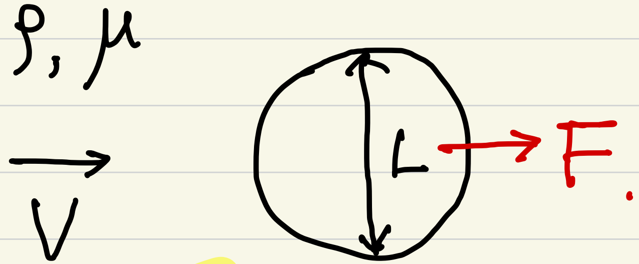
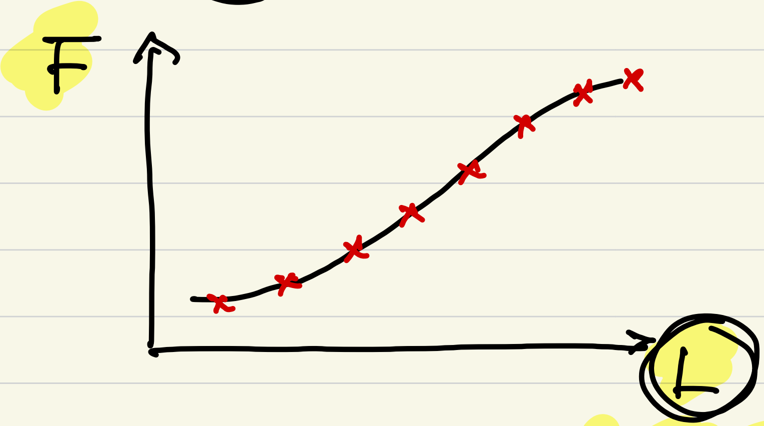
Ch.5 Dimensional Analysis and Similarity

planning, presentation, interpretation of exp/CFD.



$F = f(L, V, \rho, \mu)$



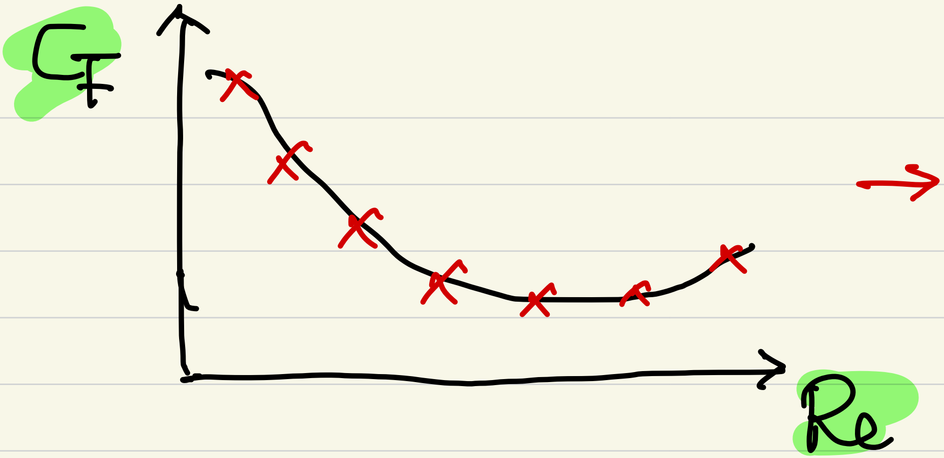
(fixed V, ρ, μ)

at least 10 data points to define a curve.

10^4 points for each L, V, ρ, μ.

But if we reduce the relation to

$$\frac{F}{\rho V^2 L^2} = g\left(\frac{\rho V L}{\mu}\right) \leftarrow \boxed{C_F = g(Re)}$$



$\checkmark$ ← dimensional analysis

→ 10 exp's only.

if $Re_{model} = \underbrace{Re_{prototype}}_{\text{given}} \rightarrow C_{F,model} = C_{F,prototype}.$

5.2. Principle of dimensional homogeneity.

→ in the same eq, each additive term should have the same dimension.

($r = r_0 + v_0 t + \frac{1}{2} g t^2$ (free falling)

$[L]$

$$\frac{p}{\rho} + \frac{1}{2}V^2 + gz = \text{constant} \quad \text{(Bernoulli eq.)}$$

$[L^2T^{-2}]$

→ dimensional variables : s, v, t, p, ρ, z · · ·

(" constants : $s_0, v_0, g,$ · · · ·

- pure constants → no dimension.

equation $\xrightarrow[\text{analysis}]{\text{dimensional}}$ equivalent non-dimensional form

5.3. Buckingham Pi - theorem

① Reduction in variables

n : dimensional variables

k : " less variables (Π)

↳ reduction $j \equiv n - k \leq$ number of dimensions necessary to describe dimensional variables.

ex) $F = f(L, V, \rho, \mu)$, $n = 5$

$[M, L, T]$

reduction $j = 5 - k \leq 3$ $\therefore k \geq 2$.

??

$C_F = g(Re)$, $k = 2$. $\Pi_1 = C_F$, $\Pi_2 = Re$

② Find the reduction

$v_1 = f(v_2, v_3, v_4, v_5)$ $n = 5$

if 3 dimensions $[M, L, T]$ are required, $k \geq 2$.

$\Rightarrow \Pi_1 = (v_2)^a (v_3)^b (v_4)^c \cdot v_i = [M^0 L^0 T^0]$ { dimen-sionless

$\Pi_2 = (v_2)^a (v_3)^b (v_4)^c \cdot v_5 = [M^0 L^0 T^0]$;

☆ form pi groups by power products of three variables that do not form a Pi

$\hookrightarrow (v_2)^a (v_3)^b (v_4)^c = [M^0 L^0 T^0]$ only when $a = b = c = 0$

· If any important variables are missing, dimensional analysis will fail.

$$F = f(L, V, \rho) \longrightarrow X$$

· choose density, velocity and length for j variables, as the first guess.

· do not put two dependent variables as

reduction variables.

ex) $F = f(\rho, L, V, \mu)$. $n=5$.

$$\left.\begin{aligned} F &= [MLT^{-2}] \\ L &= [L] \\ V &= [LT^{-1}] \\ \rho &= [ML^{-3}] \\ \mu &= [ML^{-1}T^{-1}] \end{aligned}\right\} \quad [M, L, T]$$

$5 - k \leq 3$

$\boxed{k \geq 2} \rightarrow k=2$.

$\downarrow$

j variables.

L, V, ρ.

($L^a V^b \rho^c = [M^0 L^0 T^0]$ a, b, c ?)

$$\Pi_1 = L^a V^b \rho^c F = [L]^a [LT^{-1}]^b [ML^{-3}]^c [MLT^{-2}]$$

$$= [L^{a+b-3c+1} \cdot T^{-b-2} \cdot M^{c+1}] = [L^0 T^0 M^0]$$

$\rightarrow a=-2,\ b=-2,\ c=-1$.

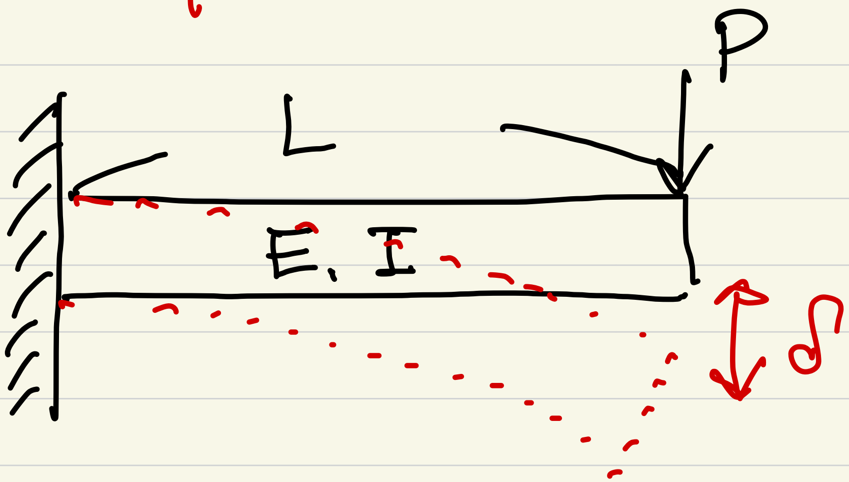
$$\therefore \Pi_1 = L^{-2}V^{-2}\rho^{-1}F = \frac{F}{\rho V^2 L^2} \ (\equiv C_F)$$

$$\Pi_2 = L^a V^b \rho^c \mu = [M^0 L^0 T^0] \qquad \underline{a=b=c=-1}$$

$$\therefore \Pi_2 = \frac{\mu}{\rho V L} = 1/Re. \qquad (\Pi_2' = 1/\Pi_2 = Re)$$

$\rightarrow C_F = g(Re)$ needs practice!

ex)



$$\underset{[L]}{\delta} = f(\overset{[MLT^{-2}]}{P}, L, \underset{[ML^{-1}T^{-2}]}{E}, \overset{[L^4]}{I})$$

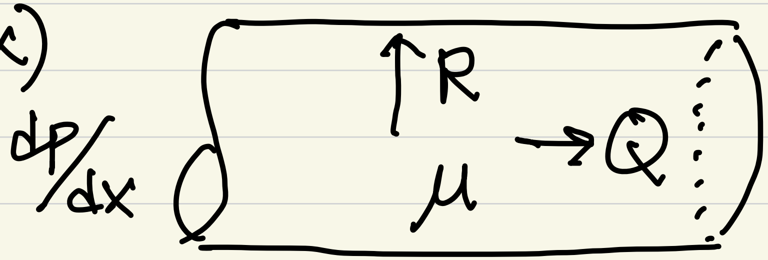
$\underline{n=5}$, $[M, L, T]$

$j = 5 - k \le 3. \qquad k \ge 2.$

can't find two π groups, because P and E contain M and T, but also same power of M and T. $\Rightarrow k = 3$

$$\left(\begin{array}{l} \Pi_1 = L^{a_1}E^{b_1}\delta \\ \Pi_2 = L^{a_2}E^{b_2}P \\ \Pi_3 = L^{a_3}E^{b_3}I \end{array}\right. \Rightarrow \frac{\delta}{L} = g\left(\frac{P}{EL^2}, \frac{I}{L^4}\right)$$

ex)



$Q = f(R, dp/dx, \mu)$, $n = 4$.

$[M, L, T]$, $4 - k \leq 3$

$\underline{k \geq 1}$

one π_i group: $\Pi_1 = R^a \mu^b (dp/dx)^c Q = [M^0 L^0 T^0]$

$$Q = AV \qquad \Pi_1 = \frac{Q\mu}{R^4 \cdot (dp/dx)} = \text{constant} = C$$

$$\dot{m} = f(R, dp/dx, \mu, \rho)$$
$$= \rho Q.$$

$$Q = C \cdot \frac{1}{\mu} \cdot R^4 \cdot (dp/dx).$$

5.4. Non-dimensionalization of the basic equations in incompressible flow.

- continuity : $\nabla \cdot \underline{V} = 0$
- N-S equation : $\rho \frac{D\underline{V}}{Dt} = -\nabla P + \mu \nabla^2 \underline{V}$

material derivative

Non-dimensionalization ??

→ characteristic scales needed! T, L, U,

5.4. Non-dimensionalization of the basic eqns in incompressible flow.

- continuity : $\nabla \cdot \underline{V} = 0$
- mtm : $\rho \frac{D\underline{V}}{Dt} = -\nabla p + \mu \nabla^2 \underline{V}$.

$\Bigg\} (*)$

material derivative.

We need characteristic scales.

↳ velocity, length, ...
(U) (L)

e.g.) $\underline{V}^* = \frac{\underline{V}}{U}$

$x^* = \frac{x}{L}, \quad y^* = \frac{y}{L}, \quad z^* = \frac{z}{L},$

$t^* = \frac{t}{(L/U)} = \frac{tU}{L}, \quad p^* = \frac{p}{\rho U^2}.$

$$\frac{\partial u}{\partial x} = \frac{\partial (U u^*)}{\partial (L x^*)} = \frac{U}{L}\frac{\partial u^*}{\partial x^*} \Rightarrow \nabla \cdot \underline{V} = 0 \rightarrow \cancel{\frac{U}{L}}\, \nabla^* \cdot \underline{V}^* = 0$$

$$\rho \frac{D\underline{V}}{Dt} = \rho \frac{D(U\underline{V}^*)}{D(t^* \cdot \frac{L}{U})} = \frac{\rho U^2}{L} \cdot \frac{D\underline{V}^*}{Dt^*}$$

$$\nabla p = \frac{1}{L}\nabla^*(p^* \cdot \rho U^2) = \frac{\rho U^2}{L}(\nabla^* p^*)$$

$$\mu \nabla^2 \underline{V} = \mu \cdot \frac{1}{L^2}\nabla^{*2}(U \cdot \underline{V}^*) = \mu \frac{U}{L^2}(\nabla^{*2}\underline{V}^*)$$

$$\boxed{\frac{\rho U^2}{L}\frac{D\underline{V}^*}{Dt^*} = -\frac{\rho U^2}{L}\nabla^* p^* + \mu \frac{U}{L^2}\nabla^{*2} \cdot \underline{V}^*}$$

$$\tau = \mu \frac{du}{dy}$$

$$\rightarrow \frac{D\underline{V}^*}{Dt^*} = -\nabla^* p^* + \frac{\mu}{\rho U L}\nabla^{*2} \cdot \underline{V}^*$$

$$= \frac{1}{Re}.$$

$$Re = \frac{\rho U L}{\mu} = \frac{\rho U^2}{\mu \frac{U}{L}}$$

$$= \frac{\text{inertia}}{\text{viscous force}}.$$

$$\frac{\partial \underline{V}^*}{\partial t^*} + (\underline{V}^* \cdot \nabla^*)\underline{V}^* = -\nabla^* P^* + \frac{1}{Re}\nabla^{*2}\underline{V}^*.$$

* Dimensional parameters.

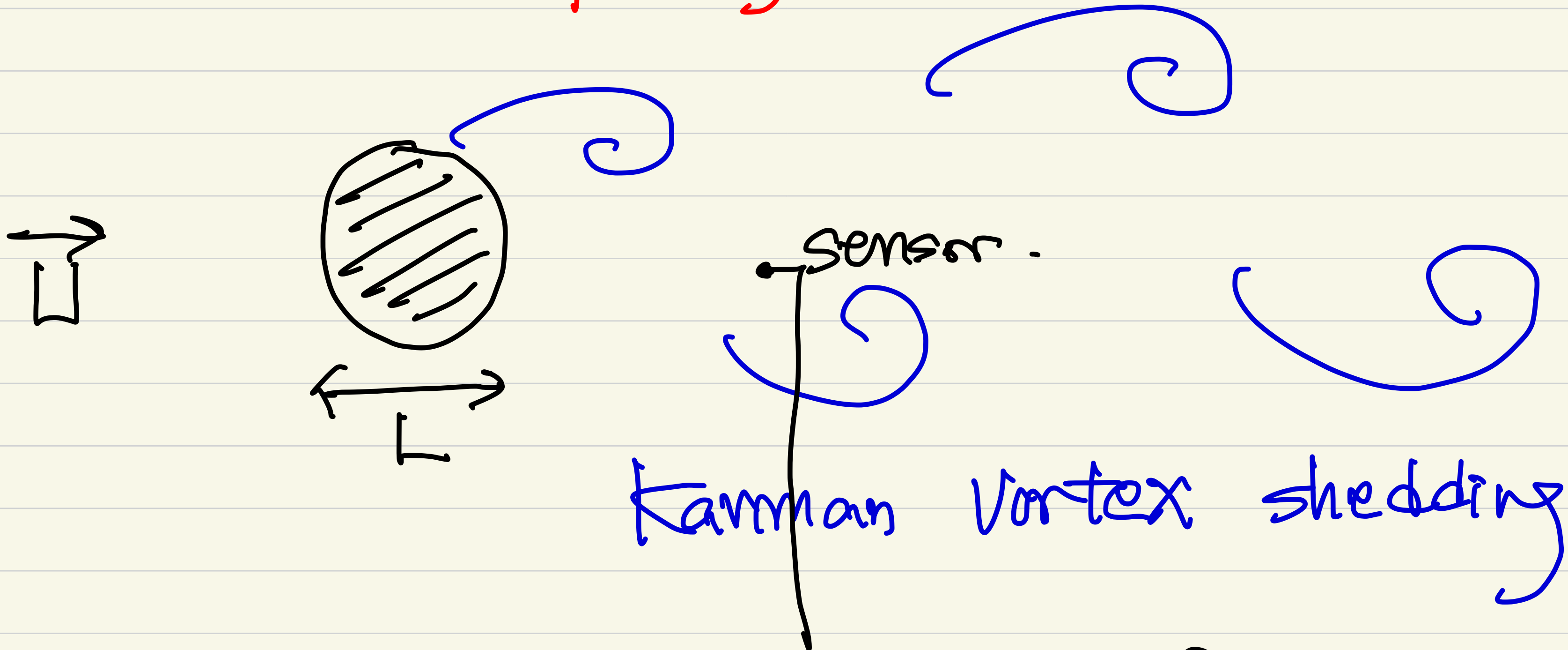
- $Re = \frac{\rho U L}{\mu}$ (Reynolds number).
- $Fr = \frac{U^2}{gL}$ (Froude number). ~ river, sea. ship
- $We = \frac{\rho U^2 L}{\sigma}$ (Weber number) ~ interfacial flow liquid droplet.

σ: surface tension

- $Ma = \frac{U}{a}$ (Mach number)

- $St = \frac{fL}{U}$ (strouhal number)

frequency.



$$u \simeq U \cdot \cos(ft).$$

$$\frac{u}{U} = u^* = \cos\left(f \cdot t^* \cdot \frac{L}{U}\right)$$

$$= \cos(St \cdot t^*).$$

- $Pr = \dfrac{\mu C_p}{k}$ (Prandtl number)

  k: thermal conductivity

$\tau = \mu \dfrac{du}{dy}$

$q = -k \dfrac{dT}{dy}$.

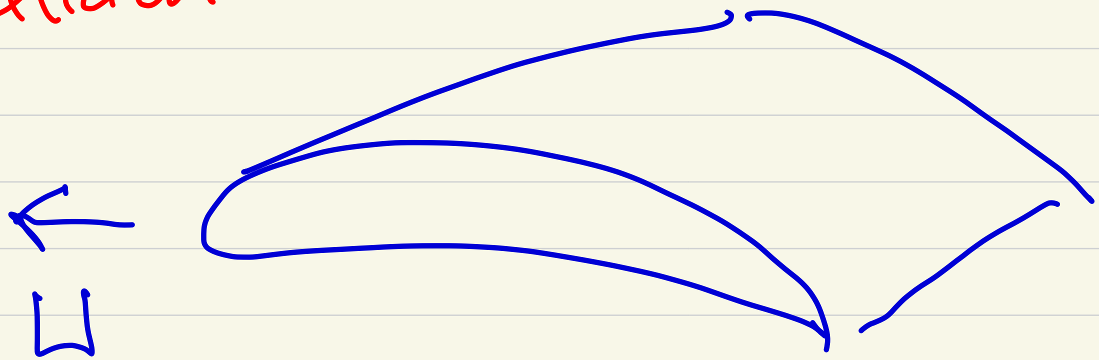
- $C_L = \dfrac{L}{\frac{1}{2}\rho U^2 A}$, $C_D = \dfrac{D}{\frac{1}{2}\rho U^2 A}$.

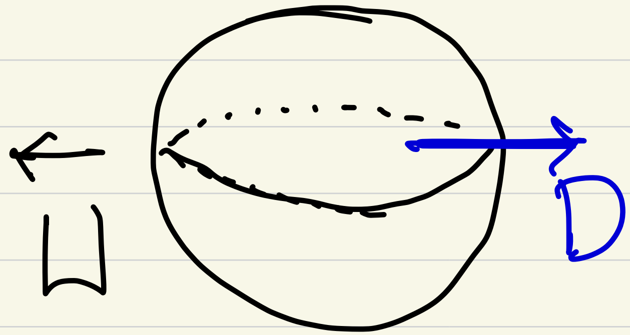
C_L: lift coefficient

L: Lift (양력)

C_D: drag coefficient

D: drag (항력).





- $C_p = \dfrac{P - P_\infty}{\frac{1}{2}\rho U^2}$ (pressure coefficient).

5.5. Modeling and similarity (상사성)

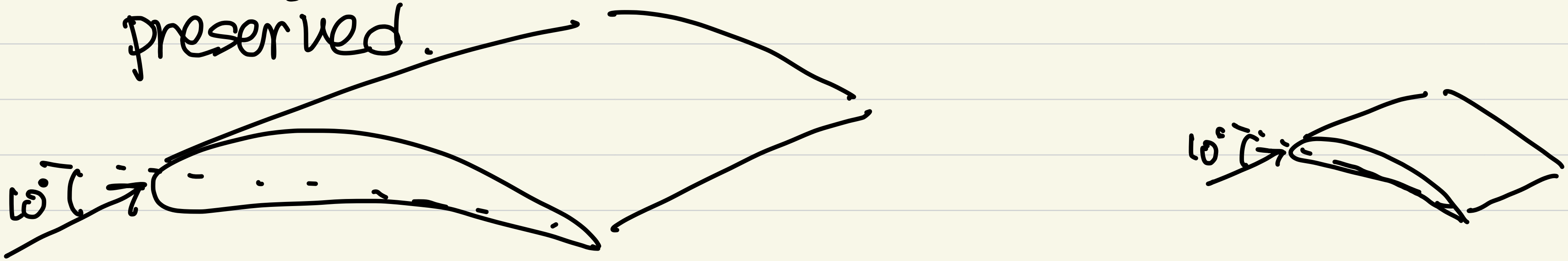
$$\Pi_1 = f(\Pi_2, \Pi_3, \cdots)$$

if $\Pi_{2m} = \Pi_{2p}$, $\Pi_{3m} = \Pi_{3p}$, $\cdots \Rightarrow \Pi_{1m} = \Pi_{1p}$.

* Geometric similarity [L].

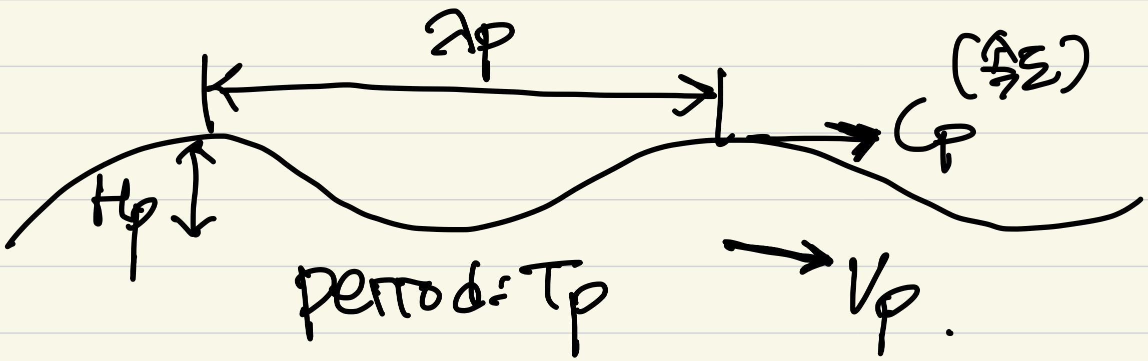
- a model and prototype are geometrically similar if and only if all body dimensions in all three coordinates have the same linear-scale ratio.

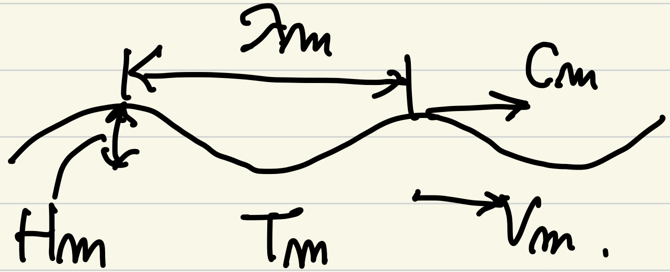
→ all angles are preserved, all flow directions are preserved.



* Kinematic similarity [L, T]

- the motions of two systems are similar → model and prototype have the same length- and time-scale ratios.





- Froude number,

$$Fr_m = \frac{V_m^2}{g L_m} = Fr_p = \frac{V_p^2}{g L_p} \quad \text{— ①}$$

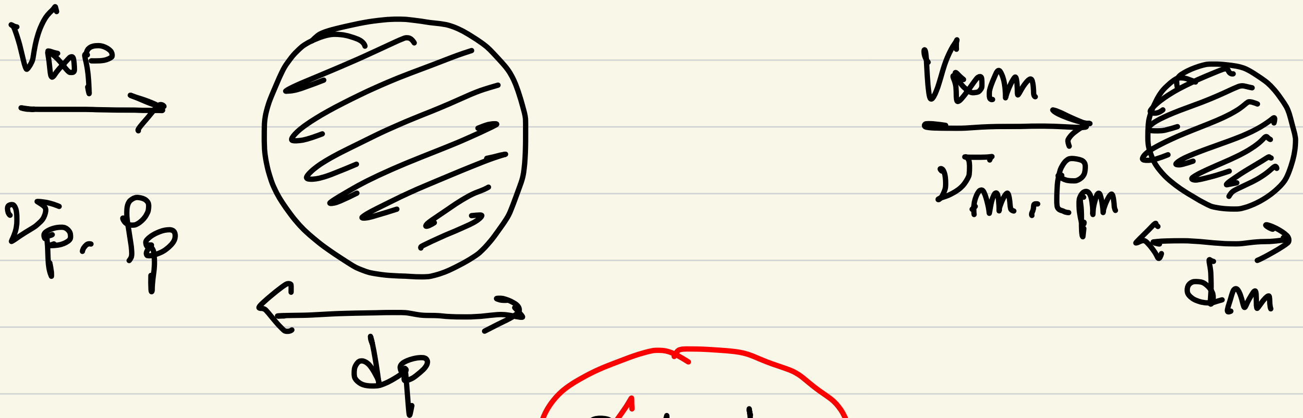
if $L_m = \alpha \cdot L_p$ — ②

① + ② : $V_m^2 = V_p^2 \cdot \frac{L_m}{L_p} \Rightarrow V_m = \sqrt{\alpha} \cdot V_p$.

$T_m = \sqrt{\alpha} \cdot T_p$.

* Dynamic similarity [M, L, T]

- same length-, time-, and mass (force) -scale ratios.



$$Re_p = \frac{\cancel{\rho_p} V_{\infty p} d_p}{\cancel{\mu_p}} = Re_m = \frac{\cancel{\rho_m} V_{\infty m} d_m}{\cancel{\mu_m}}$$

if $d_m = \alpha \cdot d_p$, same fluid, $\Rightarrow V_{\infty m} = \frac{1}{\alpha} V_{\infty p}$

Real flows $\longrightarrow$ Lab tests. (CFD, EXP)

$\Downarrow$

incompressible flow : Re.

compressible flow : Re & Ma.