Static Magnetic Fields

Introduction to Electromagnetism with Practice

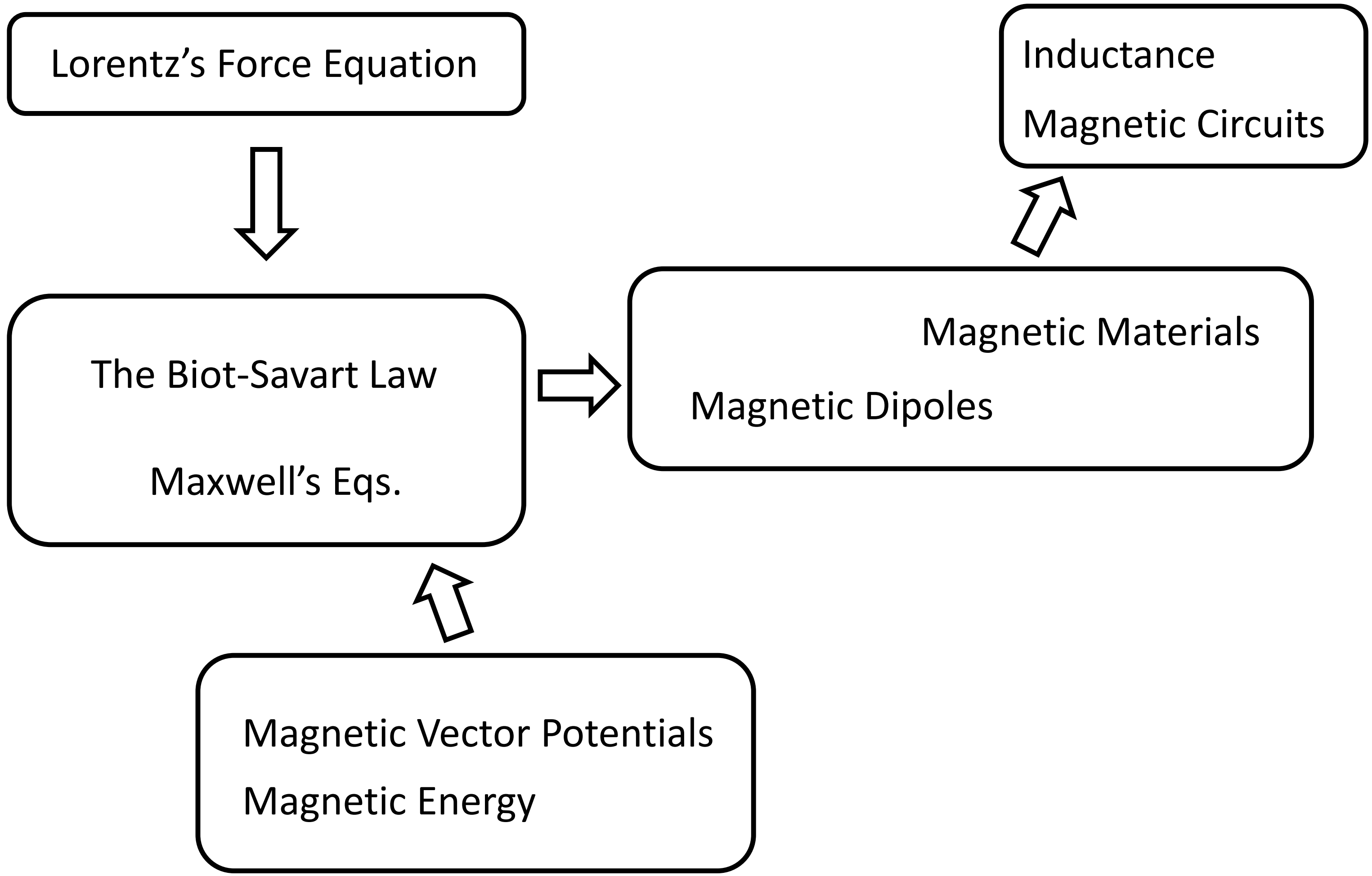
Theory & Applications

Sunkyu Yu

Dept. of Electrical and Computer Engineering

Seoul National University

Very Rough Map of Magnetostatics



Maxwell's Equations

General Form: Light in the Vacuum

$$\nabla \times \mathbf{E} = -\mu_0 \frac{\partial \mathbf{H}}{\partial t}$$

$$\nabla \times \mathbf{H} = \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t}$$

$$\nabla \cdot \mathbf{E} = 0$$

$$\nabla \cdot \mathbf{H} = 0$$

Maxwell's Equations
in the ***Vacuum***

$\mu_0 = 4\pi \times 10^{-7}$ (H/m)
$\varepsilon_0 = 8.854 \times 10^{-12}$ (F/m)

General Form: Light in Media

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$

$$\nabla \times \mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t}$$

$$\nabla \cdot \mathbf{D} = \rho$$

$$\nabla \cdot \mathbf{B} = 0$$

Maxwell's Equations
in ***General Media***

$$\mathbf{D} = \mathbf{D}(\mathbf{E}, \mathbf{H})$$

$$\mathbf{B} = \mathbf{B}(\mathbf{E}, \mathbf{H})$$

Also, functions of $\mathbf{x}, t, \mathbf{k}, \omega$

Approximated Form: Light in Simpler Media

$$\mathbf{D} = \mathbf{D}(\mathbf{E}, \mathbf{H})$$
$$\mathbf{B} = \mathbf{B}(\mathbf{E}, \mathbf{H})$$

This approx. loop is not unique!

Linear, Local (**k**-independent)

functions of **x**,*t*,*ω*

$$\mathbf{D} = \varepsilon_0 \overline{\boldsymbol{\varepsilon}}\mathbf{E} + \overline{\boldsymbol{\chi}}_{\mathrm{EH}}\mathbf{H}$$
$$\mathbf{B} = \overline{\boldsymbol{\chi}}_{\mathrm{HE}}\mathbf{E} + \mu_0 \overline{\boldsymbol{\mu}}\mathbf{H}$$

Without Bi-Isotropy/Bi-Anisotropy

$$\mathbf{D} = \varepsilon_0 \overline{\boldsymbol{\varepsilon}}\mathbf{E}$$
$$\mathbf{B} = \mu_0 \overline{\boldsymbol{\mu}}\mathbf{H}$$

Isotropic

$$\mathbf{D} = \varepsilon_0 \varepsilon_{\mathrm{r}}(\mathbf{x}, t, \omega)\mathbf{E}$$
$$\mathbf{B} = \mu_0 \mu_{\mathrm{r}}(\mathbf{x}, t, \omega)\mathbf{H}$$

Static Materials

functions of **x**,*ω*

$$\mathbf{D} = \varepsilon_0 \varepsilon_{\mathrm{r}}(\mathbf{x}, \omega)\mathbf{E}$$
$$\mathbf{B} = \mu_0 \mu_{\mathrm{r}}(\mathbf{x}, \omega)\mathbf{H}$$

Homogeneous

functions of *ω*

$$\mathbf{D} = \varepsilon_0 \varepsilon_{\mathrm{r}}(\omega)\mathbf{E}$$
$$\mathbf{B} = \mu_0 \mu_{\mathrm{r}}(\omega)\mathbf{H}$$

Linear, Local, Isotropic, Static, Homogeneous Media

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$

$$\nabla \times \mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t}$$

$$\nabla \cdot \mathbf{D} = \rho$$

$$\nabla \cdot \mathbf{B} = 0$$

Maxwell's Equations
in ***Simple Media***

$$\mathbf{D} = \varepsilon_0 \varepsilon_\mathrm{r}(\omega)\mathbf{E}$$

$$\mathbf{B} = \mu_0 \mu_\mathrm{r}(\omega)\mathbf{H}$$

In this Lecture…Focusing on Static Fields, not on Light

very, very slow variation of light fields
~ very small energy of photons

$$\nabla \times \mathbf{E} = -\cancel{\frac{\partial \mathbf{B}}{\partial t}}$$

$$\nabla \times \mathbf{H} = \mathbf{J} + \cancel{\frac{\partial \mathbf{D}}{\partial t}}$$

$$\nabla \cdot \mathbf{D} = \rho$$

$$\nabla \cdot \mathbf{B} = 0$$

Maxwell's Equations
for Static Fields

$$\mathbf{D} = \varepsilon_0 \varepsilon_r \mathbf{E}$$

$$\mathbf{B} = \mu_0 \mu_r \mathbf{H}$$

In this Chapter… Static Magnetic Fields

$$\nabla \times \mathbf{H} = \mathbf{J}$$

$$\nabla \cdot \mathbf{B} = 0$$

$$\mathbf{B} = \mu_0 \mu_{\mathrm{r}} \mathbf{H}$$

Lorentz's Force Equation

Remind: Force from Electric Fields

Electric Field

= **The *electric* force per unit charge** exerted on a small "test charge" q in the limit $q \to 0$

$$\boxed{\mathbf{E}(\mathbf{x}) = \lim_{q \to 0} \frac{\mathbf{F}_e}{q}}$$

Why $q \to 0$? To prohibit the effect on the other charges and existing fields

$$\boxed{\mathbf{F}_e = q\mathbf{E}}$$

It is the force for stationary charges

What happens when charges move?

Force from Magnetic Fields

Charge Velocity × Magnetic Field
= **The *magnetic* force per unit charge** exerted on a small "test charge" q in the limit $q \to 0$

$$\boxed{\mathbf{v} \times \mathbf{B}(\mathbf{x}) = \lim_{q \to 0} \frac{\mathbf{F}_m}{q}}$$

Why $q \to 0$? To prohibit the effect on the other charges and existing fields

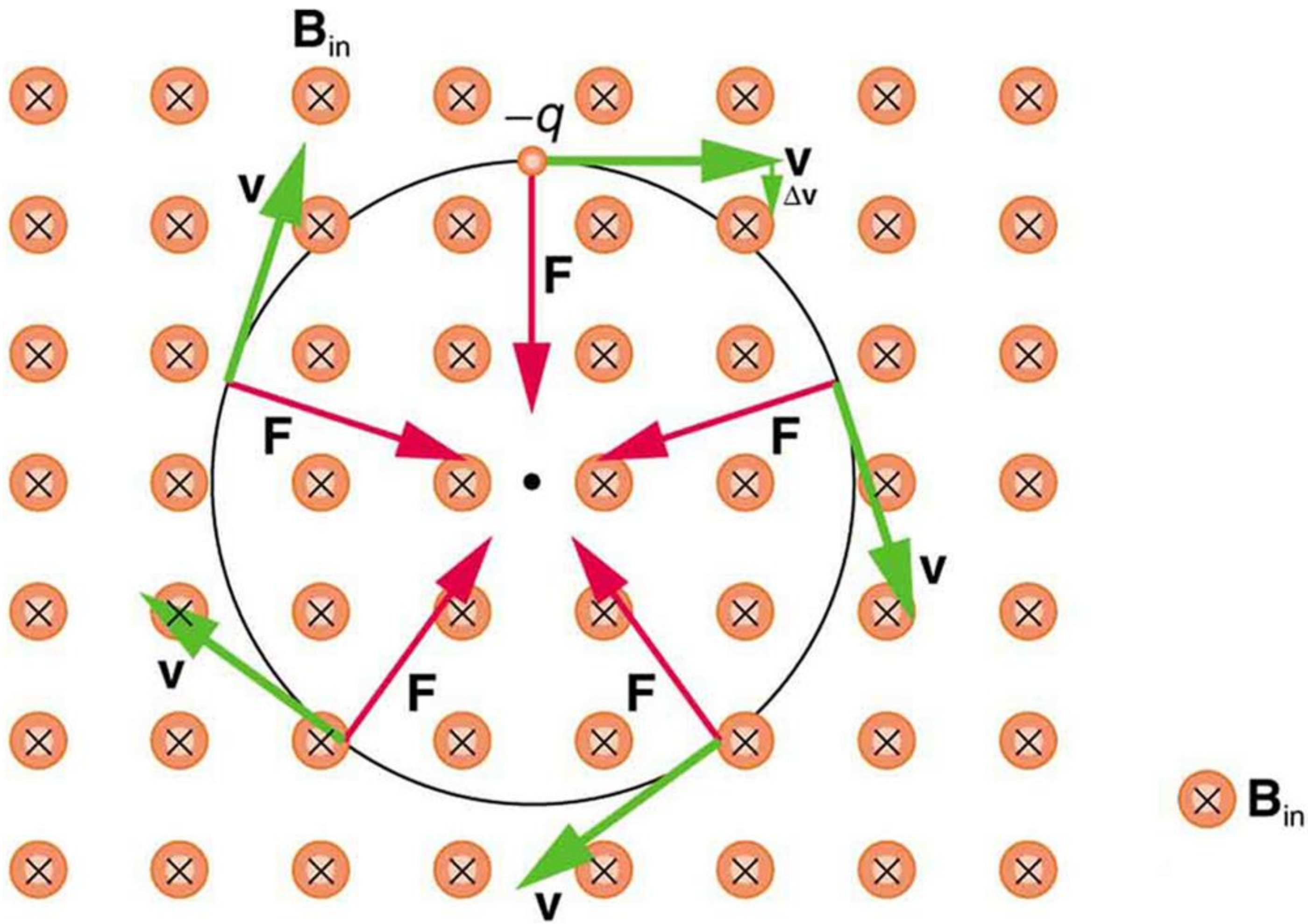
$$\boxed{\mathbf{F}_m = q\mathbf{v} \times \mathbf{B}}$$

Lorentz's Force Equation

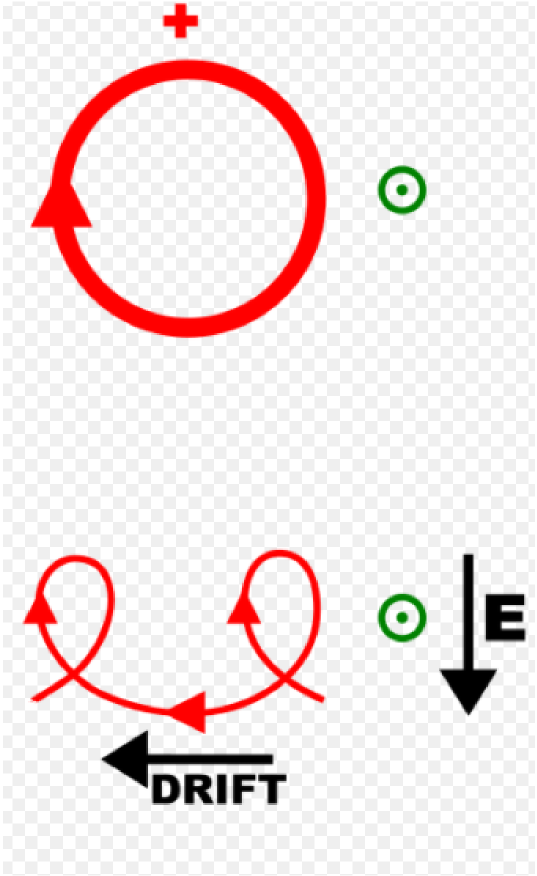
Lorentz's Force Equation

$$\boxed{\mathbf{F} = \mathbf{F}_e + \mathbf{F}_m = q\left(\mathbf{E} + \mathbf{v} \times \mathbf{B}\right)}$$

Charge Motion inside a Magnetic Field



E × B Drift



Proper E & B provides an interesting motion…

The Biot-Savart Law

The Biot-Savart Law

$$\boxed{\mathbf{F}_e = q\mathbf{E}}$$

$$\boxed{\mathbf{F}_m = q\mathbf{v} \times \mathbf{B}}$$

$$\boxed{\mathbf{E} = \frac{1}{4\pi\varepsilon_0} \int_V \rho(\mathbf{x}') \frac{\mathbf{x}-\mathbf{x}'}{|\mathbf{x}-\mathbf{x}'|^3} d^3x'}$$

$$\boxed{\mathbf{B} = \frac{\mu_0}{4\pi} \int_V \frac{\mathbf{j}(\mathbf{x}') \times (\mathbf{x}-\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|^3} d^3x'}$$

Stationary Charges

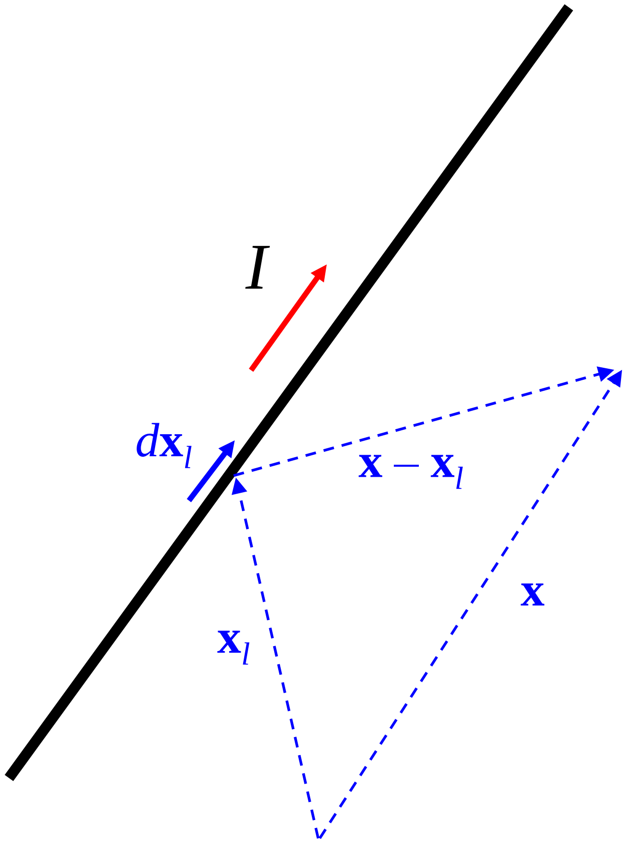
$$\boxed{\rho(\mathbf{x}, t) = \rho(\mathbf{x})}$$

$$\nabla \frac{1}{|\mathbf{x}-\mathbf{x}'|} = -\frac{\mathbf{x}-\mathbf{x}'}{|\mathbf{x}-\mathbf{x}'|^3}$$

Steady Currents

$$\boxed{\nabla \cdot \mathbf{J} = -\frac{d\rho}{dt} = 0}$$

The Biot-Savart Law for a 1D Wire – Constant Current



$$\boxed{\mathbf{B} = \frac{\mu_0}{4\pi}\int_V \frac{\mathbf{J}(\mathbf{x}')\times(\mathbf{x}-\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|^3}d^3x'}$$

$$\mathbf{J}(\mathbf{x}')d^3x' \to I d\mathbf{l}$$

$$\mathbf{B} = \frac{\mu_0 I}{4\pi}\int \frac{d\mathbf{x}_l\times(\mathbf{x}-\mathbf{x}_l)}{|\mathbf{x}-\mathbf{x}_l|^3}$$

Because the current should compose the *closed* loop, ...

$$\boxed{\mathbf{B} = \frac{\mu_0 I}{4\pi}\oint_C \frac{d\mathbf{x}_l\times(\mathbf{x}-\mathbf{x}_l)}{|\mathbf{x}-\mathbf{x}_l|^3}}$$

Example 022

EXAMPLE 6–4 A direct current I flows in a straight wire of length $2L$. Find the magnetic flux density $\mathbf{B}$ at a point located at a distance r from the wire in the bisecting plane: (a) by determining the vector magnetic potential $\mathbf{A}$ first, and (b) by applying Biot-Savart law.

By applying Biot-Savart law. From Fig. 6–5 we see that the distance vector *from* the source element dz' *to* the field point P is

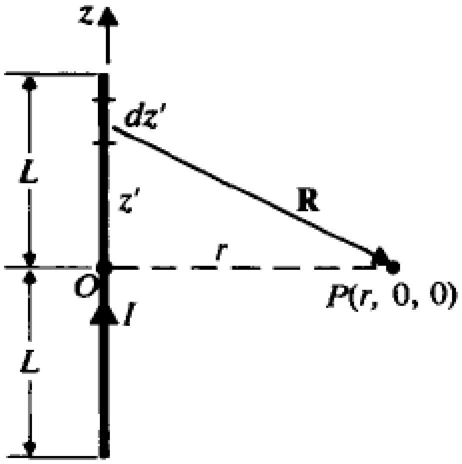
$$\mathbf{R} = \mathbf{a}_r r - \mathbf{a}_z z'$$

$$\boxed{d\ell' \times \mathbf{R} = \mathbf{a}_z\, dz' \times (\mathbf{a}_r r - \mathbf{a}_z z') = \mathbf{a}_\phi r\, dz'.}$$

Substitution in Eq. (6–33c) gives

$$\mathbf{B} = \int d\mathbf{B} = \mathbf{a}_\phi \frac{\mu_0 I}{4\pi} \int_{-L}^{L} \frac{r\, dz'}{(z'^2 + r^2)^{3/2}}$$

$$= \mathbf{a}_\phi \frac{\mu_0 I L}{2\pi r \sqrt{L^2 + r^2}},$$

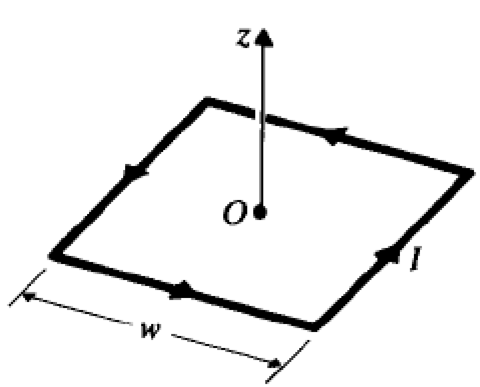


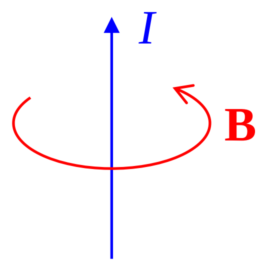
which is the same as Eq. (6–35). ∎

Example 023

EXAMPLE 6–5 Find the magnetic flux density at the center of a square loop, with side w carrying a direct current I.

Solution Assume that the loop lies in the xy-plane, as shown in Fig. 6–6. The magnetic flux density at the center of the square loop is equal to four times that caused





$$\boxed{\begin{aligned}\mathbf{B} = \int d\mathbf{B} &= \mathbf{a}_\phi \frac{\mu_0 I}{4\pi} \int_{-L}^{L} \frac{r\,dz'}{(z'^2 + r^2)^{3/2}} \\ &= \mathbf{a}_\phi \frac{\mu_0 I L}{2\pi r \sqrt{L^2 + r^2}},\end{aligned}}$$

FIGURE 6–6
A square loop carrying current I (Example 6–5).

by a single side of length w. We have, by setting $L = r = w/2$ in Eq. (6–35),

$$\mathbf{B} = \mathbf{a}_z \frac{\mu_0 I}{\sqrt{2}\pi w} \times 4 = \mathbf{a}_z \frac{2\sqrt{2}\mu_0 I}{\pi w}, \tag{6–37}$$

where the direction of $\mathbf{B}$ and that of the current in the loop follow the right-hand rule.

Example 024

EXAMPLE 6–6 Find the magnetic flux density at a point on the axis of a circular loop of radius b that carries a direct current I.

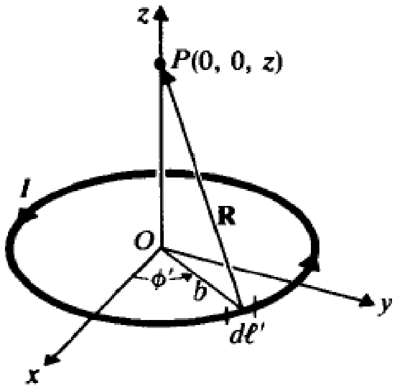


FIGURE 6–7
A circular loop carrying current I (Example 6–6).

Solution We apply Biot-Savart law to the circular loop shown in Fig. 6–7:

$$d\boldsymbol{\ell}' = \mathbf{a}_\phi b\, d\phi',$$
$$\mathbf{R} = \mathbf{a}_z z - \mathbf{a}_r b,$$
$$R = (z^2 + b^2)^{1/2}.$$

Again it is important to remember that $\mathbf{R}$ is the vector *from* the source element $d\boldsymbol{\ell}'$ *to* the field point P. We have

$$\boxed{d\boldsymbol{\ell}' \times \mathbf{R} = \mathbf{a}_\phi b\, d\phi' \times (\mathbf{a}_z z - \mathbf{a}_r b)}$$
$$= \mathbf{a}_r bz\, d\phi' + \mathbf{a}_z b^2\, d\phi'.$$

Example 024

EXAMPLE 6–6 Find the magnetic flux density at a point on the axis of a circular loop of radius b that carries a direct current I.

$$d\ell' \times \mathbf{R} = \mathbf{a}_\phi b\, d\phi' \times (\mathbf{a}_z z - \mathbf{a}_r b)$$
$$= \mathbf{a}_r bz\, d\phi' + \mathbf{a}_z b^2\, d\phi'.$$

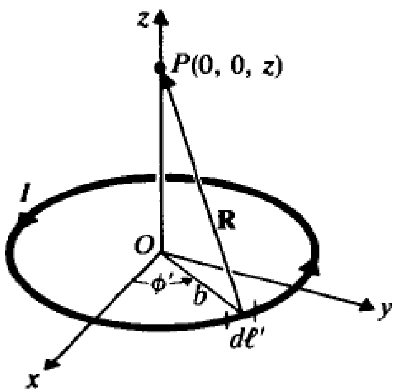


FIGURE 6–7
A circular loop carrying current I (Example 6–6).

Because of cylindrical symmetry, it is easy to see that the $\mathbf{a}_r$-component is canceled by the contribution of the element located diametrically opposite to $d\ell'$, so we need only consider the $\mathbf{a}_z$-component of this cross product.

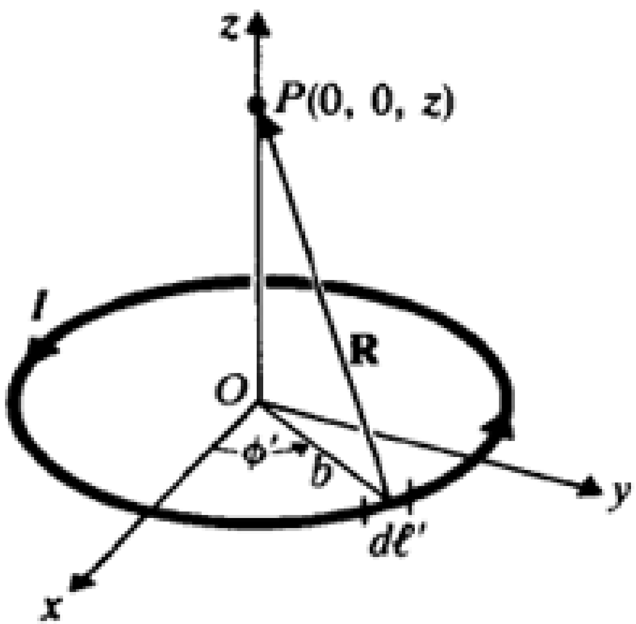
We write, from Eqs. (6–33a) and (6–33c),

$$\mathbf{B} = \frac{\mu_0 I}{4\pi} \int_0^{2\pi} \mathbf{a}_z \frac{b^2\, d\phi'}{(z^2 + b^2)^{3/2}}$$

or

$$\boxed{\mathbf{B} = \mathbf{a}_z \frac{\mu_0 I b^2}{2(z^2 + b^2)^{3/2}} \quad \text{(T)}.} \qquad (6\text{–}38)$$

Asymptotic Forms



$$\mathbf{B} = \mathbf{e}_z \frac{\mu_0 I b^2}{2\left(z^2 + b^2\right)^{3/2}}$$

b >> z

$$\mathbf{B} \sim \mathbf{e}_z \frac{\mu_0 I}{2b}$$

b << z

$$\mathbf{B} = \mathbf{e}_z \frac{\mu_0 I b^2}{2z^3}$$

Magnetostatics – Maxwell's Equations

Postulates: Differential Form

$$\mu_r = 1$$ in the Vacuum

$$\nabla \times \mathbf{H} = \mathbf{J}$$

$$\nabla \cdot \mathbf{B} = 0$$

$$\mathbf{B} = \mu_0 \mu_r \mathbf{H}$$

$$\Rightarrow$$

$$\boxed{\begin{aligned} \nabla \times \mathbf{B} &= \mu_0 \mathbf{J} \\ \nabla \cdot \mathbf{B} &= 0 \end{aligned}}$$

Postulates: Integral Form

$\mu_r = 1$ in the Vacuum

$$\nabla \times \mathbf{H} = \mathbf{J}$$

$$\nabla \cdot \mathbf{B} = 0$$

$$\mathbf{B} = \mu_0 \mu_r \mathbf{H}$$

Gauss & Stokes

$$\boxed{\begin{gathered}\oint_C \mathbf{B} \cdot d\mathbf{l} = \mu_0 I \\ \oint_S \mathbf{B} \cdot d\mathbf{S} = 0\end{gathered}}$$

Electrostatics versus Magnetostatics

$$\mathbf{E} = \frac{1}{4\pi\varepsilon_0}\int_V \rho(\mathbf{x}')\frac{\mathbf{x}-\mathbf{x}'}{|\mathbf{x}-\mathbf{x}'|^3}d^3x'$$

$$\mathbf{B} = \frac{\mu_0}{4\pi}\int_V \frac{\mathbf{j}(\mathbf{x}')\times(\mathbf{x}-\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|^3}d^3x'$$

$$\nabla\times\mathbf{E} = \mathbf{O}$$

$$\nabla\cdot\mathbf{E} = \frac{\rho}{\varepsilon_0}$$

$$\nabla\times\mathbf{B} = \mu_0\mathbf{J}$$

$$\nabla\cdot\mathbf{B} = 0$$

Then, a Natural Question should arise…

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{J}$$

$$\nabla \cdot \mathbf{B} = 0$$

$$\boxed{\mathbf{B} = \frac{\mu_0}{4\pi} \int_V \frac{\mathbf{j}(\mathbf{x}') \times (\mathbf{x} - \mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|^3} d^3x'}$$

It will be discussed in the next lecture!

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Connecting Maxwell's Eq. & The Biot-Savart Law

From the Biot-Savart to Maxwell: Divergence

The Biot-Savart Law

$$\boxed{\mathbf{B}=\frac{\mu_0}{4\pi}\int_V \frac{\mathbf{J}(\mathbf{x}')\times(\mathbf{x}-\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|^3}d^3x'}$$

$$\nabla\frac{1}{|\mathbf{x}-\mathbf{x}'|}=-\frac{\mathbf{x}-\mathbf{x}'}{|\mathbf{x}-\mathbf{x}'|^3}$$

$$\mathbf{B}=\frac{\mu_0}{4\pi}\int_V \frac{\mathbf{J}(\mathbf{x}')\times(\mathbf{x}-\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|^3}d^3x'=-\frac{\mu_0}{4\pi}\int_V \mathbf{J}(\mathbf{x}')\times\nabla\frac{1}{|\mathbf{x}-\mathbf{x}'|}d^3x'$$

$$\nabla\times(f\mathbf{A})=f(\nabla\times\mathbf{A})-\mathbf{A}\times(\nabla f) \Longrightarrow \mathbf{A}\times(\nabla f)=f(\nabla\times\mathbf{A})-\nabla\times(f\mathbf{A})$$

From the Biot-Savart to Maxwell: Divergence

$$\mathbf{B} = -\frac{\mu_0}{4\pi}\int_V \mathbf{J}(\mathbf{x}')\times\nabla\frac{1}{|\mathbf{x}-\mathbf{x}'|}d^3x' = \frac{\mu_0}{4\pi}\int_V\left[\nabla\times\frac{\mathbf{J}(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|} - \frac{\nabla\times\mathbf{J}(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|}\right]d^3x'$$

$$= \nabla\times\frac{\mu_0}{4\pi}\int_V\frac{\mathbf{J}(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|}d^3x' - \frac{\mu_0}{4\pi}\int_V\left[\frac{\nabla\times\mathbf{J}(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|}\right]d^3x'$$

Remember that ∇ is applied to **x** coordinates, not to **x**’ coordinates

$$\frac{\mu_0}{4\pi}\int_V\left[\frac{\cancelto{0}{\nabla\times\mathbf{J}(\mathbf{x}')}}{|\mathbf{x}-\mathbf{x}'|}\right]d^3x' = 0$$

$$\mathbf{B} = \nabla\times\frac{\mu_0}{4\pi}\int_V\frac{\mathbf{J}(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|}d^3x' \quad\Rightarrow\quad \boxed{\nabla\cdot\mathbf{B} = 0}$$

From the Biot-Savart to Maxwell: Curl

The Biot-Savart Law

$$\boxed{\mathbf{B} = \frac{\mu_0}{4\pi}\int_V \frac{\mathbf{J}(\mathbf{x}')\times(\mathbf{x}-\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|^3}d^3x'}$$

$$\nabla\times\mathbf{B} = \frac{\mu_0}{4\pi}\int_V \nabla\times\left(\frac{\mathbf{J}(\mathbf{x}')\times(\mathbf{x}-\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|^3}\right)d^3x'$$

$$\nabla\times(\mathbf{A}\times\mathbf{B}) = \mathbf{A}(\nabla\cdot\mathbf{B}) - \mathbf{B}(\nabla\cdot\mathbf{A}) + (\mathbf{B}\cdot\nabla)\mathbf{A} - (\mathbf{A}\cdot\nabla)\mathbf{B}$$

$$\nabla\times\left(\frac{\mathbf{J}(\mathbf{x}')\times(\mathbf{x}-\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|^3}\right) = \mathbf{J}(\mathbf{x}')\left[\nabla\cdot\frac{(\mathbf{x}-\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|^3}\right] - \frac{(\mathbf{x}-\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|^3}\cancelto{0}{\left[\nabla\cdot\mathbf{J}(\mathbf{x}')\right]}$$

$$+\cancelto{0}{\left[\frac{(\mathbf{x}-\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|^3}\cdot\nabla\right]\mathbf{J}(\mathbf{x}')} - \left[\mathbf{J}(\mathbf{x}')\cdot\nabla\right]\frac{(\mathbf{x}-\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|^3}$$

Remember that ∇ is applied to **x** coordinates, not to **x**’ coordinates

From the Biot-Savart to Maxwell: Curl

$$\nabla\times\left(\frac{\mathbf{J}(\mathbf{x}')\times(\mathbf{x}-\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|^3}\right)=\mathbf{J}(\mathbf{x}')\left[\nabla\cdot\frac{(\mathbf{x}-\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|^3}\right]-\left[\mathbf{J}(\mathbf{x}')\cdot\nabla\right]\frac{(\mathbf{x}-\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|^3}$$

Potential to Field

$$\boxed{\nabla\frac{1}{|\mathbf{x}-\mathbf{x}'|}=-\nabla'\frac{1}{|\mathbf{x}-\mathbf{x}'|}=-\frac{\mathbf{x}-\mathbf{x}'}{|\mathbf{x}-\mathbf{x}'|^3}}$$

Green Function

$$\boxed{\nabla^2\frac{1}{|\mathbf{x}-\mathbf{x}'|}=-4\pi\delta^3(\mathbf{x}-\mathbf{x}')}$$

$$\nabla\cdot\frac{\mathbf{x}-\mathbf{x}'}{|\mathbf{x}-\mathbf{x}'|^3}=-\nabla\cdot\left[\nabla\frac{1}{|\mathbf{x}-\mathbf{x}'|}\right]=-\nabla^2\frac{1}{|\mathbf{x}-\mathbf{x}'|}=4\pi\delta^3(\mathbf{x}-\mathbf{x}')$$

Let's consider this!

$$\nabla\times\left(\frac{\mathbf{J}(\mathbf{x}')\times(\mathbf{x}-\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|^3}\right)=4\pi\mathbf{J}(\mathbf{x}')\delta^3(\mathbf{x}-\mathbf{x}')-\left[\mathbf{J}(\mathbf{x}')\cdot\nabla\right]\frac{(\mathbf{x}-\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|^3}$$

From the Biot-Savart to Maxwell: Curl

$$[\mathbf{J}(\mathbf{x}')\cdot\nabla]\frac{(\mathbf{x}-\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|^3} = -[\mathbf{J}(\mathbf{x}')\cdot\nabla']\frac{(\mathbf{x}-\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|^3}$$

$$\nabla\cdot\mathbf{J} = -\frac{d\rho}{dt} = 0$$

(steady current)

$$\boxed{\nabla'\cdot[f\mathbf{J}(\mathbf{x}')] = [\mathbf{J}(\mathbf{x}')\cdot\nabla']f + f[\nabla'\cdot\mathbf{J}(\mathbf{x}')]}$$

$$\left([\mathbf{J}(\mathbf{x}')\cdot\nabla']\frac{(\mathbf{x}-\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|^3}\right)_x = \nabla'\cdot\left[\frac{(x-x')}{|\mathbf{x}-\mathbf{x}'|^3}\mathbf{J}(\mathbf{x}')\right] - \frac{(x-x')}{|\mathbf{x}-\mathbf{x}'|^3}\cancel{[\nabla'\cdot\mathbf{J}(\mathbf{x}')]}$$

$$\left(-[\mathbf{J}(\mathbf{x}')\cdot\nabla]\frac{(\mathbf{x}-\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|^3}\right)_x = \nabla'\cdot\left[\frac{(x-x')}{|\mathbf{x}-\mathbf{x}'|^3}\mathbf{J}(\mathbf{x}')\right]$$

$$\int_V \nabla'\cdot\left[\frac{(x-x')}{|\mathbf{x}-\mathbf{x}'|^3}\mathbf{J}(\mathbf{x}')\right]d^3x' = \cancel{\oint_S \frac{(x-x')}{|\mathbf{x}-\mathbf{x}'|^3}\mathbf{J}(\mathbf{x}')\cdot ds}$$

= 0: Sufficiently Large *V* for **J** = 0 at the outside

From the Biot-Savart to Maxwell: Curl

Nonzero…but neglectable at the volume integral

$$\nabla\times\left(\frac{\mathbf{J}(\mathbf{x}')\times(\mathbf{x}-\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|^3}\right)=4\pi\mathbf{J}(\mathbf{x}')\delta^3(\mathbf{x}-\mathbf{x}')-\cancel{\left[\mathbf{J}(\mathbf{x}')\cdot\nabla\right]\frac{(\mathbf{x}-\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|^3}}$$

$$\nabla\times\left(\frac{\mathbf{J}(\mathbf{x}')\times(\mathbf{x}-\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|^3}\right)=4\pi\mathbf{J}(\mathbf{x}')\delta^3(\mathbf{x}-\mathbf{x}')$$

$$\nabla\times\mathbf{B}=\frac{\mu_0}{4\pi}\int_V\nabla\times\left(\frac{\mathbf{J}(\mathbf{x}')\times(\mathbf{x}-\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|^3}\right)d^3x'$$

$$=\frac{\mu_0}{4\pi}\int_V 4\pi\mathbf{J}(\mathbf{x}')\delta^3(\mathbf{x}-\mathbf{x}')d^3x'=\mu_0\mathbf{J}(\mathbf{x})$$

$$\boxed{\nabla\times\mathbf{B}=\mu_0\mathbf{J}}$$

From the Biot-Savart to Maxwell: Conclusion

The Biot-Savart Law

$$\boxed{\mathbf{B} = \frac{\mu_0}{4\pi}\int_V \frac{\mathbf{J}(\mathbf{x}')\times(\mathbf{x}-\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|^3}\, d^3x'}$$

$$\boxed{\nabla\cdot\mathbf{B} = 0} \qquad \boxed{\nabla\times\mathbf{B} = \mu_0\mathbf{J}}$$

Not different laws, but the same laws

– The Biot-Savart Law is the Solution of Maxwell's Eqs –

Magnetic Vector Potential

Remind: Helmholtz Theorem

Helmholtz Theorem (or Helmholtz Decomposition)

An arbitrary vector field can always be decomposed into the sum of two vector fields: ***one with zero divergence*** **and** ***one with zero curl***

$$\mathbf{E} = \mathbf{E}_{\mathrm{D}} + \mathbf{E}_{\mathrm{C}}$$

Solenoidal (divergence-free)

$$\nabla \cdot \mathbf{E}_{\mathrm{D}} = 0$$

Irrotational (curl-free)

$$\nabla \times \mathbf{E}_{\mathrm{C}} = \mathbf{O}$$

Remembering Null Identities

$$\nabla \cdot \left(\nabla \times \mathbf{A} \right) = 0$$

$$\nabla \times \left(\nabla f \right) = \mathbf{O}$$

We can write **E** as follow:

$$\mathbf{E} = \nabla \times \mathbf{A} + \nabla f$$

The proper boundary condition (B.C.) allows the unique **E**

Remind: Helmholtz Theorem for Electrostatics

$$\mathbf{E} = \mathbf{E}_\mathrm{D} + \mathbf{E}_\mathrm{C}$$

$$\nabla \cdot \mathbf{E}_\mathrm{D} = 0$$

$$\nabla \times \mathbf{E}_\mathrm{C} = \mathbf{O}$$

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\varepsilon_0}$$

$$\nabla \times \mathbf{E} = \mathbf{O}$$

$$\nabla \cdot \mathbf{E} = \nabla \cdot \left(\mathbf{E}_\mathrm{D} + \mathbf{E}_\mathrm{C} \right)$$

$$= \nabla \cdot \mathbf{E}_\mathrm{C} = \frac{\rho}{\varepsilon_0}$$

$$\nabla \times \mathbf{E} = \nabla \times \left(\mathbf{E}_\mathrm{D} + \mathbf{E}_\mathrm{C} \right)$$

$$= \nabla \times \mathbf{E}_\mathrm{D} = \mathbf{O}$$

$$\mathbf{E}_\mathrm{D} = \mathbf{O}$$

Remind: Helmholtz Theorem for Electrostatics

$$\mathbf{E} = \mathbf{E}_C \qquad \nabla \times \mathbf{E}_C = \mathbf{0}$$

Let's assign $\mathbf{E}_C$ for the conventional notation

$$\mathbf{E}_C = -\nabla V$$

Electric Potential (or Scalar Potential) *V*

$$\boxed{\mathbf{E} = -\nabla V}$$

Helmholtz Theorem for Magnetostatics

$$\boxed{\nabla \cdot \mathbf{B} = 0}$$

$$\boxed{\nabla \times \mathbf{B} = \mu_0 \mathbf{J}}$$

$$\boxed{\mathbf{B} = \nabla \times \mathbf{A}}$$

Magnetic Vector Potential **A**

Gauge Condition (or Non-Uniqueness) of A

$$\boxed{\nabla \times \mathbf{A} = \mathbf{B}}$$

$$\mathbf{A}' = \mathbf{A} + \nabla \chi$$

$$\boxed{\nabla \times \mathbf{A}' = \nabla \times (\mathbf{A} + \nabla \chi) = \nabla \times \mathbf{A} = \mathbf{B}}$$

Both A & A' are magnetic vector potentials of B

➔ Non-Unique! Or Gauge-dependent!

Vector Poisson's Equation

$$\boxed{\mathbf{B} = \nabla \times \mathbf{A}}$$

$$\nabla \times \mathbf{B} = \nabla \times \left(\nabla \times \mathbf{A}\right) = \nabla\left(\nabla \cdot \mathbf{A}\right) - \nabla^2 \mathbf{A} = \mu_0 \mathbf{J}$$

If we "set"

$$\nabla \cdot \mathbf{A} = 0$$

$$\boxed{\nabla^2 \mathbf{A} = -\mu_0 \mathbf{J}}$$

Remind: Superposition for Electric Potentials

$$\mathbf{E} = -\nabla V \qquad \sum_k \mathbf{E}_k = -\nabla \sum_k V_k$$

The superposition principle is also valid for the potential *V*

$$\mathbf{E} = \frac{1}{4\pi\varepsilon_0} q \frac{\mathbf{r}}{r^3} = \frac{1}{4\pi\varepsilon_0} q \frac{\mathbf{e}_r}{r^2} \qquad \boxed{V = \frac{q}{4\pi\varepsilon_0} \frac{1}{r}}$$

$$\mathbf{E} = \frac{1}{4\pi\varepsilon_0} \sum_{k=1}^{N} q_k \frac{\mathbf{x} - \mathbf{x}_k}{\left|\mathbf{x} - \mathbf{x}_k\right|^3} \qquad \boxed{\nabla \frac{1}{\left|\mathbf{x} - \mathbf{x}'\right|} = -\frac{\mathbf{x} - \mathbf{x}'}{\left|\mathbf{x} - \mathbf{x}'\right|^3}} \qquad \boxed{V = \frac{1}{4\pi\varepsilon_0} \sum_{k=1}^{N} q_k \frac{1}{\left|\mathbf{x} - \mathbf{x}_k\right|}}$$

$$\mathbf{E} = \frac{1}{4\pi\varepsilon_0} \int_V \frac{\mathbf{x} - \mathbf{x}'}{\left|\mathbf{x} - \mathbf{x}'\right|^3} \rho(\mathbf{x}') d^3x' \qquad \boxed{V = \frac{1}{4\pi\varepsilon_0} \int_V \frac{1}{\left|\mathbf{x} - \mathbf{x}'\right|} \rho(\mathbf{x}') d^3x'}$$

Vector Poisson's Equation

$$\boxed{\nabla^2 \mathbf{A} = -\mu_0 \mathbf{J}}$$

$$A_x = \frac{\mu_0}{4\pi} \int_V \frac{1}{|\mathbf{x}-\mathbf{x}'|} J_x(\mathbf{x}') d^3x'$$

$$A_y = \frac{\mu_0}{4\pi} \int_V \frac{1}{|\mathbf{x}-\mathbf{x}'|} J_y(\mathbf{x}') d^3x'$$

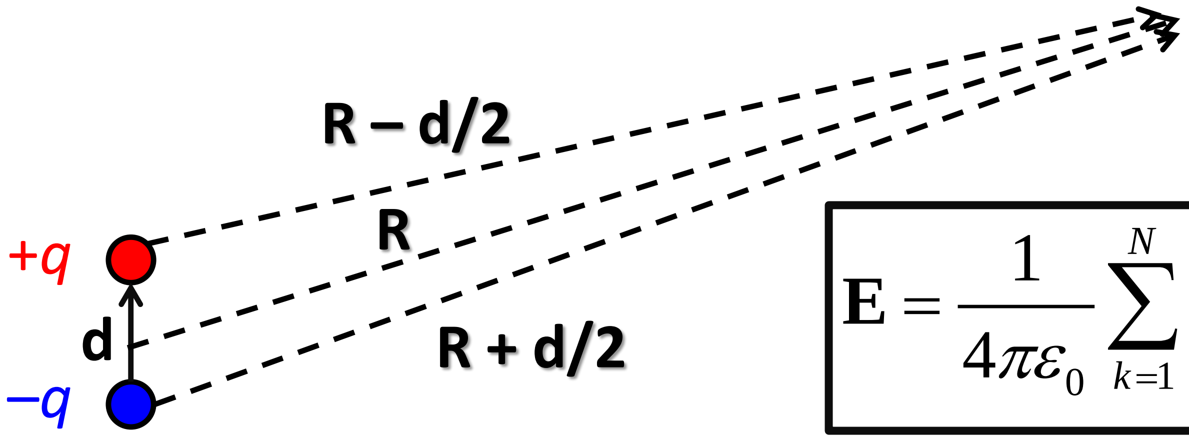
$$\boxed{\mathbf{A} = \frac{\mu_0}{4\pi} \int_V \frac{1}{|\mathbf{x}-\mathbf{x}'|} \mathbf{J}(\mathbf{x}') d^3x'}$$

$$A_z = \frac{\mu_0}{4\pi} \int_V \frac{1}{|\mathbf{x}-\mathbf{x}'|} J_z(\mathbf{x}') d^3x'$$

Magnetic Dipole

Remind: Example 001

***Electric Dipole*: Estimating an electric field far from the dipole?**



$$\mathbf{E} = \frac{1}{4\pi\varepsilon_0}\sum_{k=1}^{N} q_k \frac{\mathbf{x}-\mathbf{x}_k}{\left|\mathbf{x}-\mathbf{x}_k\right|^3}$$

$$\mathbf{E} = \frac{1}{4\pi\varepsilon_0}\left(q\frac{\mathbf{R}-\frac{\mathbf{d}}{2}}{\left|\mathbf{R}-\frac{\mathbf{d}}{2}\right|^3} - q\frac{\mathbf{R}+\frac{\mathbf{d}}{2}}{\left|\mathbf{R}+\frac{\mathbf{d}}{2}\right|^3}\right)$$

We cannot learn a lot from accurate but too complex equations!

Remind: Example 001

Electric Dipole: Estimating an electric field far from the dipole?

$$\left|\mathbf{R}-\frac{\mathbf{d}}{2}\right|^{-3}=\left|\left(\mathbf{R}-\frac{\mathbf{d}}{2}\right)\cdot\left(\mathbf{R}-\frac{\mathbf{d}}{2}\right)\right|^{-\frac{3}{2}}=\left(R^2+\frac{d^2}{4}-\mathbf{R}\cdot\mathbf{d}\right)^{-\frac{3}{2}}=\left[R^2\left(1-\frac{\mathbf{R}\cdot\mathbf{d}}{R^2}+\frac{d^2}{4R^2}\right)\right]^{-\frac{3}{2}}$$

$$=R^{-3}\left(1-\frac{\mathbf{R}\cdot\mathbf{d}}{R^2}+\frac{d^2}{4R^2}\right)^{-\frac{3}{2}}\sim R^{-3}\left(1+\frac{3}{2}\frac{\mathbf{R}\cdot\mathbf{d}}{R^2}\right)$$

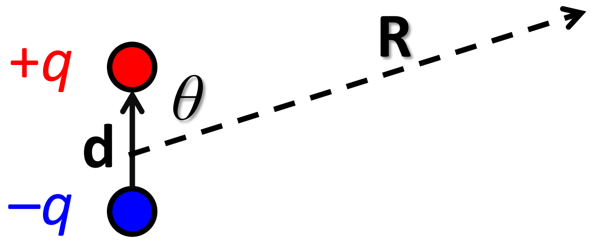
$$\mathbf{E}\sim\frac{q}{4\pi\varepsilon_0}\left[\left(\mathbf{R}-\frac{\mathbf{d}}{2}\right)R^{-3}\left(1+\frac{3}{2}\frac{\mathbf{R}\cdot\mathbf{d}}{R^2}\right)-\left(\mathbf{R}+\frac{\mathbf{d}}{2}\right)R^{-3}\left(1-\frac{3}{2}\frac{\mathbf{R}\cdot\mathbf{d}}{R^2}\right)\right]$$

$$=\frac{q}{4\pi\varepsilon_0 R^3}\left[\left(\mathbf{R}-\frac{\mathbf{d}}{2}\right)\left(1+\frac{3}{2}\frac{\mathbf{R}\cdot\mathbf{d}}{R^2}\right)-\left(\mathbf{R}+\frac{\mathbf{d}}{2}\right)\left(1-\frac{3}{2}\frac{\mathbf{R}\cdot\mathbf{d}}{R^2}\right)\right]$$

$$=\frac{q}{4\pi\varepsilon_0 R^3}\left[3\frac{\mathbf{R}\cdot\mathbf{d}}{R^2}\mathbf{R}-\mathbf{d}\right]$$

Remind: Example 001

Electric Dipole: Estimating an electric field far from the dipole?



Electric Dipole Moment

$$\mathbf{p} = q\mathbf{d}$$

$$\mathbf{E} \sim \frac{q}{4\pi\varepsilon_0 R^3}\left[3\frac{\mathbf{R}\cdot\mathbf{d}}{R^2}\mathbf{R}-\mathbf{d}\right]$$

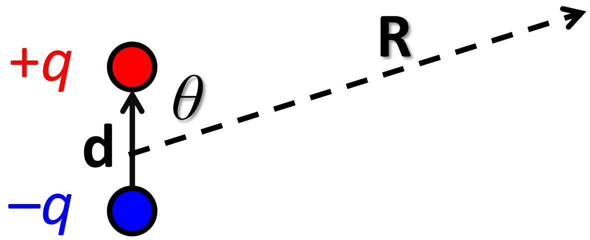
$$= \frac{1}{4\pi\varepsilon_0 R^3}\left[3\frac{\mathbf{R}\cdot\mathbf{p}}{R^2}\mathbf{R}-\mathbf{p}\right] = \frac{1}{4\pi\varepsilon_0 R^3}\left[3\frac{Rp\cos\theta}{R^2}R\mathbf{e}_r - \mathbf{e}_z p\right]$$

$$= \frac{1}{4\pi\varepsilon_0 R^3}\left[3\frac{Rp\cos\theta}{R^2}R\mathbf{e}_r - \left(\mathbf{e}_r\cos\theta - \mathbf{e}_\theta\sin\theta\right)p\right]$$

$$= \frac{p}{4\pi\varepsilon_0 R^3}\left(\mathbf{e}_r 2\cos\theta + \mathbf{e}_\theta\sin\theta\right)$$

Remind: Example 001

Electric Dipole: Estimating an electric field far from the dipole?



Electric Dipole Moment

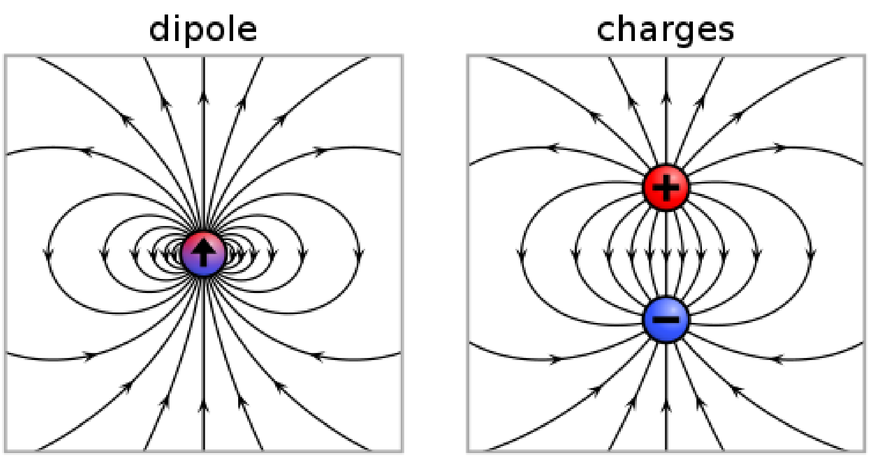
$$\mathbf{p} = q\mathbf{d}$$

$$\boxed{\mathbf{E} \sim \frac{1}{4\pi\varepsilon_0 R^3}\left[3\frac{\mathbf{R}\cdot\mathbf{p}}{R^2}\mathbf{R} - \mathbf{p}\right] = \frac{p}{4\pi\varepsilon_0 R^3}\left(\mathbf{e}_r\, 2\cos\theta + \mathbf{e}_\theta \sin\theta\right)}$$

$$\theta = 0 \qquad \mathbf{E} \sim \frac{2p}{4\pi\varepsilon_0 R^3}\mathbf{e}_r$$

$$\theta = \frac{\pi}{2} \qquad \mathbf{E} \sim \frac{p}{4\pi\varepsilon_0 R^3}\mathbf{e}_\theta$$

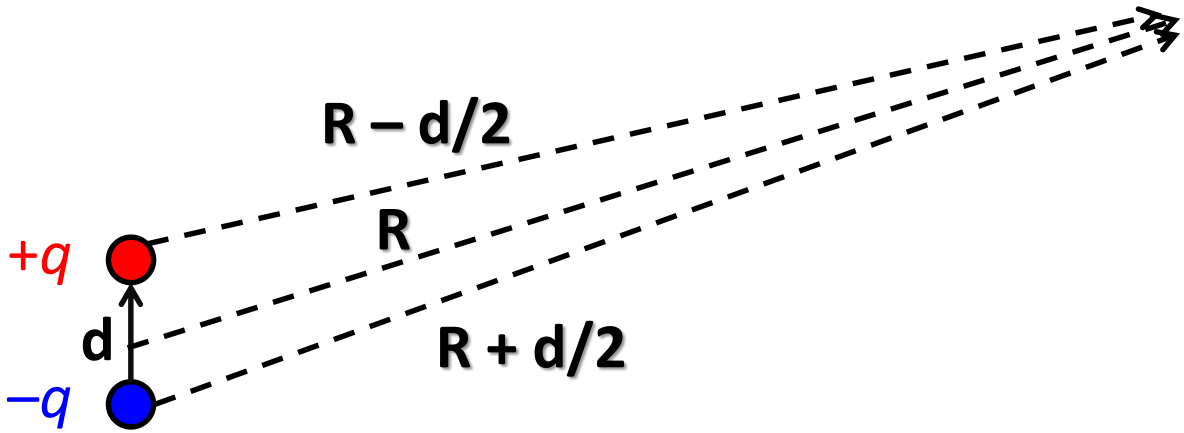
$$\theta = \pi \qquad \mathbf{E} \sim -\frac{2p}{4\pi\varepsilon_0 R^3}\mathbf{e}_r$$



WIKI

Remind: Example 006: Revisiting a Dipole

***Electric Dipole*: Estimating an electric field far from the dipole?**



$$\boxed{V = \frac{1}{4\pi\varepsilon_0}\sum_{k=1}^{N} q_k \frac{1}{\left|\mathbf{x}-\mathbf{x}_k\right|}}$$

$$V = \frac{1}{4\pi\varepsilon_0}\sum_{k=1}^{N} q_k \frac{1}{\left|\mathbf{x}-\mathbf{x}_k\right|} = \frac{q}{4\pi\varepsilon_0}\left(\frac{1}{\left|\mathbf{R}-\mathbf{d}/2\right|} - \frac{1}{\left|\mathbf{R}+\mathbf{d}/2\right|}\right)$$

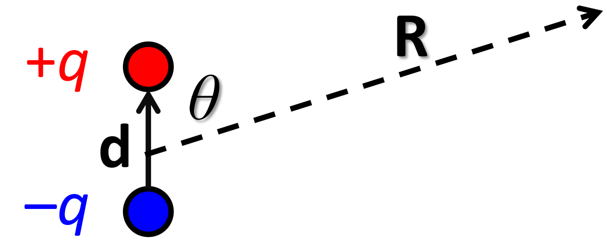
Remind: Example 006: Revisiting a Dipole

$$\left|\mathbf{R}-\mathbf{d}/2\right|^{-1}=\left[\left(\mathbf{R}-\mathbf{d}/2\right)\cdot\left(\mathbf{R}-\mathbf{d}/2\right)\right]^{-1/2}$$
$$=\left[\left|\mathbf{R}\right|^{2}-\mathbf{R}\cdot\mathbf{d}+\left|\mathbf{d}\right|^{2}/4\right]^{-1/2}$$
$$=\frac{1}{\left|\mathbf{R}\right|}\left[1-\frac{\mathbf{R}\cdot\mathbf{d}}{\left|\mathbf{R}\right|^{2}}+\frac{\left|\mathbf{d}\right|^{2}}{4\left|\mathbf{R}\right|^{2}}\right]^{-1/2}\sim\frac{1}{\left|\mathbf{R}\right|}\left(1+\frac{\mathbf{R}\cdot\mathbf{d}}{2\left|\mathbf{R}\right|^{2}}\right)$$

$$V=\frac{q}{4\pi\varepsilon_0}\left(\frac{1}{\left|\mathbf{R}-\mathbf{d}/2\right|}-\frac{1}{\left|\mathbf{R}+\mathbf{d}/2\right|}\right)=\frac{q}{4\pi\varepsilon_0}\frac{\mathbf{R}\cdot\mathbf{d}}{\left|\mathbf{R}\right|^{3}}$$

Remind: Example 006: Revisiting a Dipole

$$V = \frac{q}{4\pi\varepsilon_0}\frac{\mathbf{R}\cdot\mathbf{d}}{|\mathbf{R}|^3} = \frac{q}{4\pi\varepsilon_0}\frac{d\cos\theta}{R^2}$$



Electric Dipole Moment

$$\mathbf{p} = q\mathbf{d}$$

$$V = \frac{p}{4\pi\varepsilon_0}\frac{\cos\theta}{R^2}$$

$$\mathbf{E} = -\nabla V = -\mathbf{e}_r\frac{\partial}{\partial R}\left(\frac{p\cos\theta}{4\pi\varepsilon_0 R^2}\right) - \mathbf{e}_\theta\frac{\partial}{R\partial\theta}\left(\frac{p\cos\theta}{4\pi\varepsilon_0 R^2}\right)$$

$$= \mathbf{e}_r 2\frac{p\cos\theta}{4\pi\varepsilon_0 R^3} + \mathbf{e}_\theta\frac{p\sin\theta}{4\pi\varepsilon_0 R^3} = \frac{p}{4\pi\varepsilon_0 R^3}\left(\mathbf{e}_r 2\cos\theta + \mathbf{e}_\theta\sin\theta\right)$$

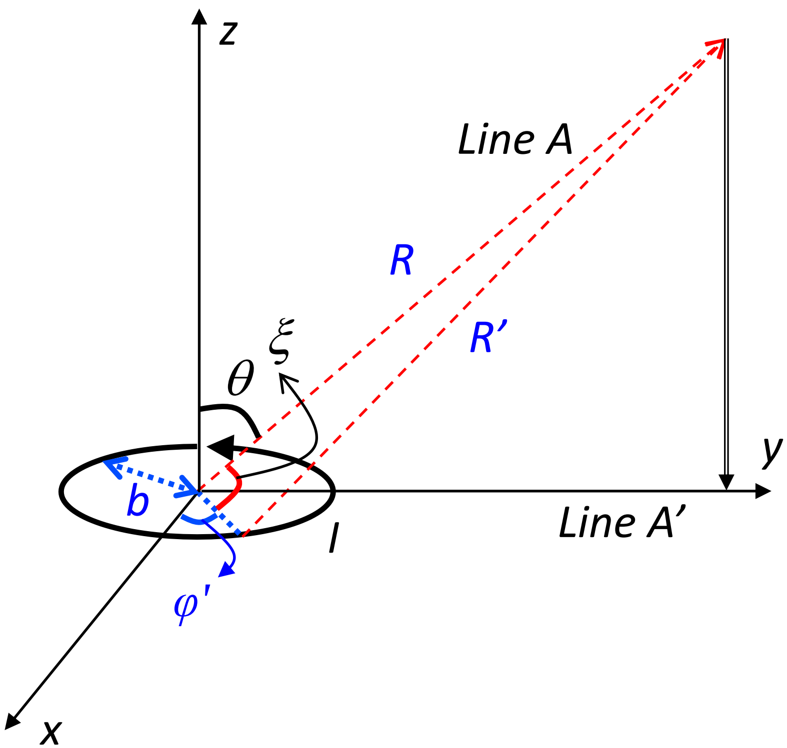
Same result through a simpler process!

Example 025

$$V=\frac{1}{4\pi\varepsilon_0}\sum_{k=1}^{N} q_k \frac{1}{|\mathbf{x}-\mathbf{x}_k|}$$

$$\mathbf{A}=\frac{\mu_0}{4\pi}\int_V \frac{1}{|\mathbf{x}-\mathbf{x}'|}\mathbf{J}(\mathbf{x}')d^3x'$$

$$\mathbf{A}=\frac{\mu_0 I}{4\pi}\oint_{C'}\frac{1}{R'}d\mathbf{l}'$$



$$d\mathbf{l}'=\mathbf{e}_\varphi' b\,d\varphi'=\left(-\mathbf{e}_x\sin\varphi'+\underline{\underline{\mathbf{e}_y}}\cos\varphi'\right)b\,d\varphi'$$

Cancelled

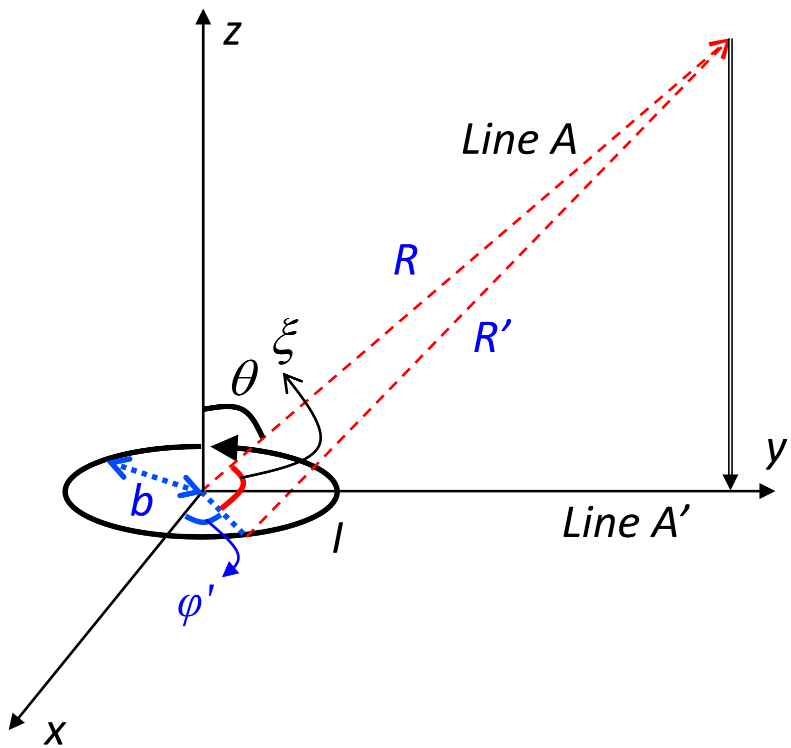
$$\mathbf{A}=-\mathbf{e}_x\frac{\mu_0 I}{4\pi}\int_0^{2\pi}\frac{b\sin\varphi'}{R'}d\varphi'$$

$$=\mathbf{e}_\varphi\frac{\mu_0 I}{4\pi}\int_0^{2\pi}\frac{b\sin\varphi'}{R'}d\varphi'$$

Example 025

$$\mathbf{A} = \mathbf{e}_{\varphi} \frac{\mu_0 I}{4\pi} \int_0^{2\pi} \frac{b \sin\varphi'}{R'} d\varphi'$$

$$R'^2 = R^2 + b^2 - 2bR\cos\xi$$
$$= R^2 + b^2 - 2bR\sin\theta\sin\varphi'$$



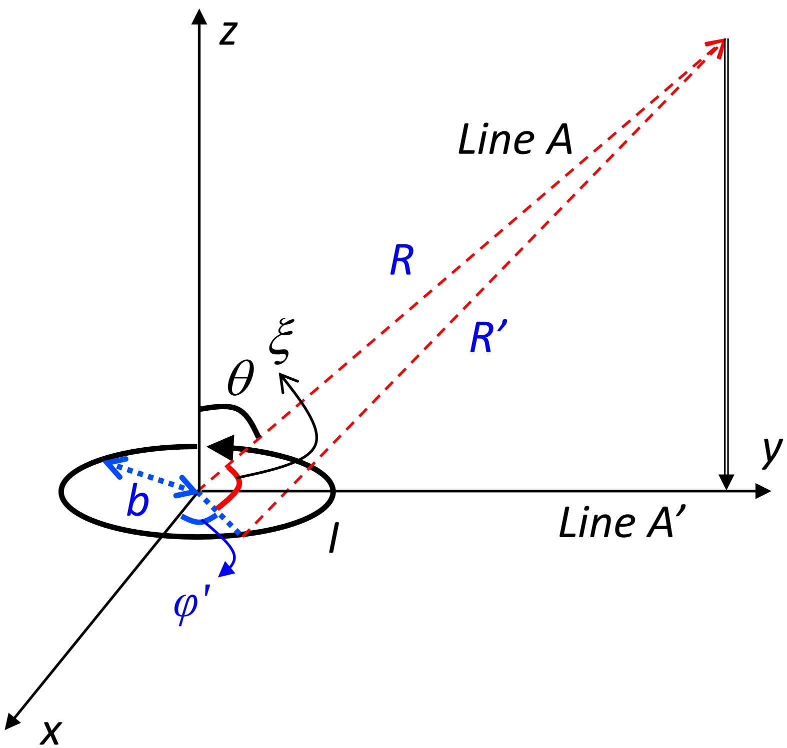
$$\frac{1}{R'} = \frac{1}{R}\left(1 - \frac{2b}{R}\sin\theta\sin\varphi' + \left(\frac{b}{R}\right)^2\right)^{-1/2}$$

$$\sim \frac{1}{R}\left(1 + \frac{b}{R}\sin\theta\sin\varphi'\right)$$

$$\mathbf{A} = \mathbf{e}_{\varphi} \frac{\mu_0 I}{4\pi} \frac{b}{R} \int_0^{2\pi} \left(1 + \frac{b}{R}\sin\theta\sin\varphi'\right) \sin\varphi' d\varphi'$$

Example 025

$$\mathbf{A} = \mathbf{e}_\varphi \frac{\mu_0 I}{4\pi} \frac{b}{R} \int_0^{2\pi} \left(1 + \frac{b}{R} \sin\theta \sin\varphi'\right) \sin\varphi' \, d\varphi'$$



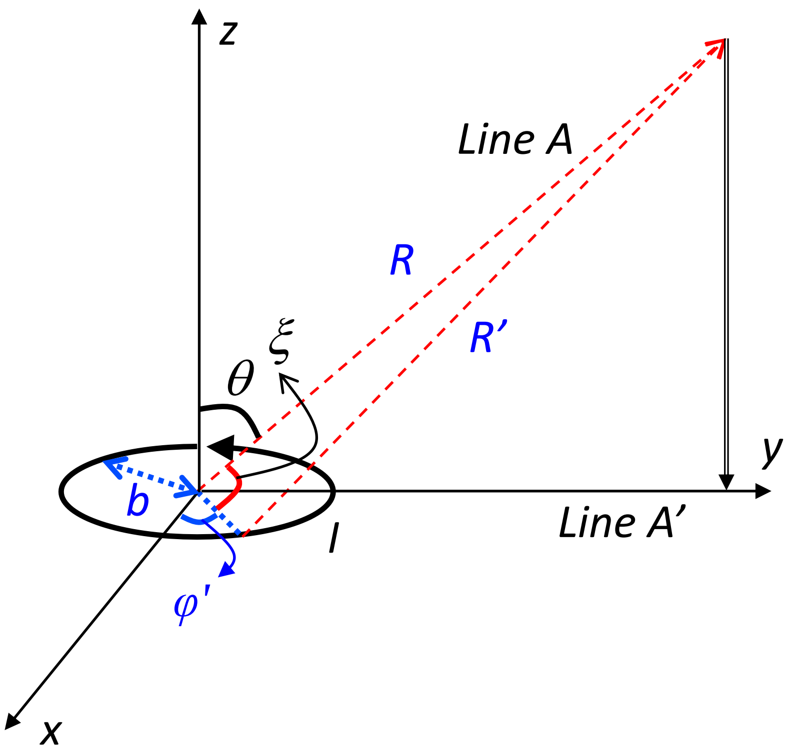
$$\mathbf{A} = \mathbf{e}_\varphi \frac{\mu_0 I}{4\pi} \frac{\pi b^2}{R^2} \sin\theta$$

$$\mathbf{m} = \mathbf{e}_z \pi b^2 I$$

$$\mathbf{e}_z \times \mathbf{e}_r = \mathbf{e}_\varphi \sin\theta$$

$$\boxed{\mathbf{A} = \frac{\mu_0}{4\pi} \frac{\mathbf{m} \times \mathbf{e}_R}{R^2}}$$

Example 025



$$\mathbf{m} = \mathbf{e}_z I \pi b^2$$

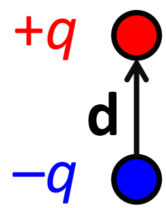
$$\mathbf{A} = \mathbf{e}_\varphi \frac{\mu_0 I}{4\pi} \frac{\pi b^2}{R^2} \sin\theta$$

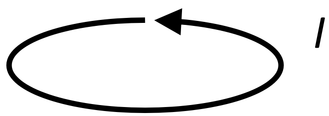
$$\boxed{\mathbf{A} = \frac{\mu_0}{4\pi} \frac{\mathbf{m} \times \mathbf{e}_r}{R^2}}$$

$$\mathbf{B} = \frac{\mu_0 I}{4\pi} \frac{\pi b^2}{R^3} \left(\mathbf{e}_r 2\cos\theta + \mathbf{e}_\theta \sin\theta \right)$$

$$\boxed{\mathbf{B} = \frac{\mu_0}{4\pi} \frac{m}{R^3} \left(\mathbf{e}_r 2\cos\theta + \mathbf{e}_\theta \sin\theta \right)}$$

Electric Dipole VS Magnetic Dipole





$$\mathbf{p} = q\mathbf{d}$$

$$\mathbf{m} = \mathbf{e}_z I \pi b^2$$

$$\boxed{V = \frac{1}{4\pi\varepsilon_0}\frac{\mathbf{p}\cdot\mathbf{e}_r}{R^2}}$$

$$\boxed{\mathbf{A} = \frac{\mu_0}{4\pi}\frac{\mathbf{m}\times\mathbf{e}_r}{R^2}}$$

$$\boxed{\mathbf{E} = \frac{p}{4\pi\varepsilon_0 R^3}\left(\mathbf{e}_r 2\cos\theta + \mathbf{e}_\theta \sin\theta\right)}$$

$$\boxed{\mathbf{B} = \frac{\mu_0 m}{4\pi R^3}\left(\mathbf{e}_r 2\cos\theta + \mathbf{e}_\theta \sin\theta\right)}$$