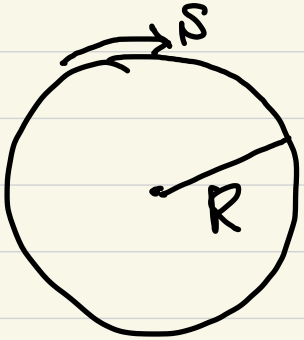
· We can use Eötvös number, $Eo \left(\equiv \frac{g|1-\rho_p/\rho_f|R^2}{\sigma/\rho_f}\right)$ for bubbles. For droplets, we call the same definition as Bond number (Bo).

The spherical bubble deform when $\boxed{Re > Mo^{-1/4},}$ to ellipsoid, and then eventually be spherical cap bubble at higher Re.

⊙ Marangoni effect.

Gradient of surface tension (σ), caused by the gradients in the ① temperature, ② solvent concentration, electrical potential, ⋯ .

$\partial\sigma/\partial T < 0$.



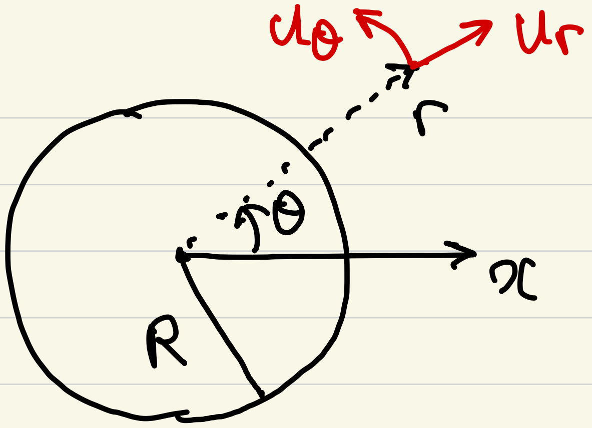
$$\frac{d\sigma}{ds} = \boxed{\frac{d\sigma}{dT}} \cdot \frac{dT}{ds}.$$

ds → tangential to the surface.

→ shear stress should act in the negative "s" direction to be in equilibrium with $d\sigma/ds$.

⇙ modification of BC.

⇖ by external condition.



Steady translation of a spherical bubble in a viscous fluid w/ $\underline{\partial T/\partial x}$.

two cases

① $T = \left(\underline{\frac{dT}{dx}}\right)x$. $\Rightarrow \left(\frac{1}{r}\frac{\partial\sigma}{\partial\theta}\right)_{r=R} = -\sin\theta \frac{d\sigma}{dT}\cdot\frac{dT}{dx}$.

☆② $\nabla^2 T = 0$. (thermal conduction dominates and there is no heat transfer across the surface).

⇓

thermo-capillary (more realistic)

$$\left(\frac{1}{r}\boxed{\frac{\partial\sigma}{\partial\theta}}\right)_{r=R} = \underline{-\frac{3}{2}}\sin\theta\frac{d\sigma}{dT}\frac{dT}{dx}.$$

(Young et al. 1959).

the tangential stress BC (in Hadamard-Rybczynski problem) becomes

$$\rho_\ell \nu_\ell \left(\frac{\partial u_\theta}{\partial r} - \frac{u_\theta}{r}\right)_{r=R} + \frac{1}{R}\left(\frac{\partial \sigma}{\partial \theta}\right)_{r=R} = 0$$

↳ implemented to HR solution.

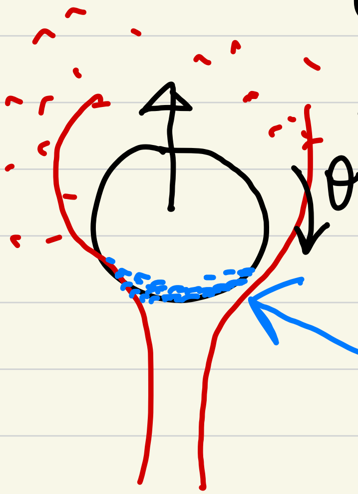
Then, $$F_D = \underbrace{-4\pi \rho_\ell \nu_\ell U_r R}_{\text{HR drag.}} \ominus \underbrace{2\pi R^2 \frac{d\sigma}{dx}}_{\text{added. Marangoni force.}}$$

(acting in the direction of decreasing σ).

(increasing T).

↙ thermo-capillary effect.

- Effect of surface contamination.



- convection may cause contaminants to accumulate on the downstream side of the bubble.

$\Rightarrow$ positive $\frac{\partial\sigma}{\partial\theta}$. $\Rightarrow$ shear stress acts opposite to the flow.

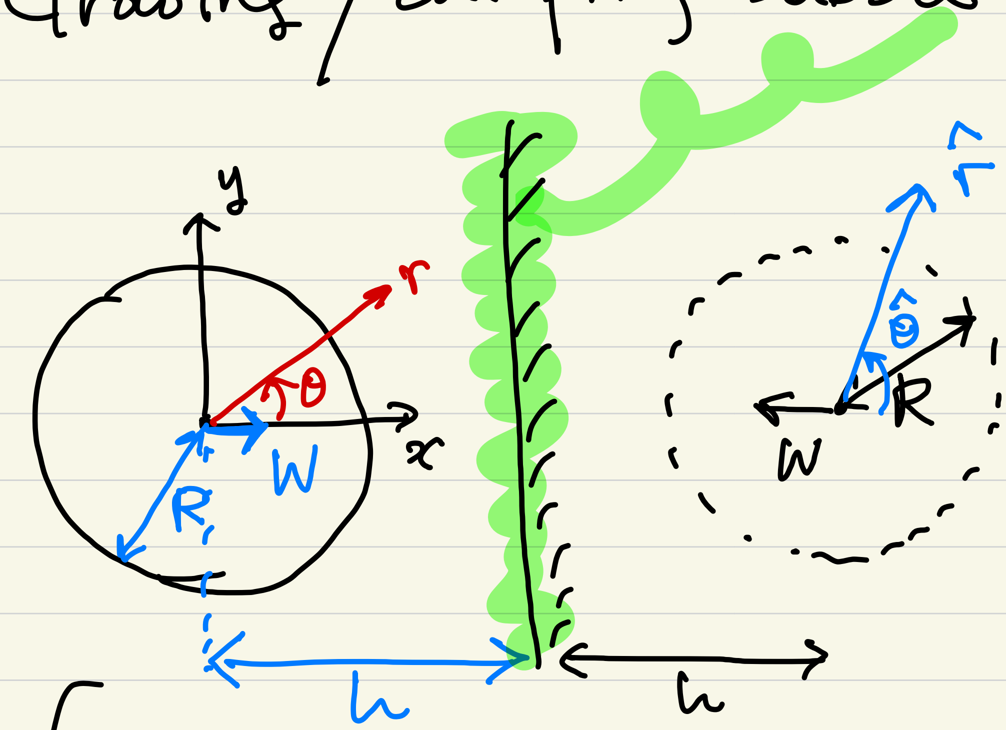
$\Rightarrow$ immobilize the surface. (H-R solution $\rightarrow$ Stokes ").

⊙ Growing / Collapsing bubble (at high Re).

: A bubble growing / collapsing closer to a boundary, may undergo translation due to the induced asymmetry. ⟸ "underwater explosion"

(Herring, 1941, Davies & Taylor 1942, 1943).

① Growing / collapsing bubble (high Re).



Herring (1941)

Davies & Taylor (1942, 1943)

only perpendicular to wall, inviscid, irrotational.

(zero vorticity)

($\nabla \times \bar{v} = 0$

↳ $\bar{v} = \nabla \phi$.)

↓

velocity potential. ϕ.

↓

$$\phi = -\frac{R^2\dot{R}}{r} - \frac{WR^3\cos\theta}{2r^2} \underset{\downarrow}{+} \left[-\frac{R^2\dot{R}}{\hat{r}} + \frac{WR^3\cos\hat{\theta}}{2\hat{r}^2} - \frac{R^5\dot{R}\cos\theta}{8h^2r^2} \right]$$

① ② ③ ④ ⑤

($\dot{R} = dR/dt$)

$+$: solid boundary
($-$: free-surf. ")

①, ③ : source/sink from the spherical bubble

②, ④ : dipole due to the bubble translation.

⑤ : convection term to satisfy BC's at the boundary.

Using the unsteady Bernoulli eq, to obtain pressure. (Davies & Taylor, 1943)

$$F_x = -\frac{2\pi}{3}\left\{\frac{d}{dt}(R^3 W) \pm \frac{3}{4}\frac{R^2}{h^2}\frac{d}{dt}\left(R^3\frac{dR}{dt}\right)\right\}$$

$W(t)$

↳ eq. of motion : $\frac{d}{dt}(R^3 W) \pm \frac{3}{4}\frac{R^2}{h^2}\frac{d}{dt}\left(R^3\frac{dR}{dt}\right) + \frac{4\pi R^3 g_x}{3} = 0$