Seoul National University
Dept. of Civil and Environmental Engineering

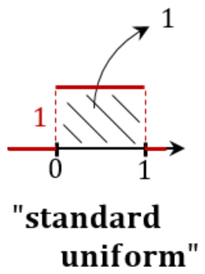
Instructor: Junho Song
junhosong@snu.ac.kr

457.646 Topics in Structural Reliability
In-Class Material: Class 21

VI. Simulation Methods

- Supplementary reading material: Ch3 of Melchers & Beck (2017)

◎ Simulating uniform random variable $U(0,1)$



→ Basic in generation of random numbers

→ (   ) sequence from a seed number

→ Desirable to have a (   ) period and (   ) sampling

※ Matlab : rand(   )

→ could choose a random number generation algorithm

→ default: Mersenne Twister (Matsumoto & Nishimura 1997)

→ Period: $2^{19936}-1$

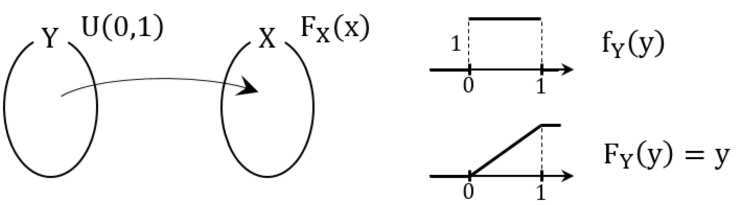
→ "Very fast"

Demo

```
Xv =[100 1000 10000]
for i=1:3
X=rand(Xv (i),1);
subplot(3,1,i)
hist(X,sqrt(Xv(i)));
end
```

◎ Generate random numbers according to CDF

Consider $Y \sim U(0,1)$



$$F_Y(y) \quad = \quad F_X(x)$$
$$\quad = \quad F_X(x)$$

$\therefore\ x =$

(1) Generate y_i, $i=1,\cdots,N$

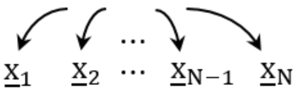
per ↖ $U(0,1)$

(2) Find corresponding x_i, $x_i =$, $i=1,\cdots,N$

◎ Generate general dependent variables

$\mathbf{X}=\{X_1,\cdots,X_n\}^T$ defined by $\begin{cases} \text{joint PDF } f_{\mathbf{X}}(\mathbf{x}) \\ \text{joint CDF } F_{\mathbf{X}}(\mathbf{x}) \end{cases}$

cf. Rosenblatt



$$\begin{cases} y_1 = F_{X_1}(x_1) \\ y_2 = F_{X_2|X_1}(x_2|x_1) \\ \vdots \\ y_n = F_{X_n|X_1\cdots X_{n-1}}(x_n|x_1\cdots x_{n-1}) \end{cases}$$

$$\begin{cases} x_1 = F_{X_1}^{-1}(y_1) \\ x_2 = F_{X_2|X_1}^{-1}(y_2|x_1) \\ \vdots \\ x_n = F_{X_n|X_1\cdots X_{n-1}}^{-1}(y_n|x_1\cdots x_{n-1}) \end{cases}$$

(1) Simulate $\{y_1,\cdots,y_n\}^T$

(2) Find $\{x_1,\cdots,x_n\}^T$

Using (←)

◎ Simulation of normally distributed RV's *(Box & Muller 1958)

→ **homework**

※ **Matlab mvrnd($\mathbf{M},\mathbf{\Sigma},N$)**

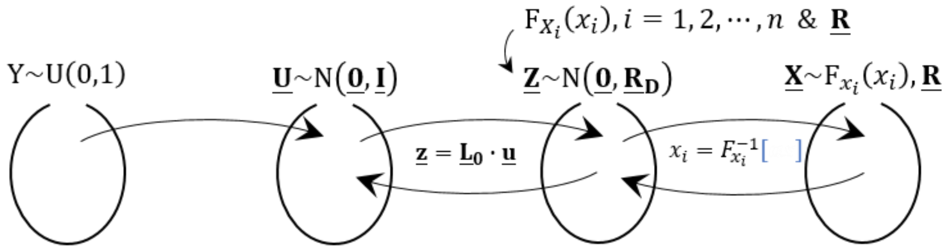
Generate N samples of $\mathbf{X}\sim N(\mathbf{M},\mathbf{\Sigma})$

cf. normrnd

$\mathbf{u}\sim N(\mathbf{0},\mathbf{I})$

$\mathbf{X}=\mathbf{DLu}+\mathbf{M}$

◎ Generate random numbers from Nataf distribution



 Instructor: Junho Song
junhosong@snu.ac.kr

i. Find $\mathbf{R}_0$ (Liu & ADK, 1986)

ii. Generate $\mathbf{u}$ from $N(\mathbf{0},\mathbf{I})$ (or $\mathbf{y}$ from $U(0,1)$ & transform)

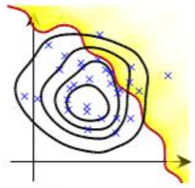
iii. Compute $\mathbf{Z} \sim \mathbf{L}_0\mathbf{u}$ (or $\mathbf{Z} \sim N(\mathbf{0},\mathbf{R}_0)$)

iv. Compute $x_i = F_{X_i}^{-1}(\quad)$, i=1,⋯,n

◎ <u>Monte Carlo</u> Simulation

A secret project at Los Alamos on nuclear weapons (performed by Von Neumann & Ulam) was named "Monte Carlo" project by Metropolis (after the City in Monaco).

$$P_f = \int_{(\cup\cap)g(\mathbf{x})\le 0} f_{\mathbf{X}}(\mathbf{x})d\mathbf{x}$$



$$= \int \qquad f_{\mathbf{X}}(\mathbf{x})d\mathbf{x}$$

$$I(\mathbf{x}) = \begin{cases} & g(\mathbf{x}) \le 0 \\ & \text{otherwise} \end{cases}$$

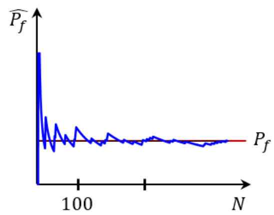
Index Function

$=$ average of index function value (w.r.t $\mathbf{X} \sim F_{\mathbf{X}}(\mathbf{x})$)

Simulate $\mathbf{x}_i$, i=1,⋯,N according to $f_{\mathbf{X}}(\mathbf{x})$

Let $q_i = I(\mathbf{x}_i)$, i=1,⋯,N

$$P_f = \lim_{N\to\infty}$$



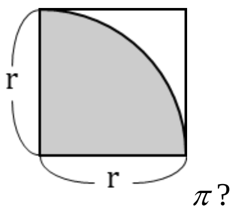
$\hat{P}_f =$ Estimation of P_f using N sample

Compare

mean (rand(3,1))

mean (rand(100000,1))

"MCS is an extremely bad method. It should be used only when all alternative methods are worse" –Alan Sokal (1996)



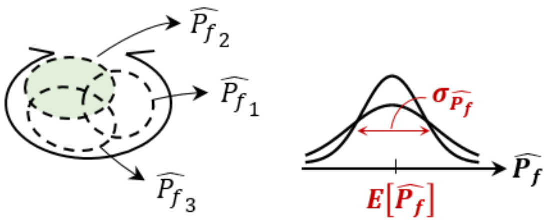
π ?

$$\frac{\frac{1}{4}\pi r^2}{r^2} = \frac{1}{4}\pi$$

$$\therefore \pi = 4 \times \frac{\#\ \text{◔}}{\#\ \square}$$

Instructor: Junho Song
junhosong@snu.ac.kr

Note: $\hat{P}_f$ is random



↓

How much variability? $\delta_{\hat{P}_f}$

q_i : Bernoulli random variable

$\begin{cases} 1 & \text{with} \quad p= \\ 0 & \quad\quad\quad 1\text{-}p= \end{cases}$

$E[q_i] =$

$Var[q_i] =$

$=$

$=$

- $E[\hat{P}_f] =$

  "unbiased" estimator of true P_f

- $Var[\hat{P}_f] =$

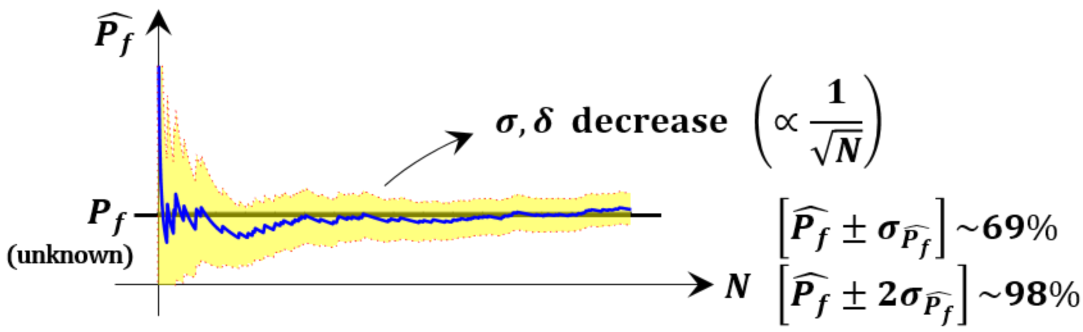
  $=$

  $=$

$\Rightarrow \quad \delta_{\hat{P}_f} = \qquad\qquad = \frac{1}{\sqrt{N}}\sqrt{\frac{1-P_f}{P_f}}$

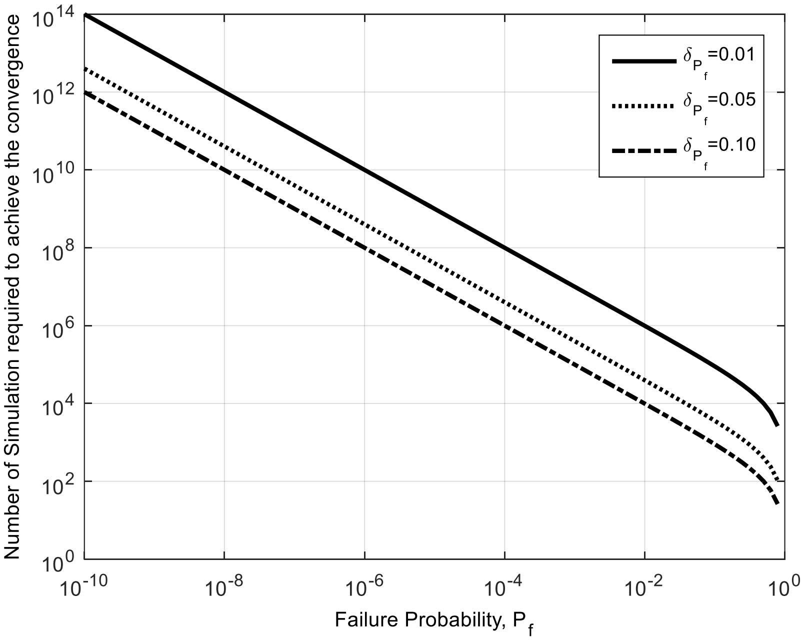
Quantifies variation of $\hat{P}_f$

Used as a measure of convergence



See MCS.m

Instructor: Junho Song
junhosong@snu.ac.kr



※ Minimum No. of Simulation to achieve $\bar{\delta}$

Target c.o.v $\bar{\delta} = \frac{1}{\sqrt{N_{\bar{\delta}}}}\sqrt{\frac{1-P_f}{P_f}}$

$\therefore \quad N_{\bar{\delta}} = \frac{1-P_f}{\bar{\delta}^2 \cdot P_f}$

e.g $P_f = 0.01$

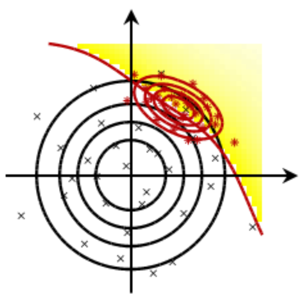
$\bar{\delta}$	$N_{\bar{\delta}}$
0.01	$\simeq 10^6$
0.05	$\simeq 4.0 \times 10^4$
0.10	$\simeq 1.0 \times 10^4$

※ How to improve accuracy of simulation

$$\delta_{P_f} = \frac{\sqrt{Var[\hat{P}_f]}}{E[\hat{P}_f]} = \frac{\frac{1}{\sqrt{N}}\sqrt{Var[q_i]}}{E[q_i]} = \frac{1}{\sqrt{N}} \cdot \delta_{q_i}$$

① Increase N

② Decrease δ_{q_i}



Seoul National University
Dept. of Civil and Environmental Engineering

Instructor: Junho Song
junhosong@snu.ac.kr

457.646 Topics in Structural Reliability
In-Class Material: Class 22

◎ Importance sampling

Need to compute integral (in general)

$$I_t = \int g(\mathbf{x})d\mathbf{x}$$
$$= \int \left[\frac{g(\mathbf{x})}{h(\mathbf{x})} \right] h(\mathbf{x}) d\mathbf{x}$$
$$=$$

$g(\mathbf{x})$: general function
$h(\mathbf{x})$: sampling PDF having non-
values where g($\mathbf{x}$) is non-

Procedure:

i. Sample $\mathbf{x}_i$, $i = 1, \cdots, N$ according to

ii. Compute $q_i =$

iii. Estimate $\hat{I}_t = \frac{1}{N}\sum_{i=1}^{N} q_i$

To have accuracy (& efficiency), the variance in q must be small. If $g(\mathbf{x}) \geq 0$, $h(\mathbf{x})$ $g(\mathbf{x})$ is the best choice.

Rubinstein (1981) proved that the best sampling density minimizing the variance is $h^*(\mathbf{x}) = g(\mathbf{x})/I_t$

※ Application to reliability problem:

$$P_f = \int_x I(\mathbf{x}) f_X(\mathbf{x}) d\mathbf{x}$$
$$= \int_x \left[\frac{I(\mathbf{x})}{\quad} f_X(\mathbf{x}) \right] \cdot \qquad d\mathbf{x}$$
$$= E[\qquad\qquad]$$ relative to

$q_i =$

Sampling density

(non-zero) where $g = I \cdot f \neq 0$

Find $h(\mathbf{x})$ such that

$Var[\frac{I_f}{h}]_h$ $\qquad Var[I]_f$

Best sampling density for the reliability problem:

$$h^*(\mathbf{x}) = \frac{I(\mathbf{x}) f(\mathbf{x})}{P_f}$$

Instructor: Junho Song
junhosong@snu.ac.kr

c.o.v of $\hat{P}_f$, $\delta_{\hat{P}_f}$ for importance sampling?

$$\hat{P}_f = \frac{1}{N}\sum_{i=1}^{N} q_i$$

$$X,\ x_1\ ;\cdots\ x_N$$

$$\bar{X} = \frac{1}{N}\{x_1 + \cdots + x_N\}$$

$$\mu_{\hat{P}_f} = \frac{1}{N}\sum_{i=1}^{N} E[q_i] \rightarrow \bar{\hat{P}}_f = \frac{1}{N}\sum_{i=1}^{N}\bar{Q}$$

↖ Estimate on the of $\hat{P}_f$ (=sample mean of)

$$\sigma^2_{\hat{P}_f} = \frac{1}{N^2}\sum_{i=1}^{N} Var[q_i] \rightarrow S^2_{\hat{P}_f} = \frac{1}{N^2}\sum_{i=1}^{N} S_q^2 = \frac{1}{N}S_q^2$$

↖ Estimate on the variance of $\hat{P}_f = \frac{1}{N} \times$ sample variance of q_i's

$$\delta_{\hat{P}_f} = \frac{\sqrt{S_{\hat{P}_f}}}{\bar{\hat{P}}_f} = \frac{\frac{1}{\sqrt{N}}S_q}{\bar{Q}}$$

$\left(\frac{If}{h}, \frac{If}{h}, \cdots, \frac{If}{h}\right)$ $\quad x_i \leftarrow h(\mathbf{x})$

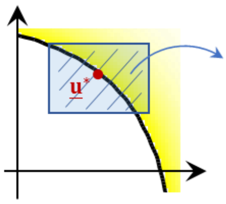
Importance sampling $P_f = \int_x \left[\frac{If}{h}\right] \cdot h(\mathbf{x}) d\mathbf{x} = E\left[\frac{If}{h}\right]$

$$Var\left[\frac{If}{h}\right]_h << Var[I]_f$$

How to find a good importance sampling density?

◎ Selection of sampling density

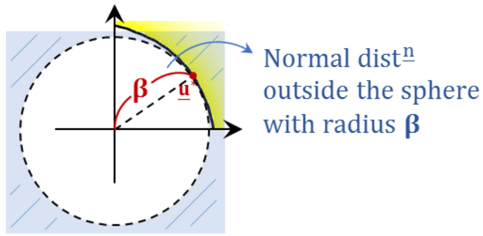
① Shinozuka (1983)



→ not good because zero density assigned to failure cases $q_i = \frac{I \cdot f}{h}$

Instructor: Junho Song
junhosong@snu.ac.kr

② Harbitz (1986)



$$h(\mathbf{u}) \begin{cases} c \cdot \varphi_n(\mathbf{u}) & \|\mathbf{u}\| \quad \beta \\ 0 & \text{otherwise} \end{cases}$$

$$c \int_{\|\mathbf{u}\| \geq \beta} \varphi_n(\mathbf{u}) d\mathbf{u} = c \cdot P(\|\mathbf{u}\| \geq \beta) =$$

$$P(\|\mathbf{u}\| \geq \beta) = 1 - P(\|\mathbf{u}\| \leq \beta)$$

$$= 1 - P(\|\mathbf{u}\|^2 \leq \beta^2) = 1 - P(u_1^2 + \cdots + u_n^2 \leq \beta^2)$$

$$= 1 - X_n^2(\beta^2)$$

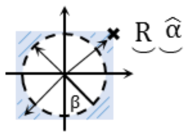
Chi-square distribution *n* degree of freedom

$\therefore \; c=$

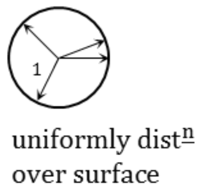
$$q_i = \frac{I \cdot f}{h} = \qquad\qquad =$$

How to simulate according to $h(\mathbf{u})$?

i. Simulate $\mathbf{u}_i \sim N(\mathbf{0}, \mathbf{I})$



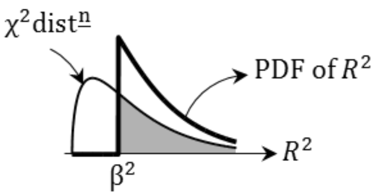
ii. Compute $\hat{\alpha}_i = \frac{\mathbf{u}_i}{\|\mathbf{u}_i\|}$



iii. Simulate R^2

$$R^2 = u_1^2 + \cdots + u_n^2 \quad (\sim X_n^2(\;))$$

But truncate $R^2 < \beta^2$

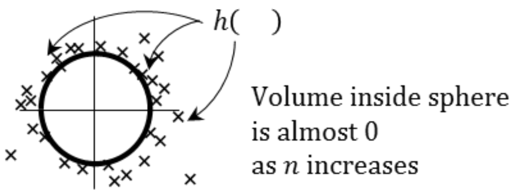


$$F_{R^2}(r^2) = \frac{X_n^2(r^2)}{1 - X_n^2(\beta^2)}$$

iv. Compute $R \cdot \hat{\alpha}$

Note: Not effective as $n \uparrow$

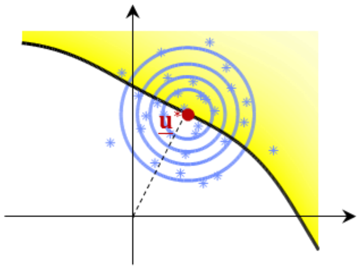
$$1 - X_n^2(\beta^2) \cong 1$$



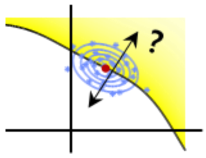
Instructor: Junho Song
junhosong@snu.ac.kr

③ Melchers (1989)

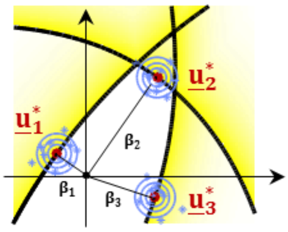
$h(\mathbf{u}) = N(\qquad)$



e.g. $\mathbf{\Sigma} = \sigma^2 \mathrm{I}$ (FERUM's Importance Sampling Option)



④ Series system

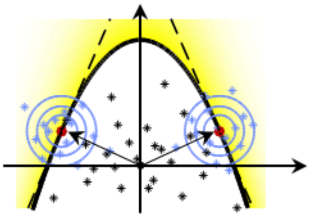


$h(\mathbf{u}) = \sum_i w_i h_i(\mathbf{u})$ where $h_i(\mathbf{u}) \leftarrow N(\mathbf{u}_i^*, \mathbf{\Sigma})$

w_i : weight ($\propto \beta_i^{-m},\ m > 0$)

◎ Challenges

① $h(\mathbf{x}) = 0$ where $\mathrm{I}(\mathbf{x}) \neq 0 \Rightarrow$ does not converge

② 

Multiple design points?

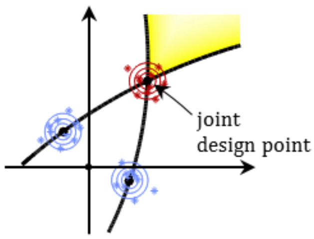
$h(\mathbf{u}) \Rightarrow wh(\mathbf{u}) + w_0 \varphi(\mathbf{0}, \mathbf{I})$

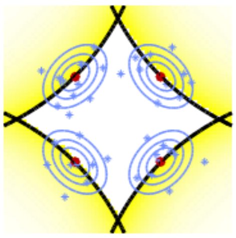
ADK & Dakessaian (1998)

③ System problems

i. Where?

ii. Cost of finding the important points





④ High-dimensional reliability problems ("curse of dimensionality")

Most of the density exist in the important ring $R \cong \sqrt{n}$ (Katafygiotis & Zuev 2008)

Instructor: Junho Song
junhosong@snu.ac.kr

※ Adaptive Importance Sampling methods can be used to overcome these challenges

(1) Kurtz, N., and J. Song (2013). Cross-entropy-based adaptive importance sampling using Gaussian mixture. *Structural Safety*. Vol. 42, 35-44.

(2) Wang, Z., and J. Song (2016). Cross-entropy-based adaptive importance sampling using von Mises–Fisher mixture for high dimensional reliability analysis. *Structural Safety*. Vol. 59, 42-52.

➔ The adaptive sampling method can be used to identify critical subdomains and Generalized Reliability Importance Measures (Kim and Song, 2018)

Cross-entropy-based adaptive importance sampling (CE-AIS) using Gaussian Mixture (GM) and von Mises Fisher Mixture (vMFM)

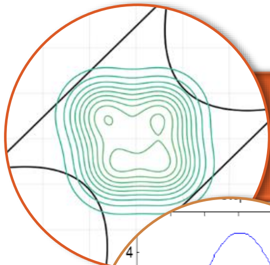
Junho SONG

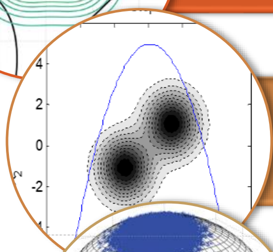
Department of Civil and Environmental Engineering
Seoul National University, S. Korea

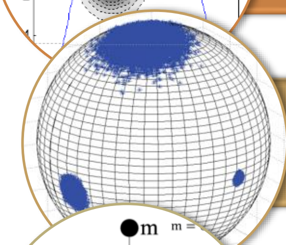
Seoul National University
Department of Civil & Environmental Engineering

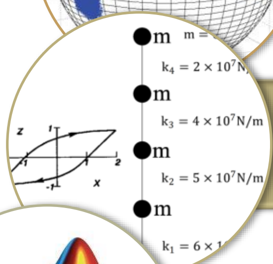
SSRG
Structural & System Reliability Group
SEOUL NATIONAL UNIVERSITY

Contents

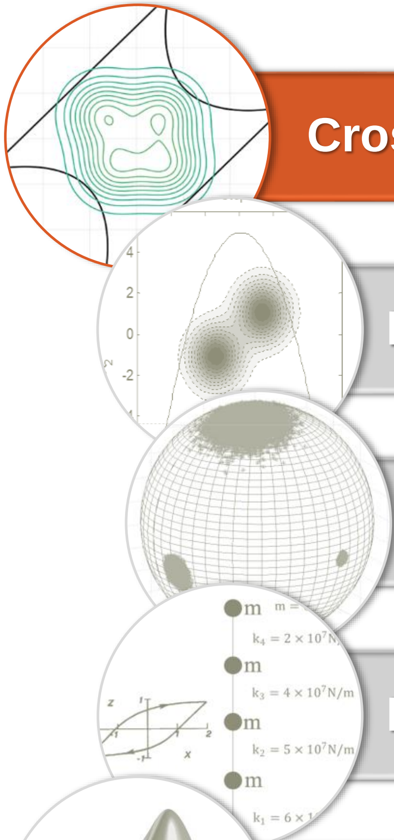






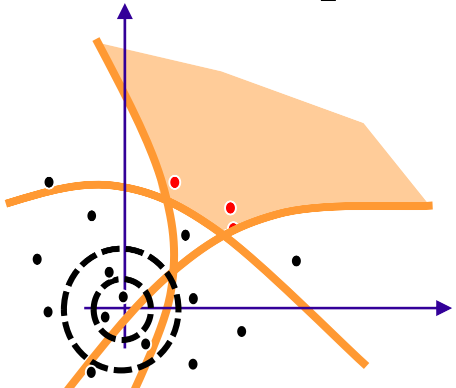


Contents



Reliability analysis by importance sampling

Monte Carlo simulations

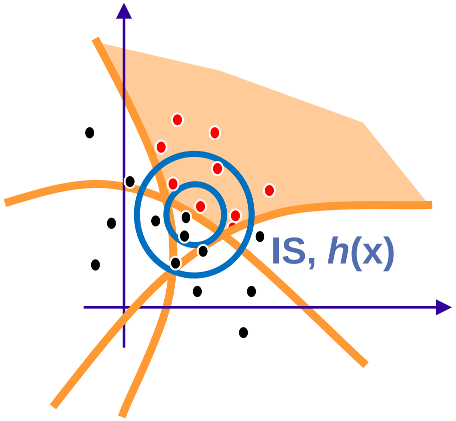


$$P_f = \int_D f_{\mathbf{X}}(\mathbf{x})d\mathbf{x} = \int_{\mathbf{x}} I(\mathbf{x}) f_{\mathbf{X}}(\mathbf{x})d\mathbf{x} = \mathrm{E}_f[I(\mathbf{x})]$$

$$\cong \frac{1}{N}\sum_{i=1}^{N} I(\mathbf{x}_i)$$

- May take a long time to get a converged result for rare events

Importance sampling



$$P_f = \int_{\mathbf{x}} \left[\frac{I(\mathbf{x}) f_{\mathbf{X}}(\mathbf{x})}{h_{\mathbf{X}}(\mathbf{x})}\right] h_{\mathbf{X}}(\mathbf{x})d\mathbf{x} = \mathrm{E}_h\left[\frac{I(\mathbf{x})\cdot f_{\mathbf{X}}(\mathbf{x})}{h_{\mathbf{X}}(\mathbf{x})}\right]$$

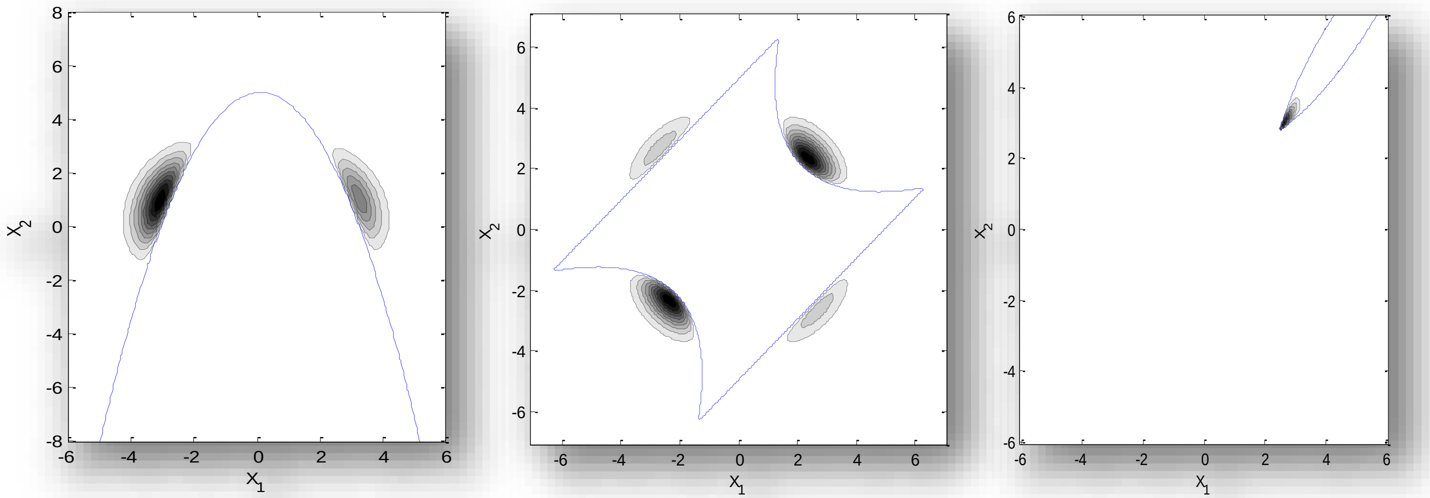
$$\cong \frac{1}{N}\sum_{i=1}^{N} q(\mathbf{x}_i)$$

- May improve efficiency by choosing a new sampling density function *h*(**x**) that reduces the variance of $(I \cdot f)/h$
- Challenging to find "good" *h*(**x**), e.g. "design point" by FORM
- Component problems with multiple/competing design points or system problems → **General & systematic procedure to find good *h*(x)?**

"Best" importance sampling density

Importance sampling density minimizing variance

$$p^*(\mathbf{x}) = \frac{|H(\mathbf{x})|}{\int |H(\mathbf{x})| d\mathbf{x}} = \frac{I(\mathbf{x}) f_{\mathbf{X}}(\mathbf{x})}{P_f}$$



- Can't use directly… if we already know P_f, we do not need MCS or IS.
- Still helpful for improving efficiency, if $h(\mathbf{x})$ is chosen in order to have **a shape similar to that of** $I(\mathbf{x})f_X(\mathbf{x})$

Adaptive importance sampling by minimizing cross entropy

Kullback-Leibler "Cross Entropy" (CE)

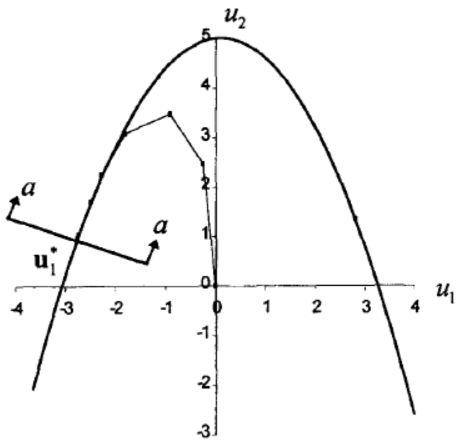
$$D(p^*, h) = \int p^*(\mathbf{x}) \ln p^*(\mathbf{x}) d\mathbf{x} - \int p^*(\mathbf{x}) \ln h(\mathbf{x}) d\mathbf{x}$$

- "Distance" between "best" IS density $p^*(\mathbf{x})$ and current one $h(\mathbf{x})$
- One can find a good $h(\mathbf{x})$ by minimizing Kullback-Leibler CE, i.e.

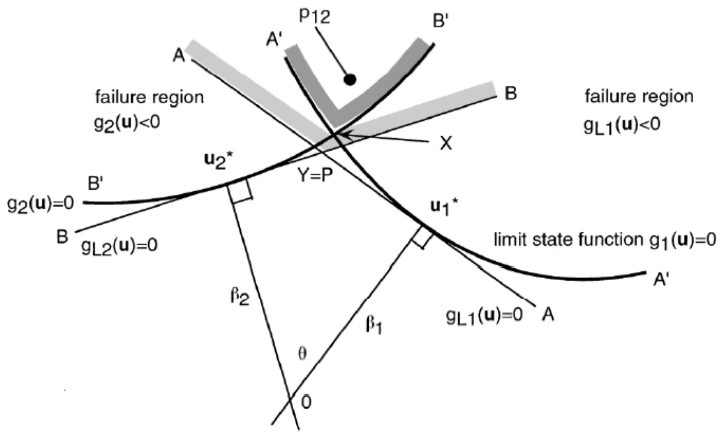
$$\begin{aligned}\arg\min_{\mathbf{v}} D(p^*, h(\mathbf{v})) &= \arg\min_{\mathbf{v}} \left[\int p^*(\mathbf{x}) \ln p^*(\mathbf{x}) d\mathbf{x} - \int p^*(\mathbf{x}) \ln h(\mathbf{x}; \mathbf{v}) d\mathbf{x} \right] \\ &= \arg\max_{\mathbf{v}} \int p^*(\mathbf{x}) \ln h(\mathbf{x}; \mathbf{v}) d\mathbf{x} \\ &= \arg\max_{\mathbf{v}} \int I(\mathbf{x}) f_{\mathrm{X}}(\mathbf{x}) \ln h(\mathbf{x}; \mathbf{v}) d\mathbf{x}\end{aligned}$$

- Finds the optimal values of the distribution parameter(s) **v approximately by small-size pre-sampling**, then performs final importance sampling
- Rubinstein & Kroese (2004) used **uni-modal parametric distribution** for $h(\mathbf{x};\mathbf{v})$ and provided **updating rules** to find optimal **v** through sampling

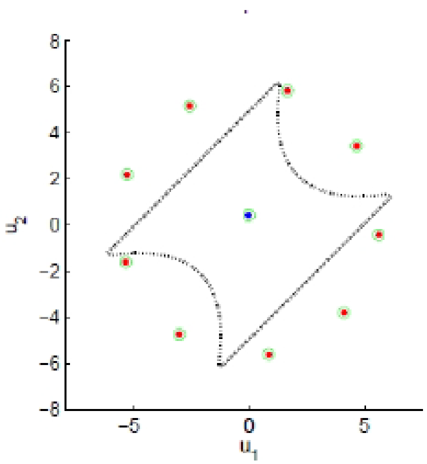
Limitations of Importance Sampling Using Uni-modal Sampling Density



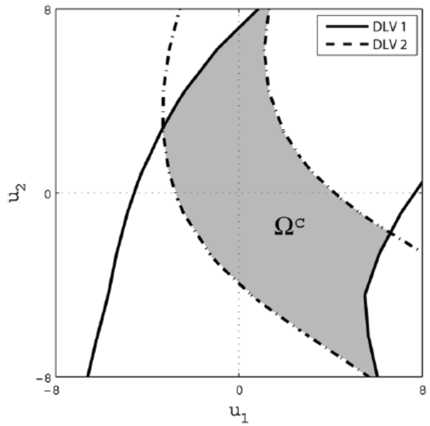
Component problems with multiple or competing design points (Der Kiureghian & Dakessian, 1998)



Parallel system problems and “joint design point” (Melchers & Ahammed, 2001)



Complex failure domain by a series system (Waarts, 2000)

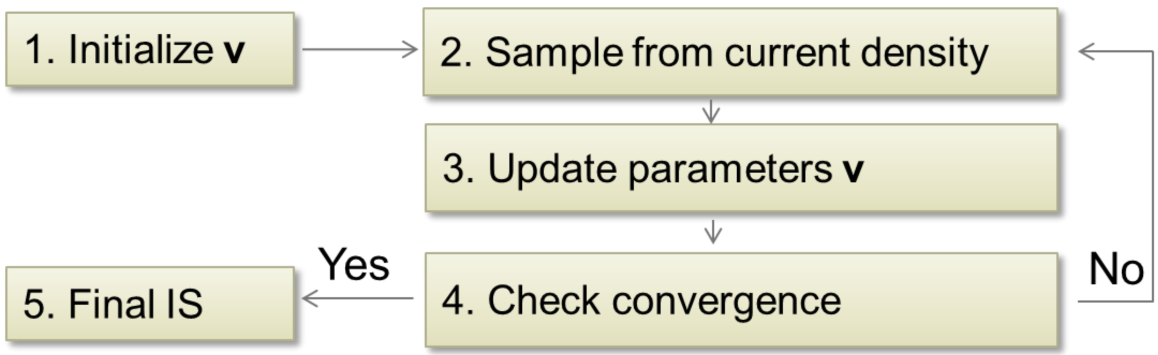


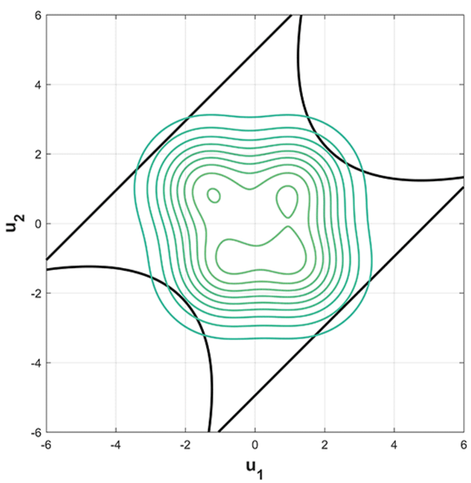
Reliability-based evaluation of DLV method (Sim et al. 2008)

CE-AIS with Gaussian Mixture (Kurtz & Song 2013)

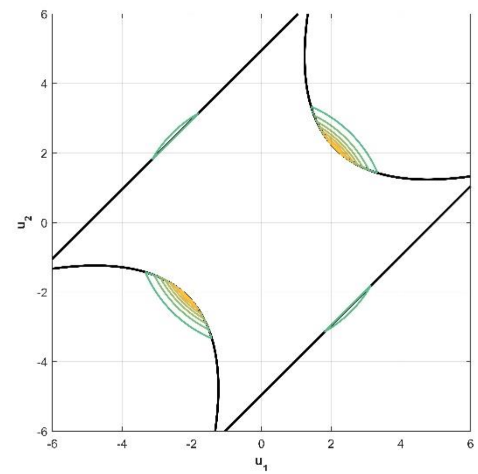
Kurtz, N., and J. Song (2013). Cross-entropy-based adaptive importance sampling using Gaussian mixture. *Structural Safety*. 42:35-44.

- **CE-AIS-GM Algorithm** $h(\mathbf{x};\mathbf{v}) = \sum_{k=1}^{K} \pi_k N(\mathbf{x} \mid \boldsymbol{\mu}_k, \boldsymbol{\Sigma}_k)$



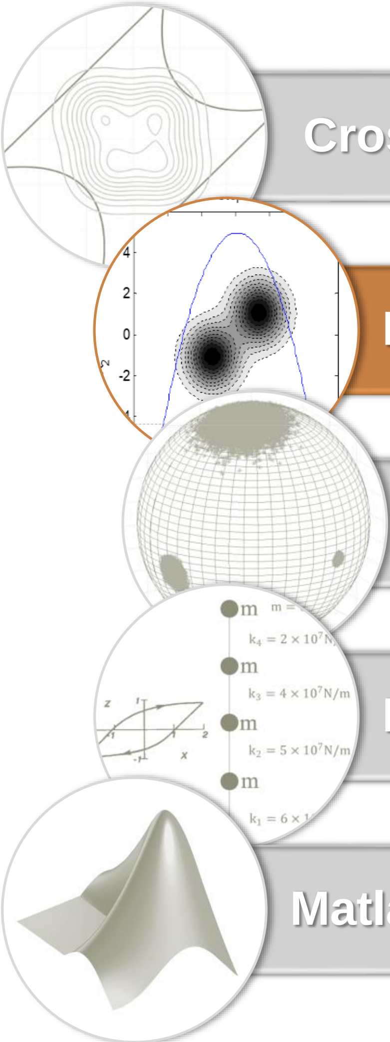


Near Optimal Density

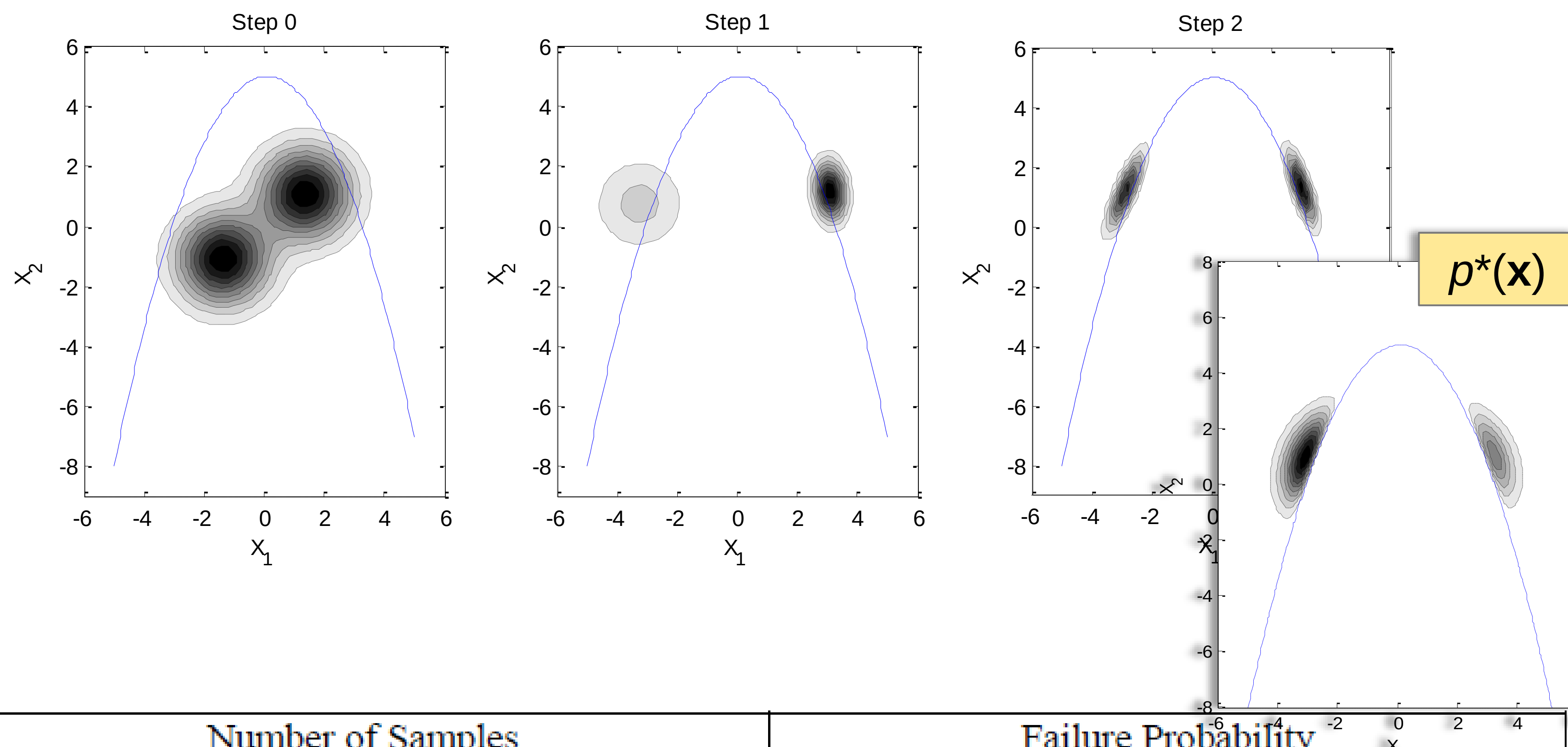


Optimal Density

Contents

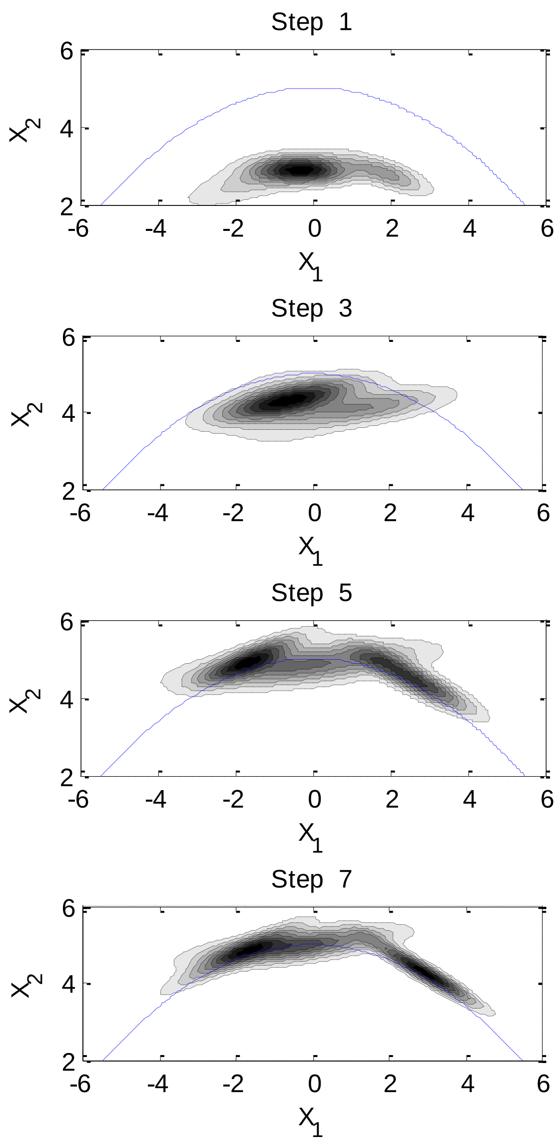


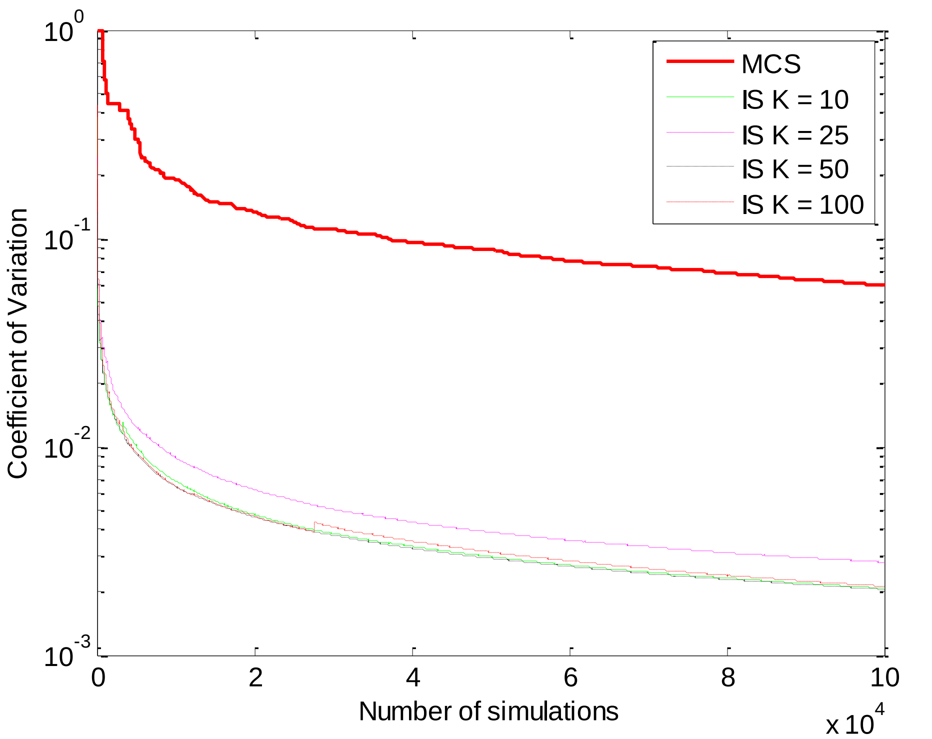
Component Problem with Multiple Design Points (Der Kiureghian & Dakessian 1998)

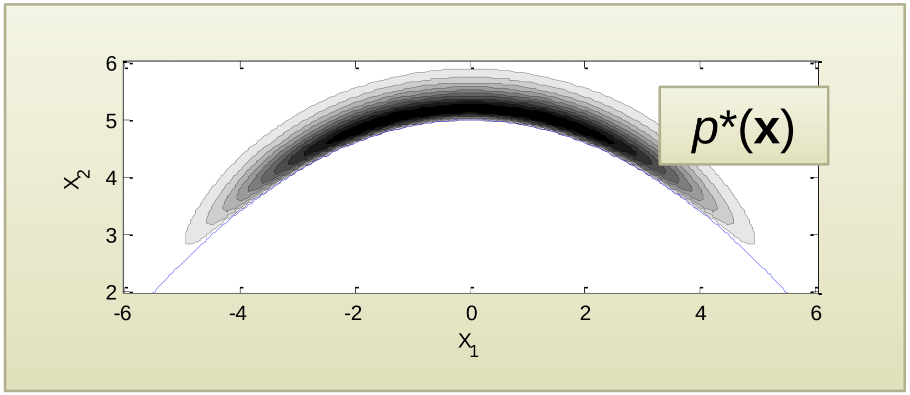


c.o.v. (%)	Number of Samples			Failure Probability		
	MCS	CE-AIS-SG	CE-AIS-GM	MCS	CE-AIS-SG	CE-AIS-GM
10	32,000	400+3,000	400+403	3.16×10^{-3}	2.85×10^{-3}	2.61×10^{-3}
5	1.28×10^{5}	400+8,000	400+434	3.12×10^{-3}	2.98×10^{-3}	2.49×10^{-3}
3	3.64×10^{5}	400+23,000	400+1,390	3.06×10^{-3}	2.98×10^{-3}	2.85×10^{-3}

Component Problem with Competing Design Points (Slight modifications to Der Kiureghian & Dakessian 1998)

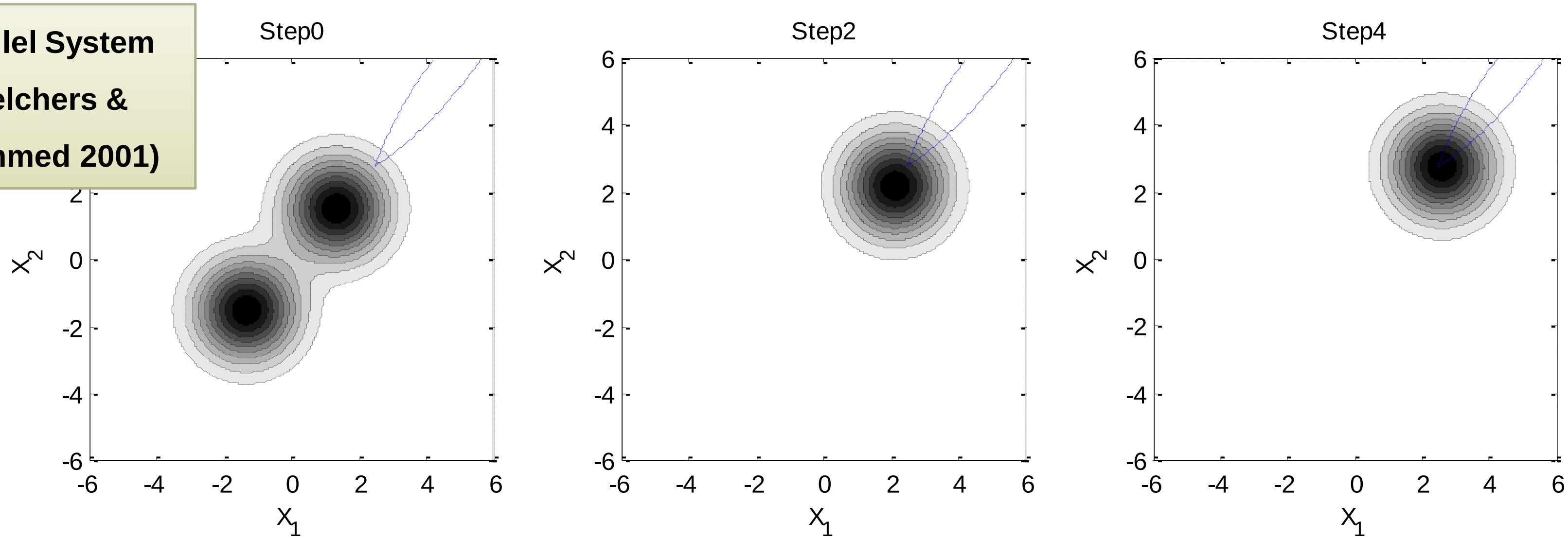






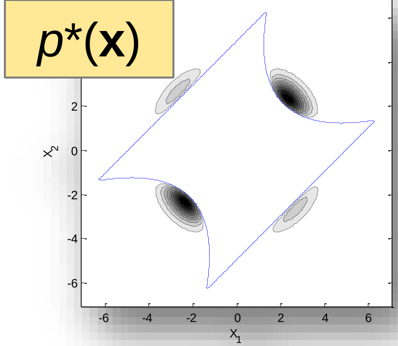
System Problems: Parallel

Parallel System (Melchers & Ahammed 2001)



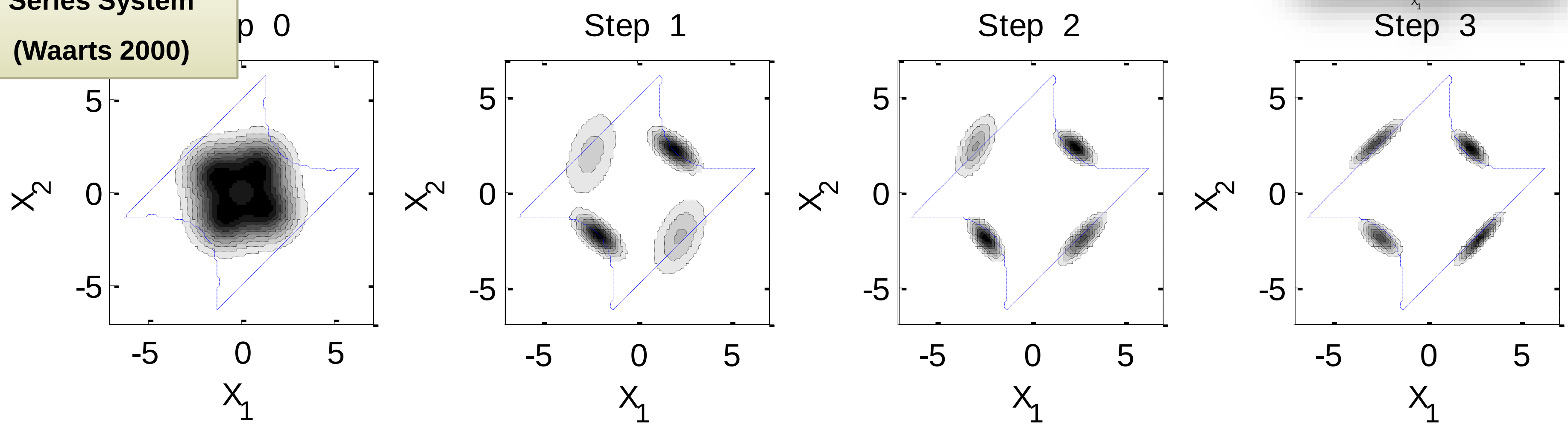
c.o.v. (%)	Number of Samples			Failure Probability		
	MCS	CE-AIS-SG	CE-AIS-GM	MCS	CE-AIS-SG	CE-AIS-GM
10	1.59×10^{7}	500+4,500	600+5,240	6.28×10^{-6}	7.80×10^{-6}	6.30×10^{-6}
5	5.72×10^{7}	500+18,500	600+20,300	7.00×10^{-6}	6.70×10^{-6}	6.28×10^{-6}
3	1.77×10^{8}	500+50,000	600+53,600	6.29×10^{-6}	6.48×10^{-6}	6.14×10^{-6}

System Problems: Series



Series System

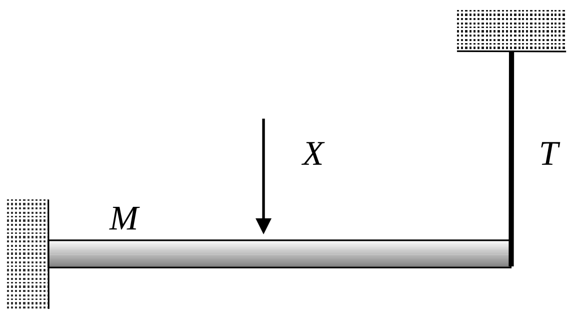
(Waarts 2000)



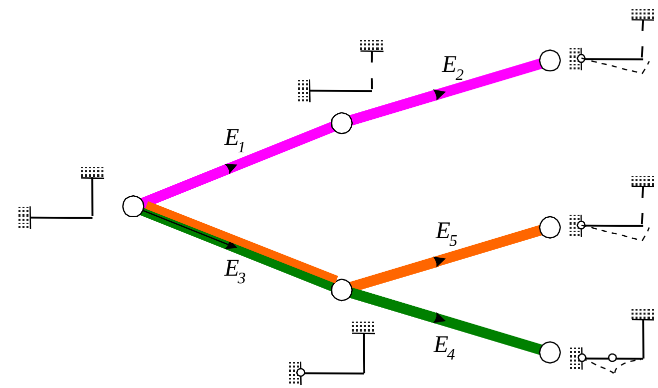
c.o.v. (%)	Number of Samples			Failure Probability		
	MCS	CE-AIS-SG	CE-AIS-GM	MCS	CE-AIS-SG	CE-AIS-GM
10	60,000	4,000+500	3,000+30	1.83×10^{-3}	7.97×10^{-4}	1.50×10^{-3}
5	1.90×10^{5}	4,000+1,500	3,000+348	2.12×10^{-3}	8.80×10^{-4}	2.12×10^{-3}
3	5.20×10^{5}	4,000+2,500	3,000+943	2.16×10^{-3}	8.72×10^{-4}	2.15×10^{-3}

System Problems: General

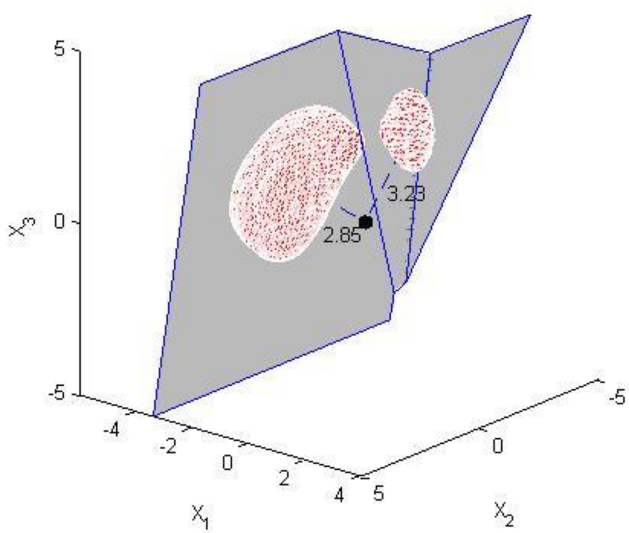
(Song & Der Kiureghian 2003)



$$E_{\text{system}} = \underbrace{(E_1 \cap E_2)}_{\text{Scenario 1}} \cup \underbrace{(E_3 \cap E_4)}_{\text{Scenario 2}} \cup \underbrace{(E_3 \cap E_5)}_{\text{Scenario 3}}$$



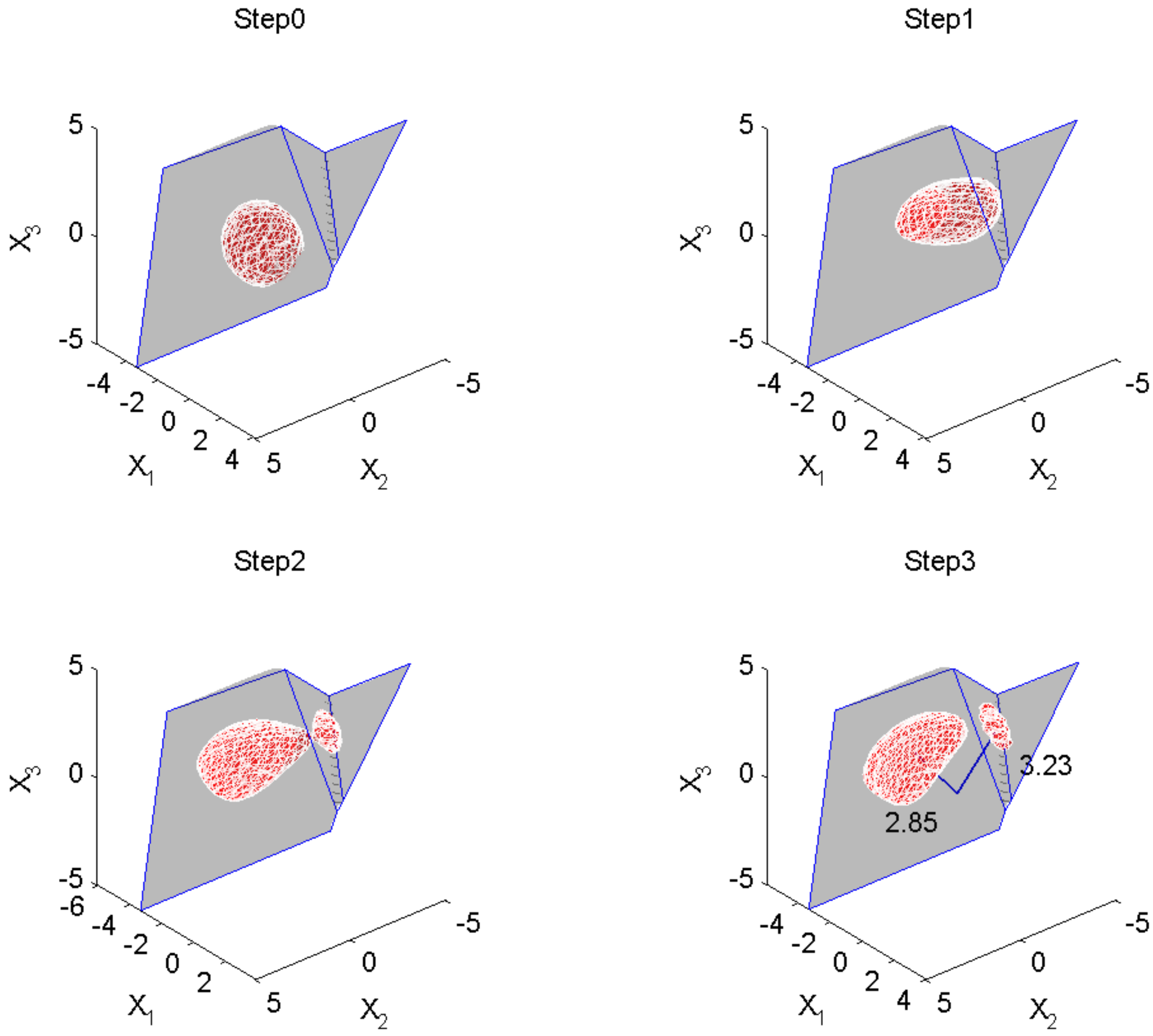
* Component failure events and failure paths



C.O.V. (%)	MCS	AIS-CE-GM
10	20,000	4,000+48
5	60,000	4,000+310
3	150,000	4,000+1,000

* Failure Probability: 7.82×10^{-3}

System Problems: General



Evaluation of the Proposed Method

Engelund & Rackwitz (1993), A Benchmark Study on Importance Sampling Techniques in Structural Reliability, *Structural Safety*, 12: 255-276

2. Evaluation criteria

In order to judge the different importance sampling methods the following quality criteria have been selected.

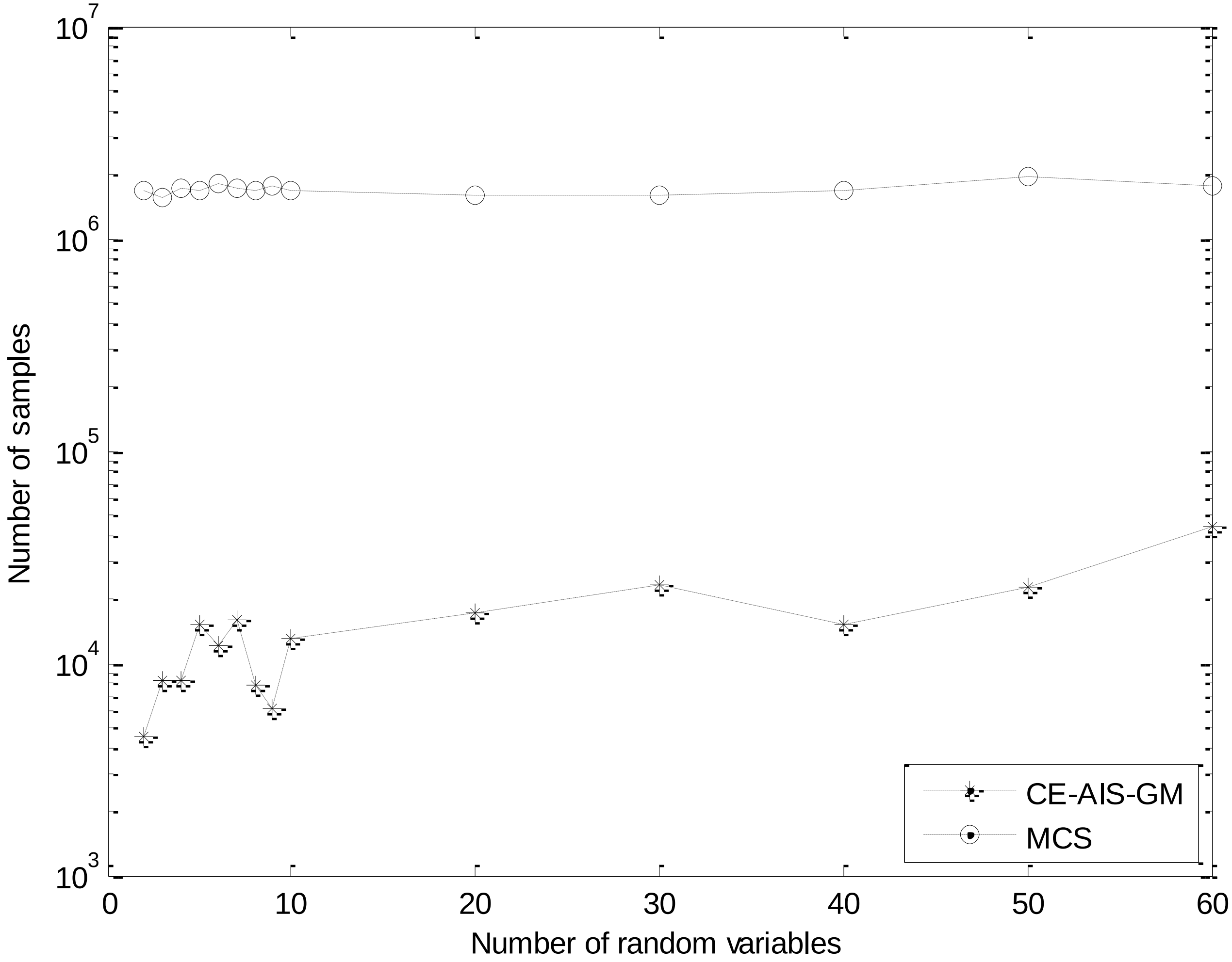
- Basic variable (x-) or standard (u-) space formulation
- Robustness (against multiple critical points and noisy failure boundaries)
- Capabilities to handle equalities, unions and intersections
- Continuity of limit state function and/or joint distribution function of X
- Efficiency and accuracy (convergence properties) especially with respect to
  - Space dimension
  - Probability level
  - Curvatures of limit state function

There is an ongoing debate about which of the alternative computation schemes in the x- or u-space, respectively, is preferable. The u-space formulations appear to be more widely used at

(1) Space Dimension

$$g(\mathbf{x}) = \beta\sqrt{N_{RV}} - \sum_{i=1}^{N_{RV}} x_i$$

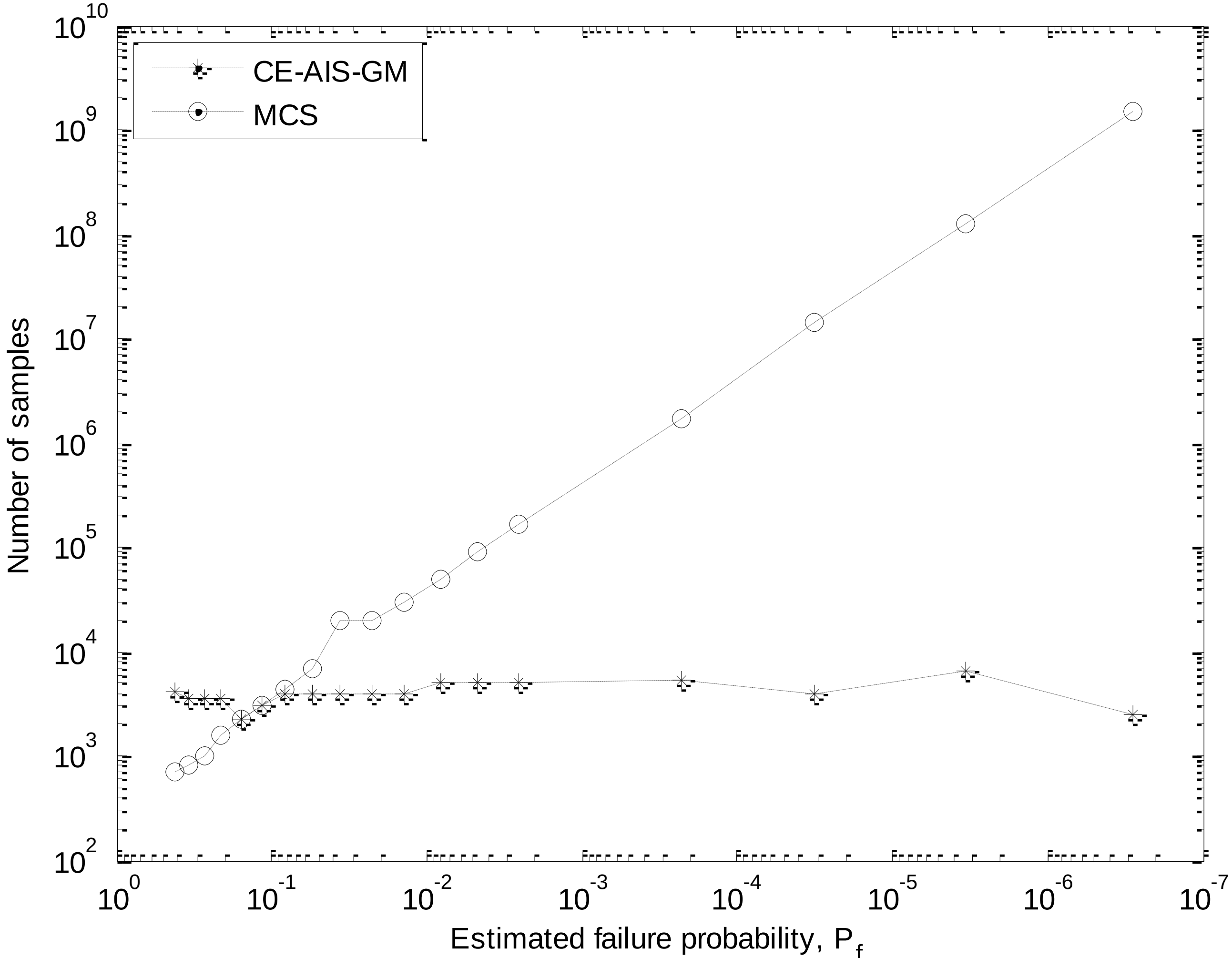
Engelund & Rackwitz (1993)



(2) Level of Probability

$$g(\mathbf{x}) = \beta\sqrt{N_{RV}} - \sum_{i=1}^{N_{RV}} x_i$$

Engelund & Rackwitz (1993); N_{RV} = 5, β varied



(3) Curvature of Limit State Function

$$g(\mathbf{x}) = \beta - x_{N_{RV}} + \frac{1}{2}\sum_{i=1}^{N_{RV}-1} \kappa x_i^2$$

N_{RV} = 5, β and κ controlled together to have P_f = 1.32x10^{-3}

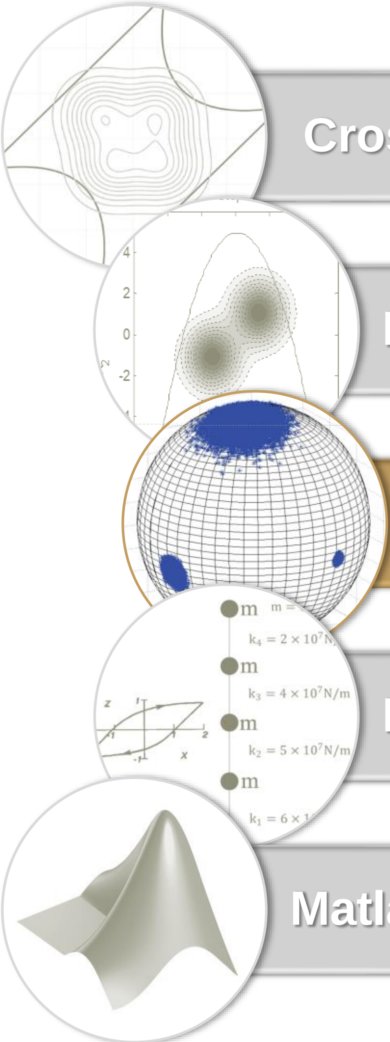
Curvature k	Number of Samples		Failure Probability	
	MCS	CE-AIS-GM	MCS	CE-AIS-GM
1.154 (convex)	308,300	5,000+700	1.30×10^{-3}	1.28×10^{-3}
0 (neutral)	292,400	4,000+600	1.39×10^{-3}	1.36×10^{-3}
–0.291 (concave)	306,400	6,000+1,000	1.31×10^{-3}	1.36×10^{-3}

(4) Robustness against Noisy Limit State

$$g(\mathbf{x}) = b - x_2 - \kappa(x_1 - e)^2 + 0.001\sum_{i=1}^{2} \sin(100 \cdot x_i)$$

c.o.v. (%)	Number of Samples			Failure Probability		
	MCS	CE-AIS-SG	CE-AIS-GM	MCS	CE-AIS-SG	CE-AIS-GM
10	23,400	2,000+100	3,000+100	2.99×10^{-3}	1.08×10^{-3}	2.74×10^{-3}
5	1.30×10^{5}	2,000+300	3,000+300	3.07×10^{-3}	1.08×10^{-3}	2.94×10^{-3}
3	3.60×10^{5}	2,000+900	3,000+900	3.07×10^{-3}	1.09×10^{-3}	3.05×10^{-3}

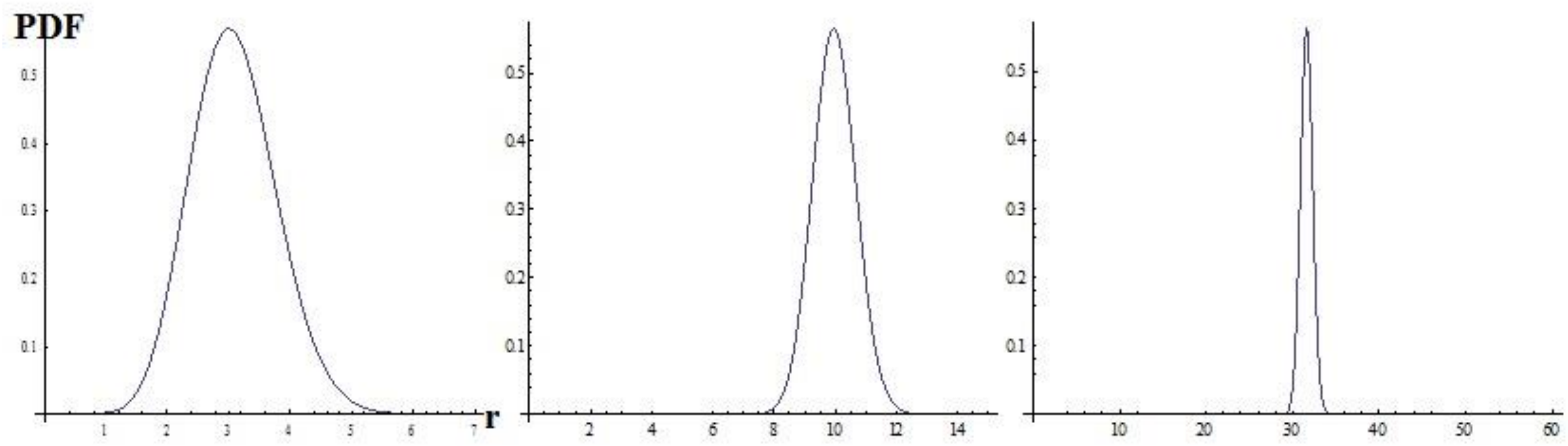
Contents



Reliability analysis in *high* dimensional space

PDF of "distance" from the origin: $r = \sqrt{\sum_{i=1}^{n} u_i^2}$ **,** $r \sim \chi_n$

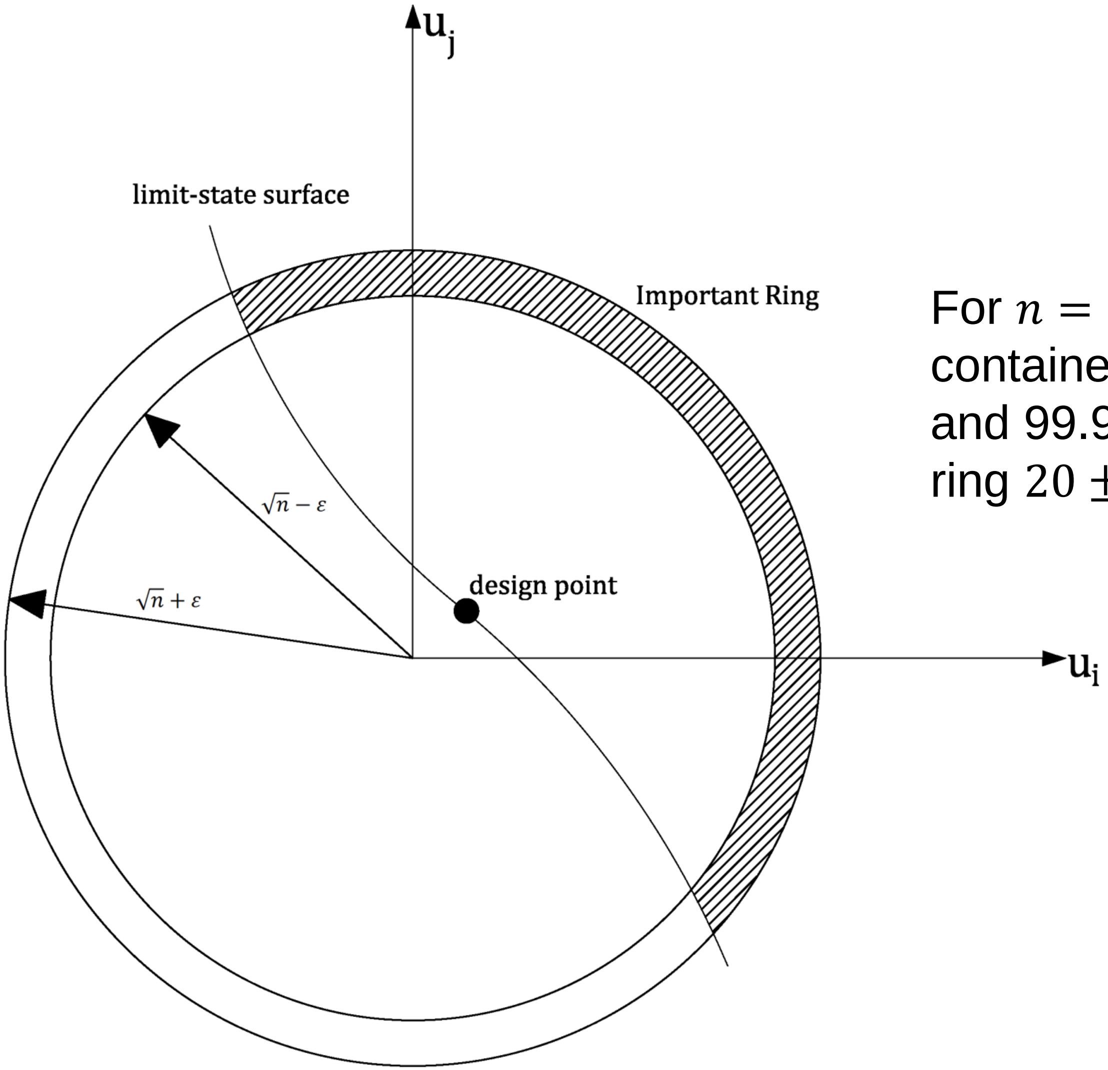
***n*=10** ***n*=100** ***n*=1,000**



$n \to \infty$**,** $\chi_n \to \mathcal{N}(\sqrt{n}, 1/2)$

As the dimension grows, χ_n **becomes relatively narrower, a random distance *r* will 'surely' have** $r \in [\sqrt{n} - \varepsilon, \sqrt{n} + \varepsilon]$

“Important ring”

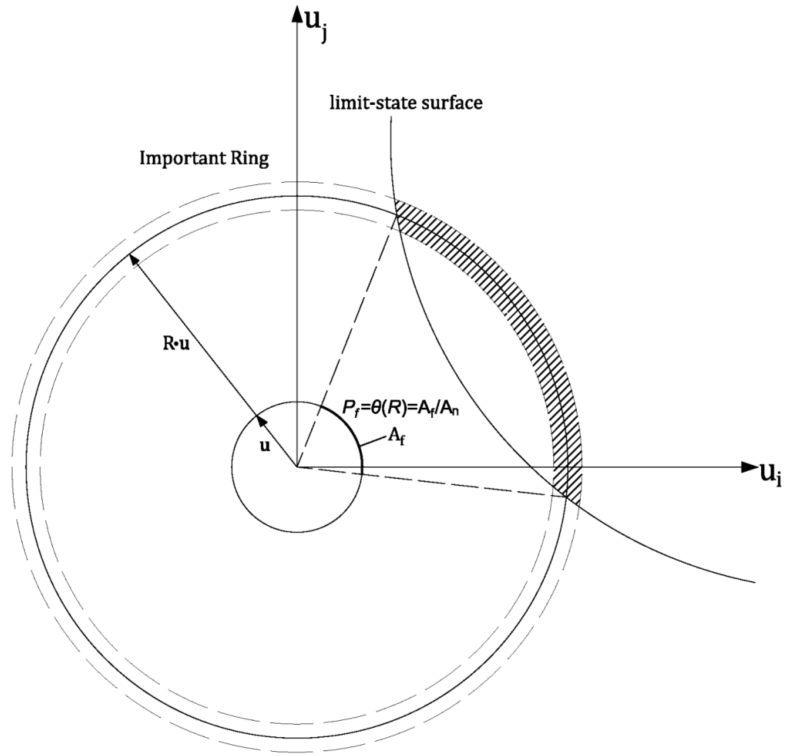


For $n = 400$, 95% probability is contained within the ring 20 ± 1, and 99.99% is contained within the ring 20 ± 2.

Importance sampling strategy for large *n*

$$P_f = \int_0^\infty \theta(r) f_\chi(r) dr \cong \frac{1}{M}\sum_{i=1}^{M} \theta(r_i)$$

where $\theta(r) = A_f(r)/A_n$



For large n, $R \cong \sqrt{n}$

- MCS: $P_f \cong \theta(\sqrt{n}) = \int I_R(\bar{u})\, f_U(\bar{u}) d\bar{u} \cong \frac{1}{N}\sum_{i=1}^{N} I_R(\bar{u}_i) f_U(\bar{u}_i)$
- Importance Sampling: $P_f \cong \frac{1}{N}\sum_{i=1}^{N} \frac{I_R(\bar{u}_i) f_U(\bar{u}_i)}{h(\bar{u}_i; \mathbf{v})}$

Near-optimal sampling parameter $\mathbf{v}$ could be found by pre-sampling of the CE-AIS technique

CE-AIS using von Mises-Fisher Mixture (CE-AIS-vMFM, Wang & Song 2016)

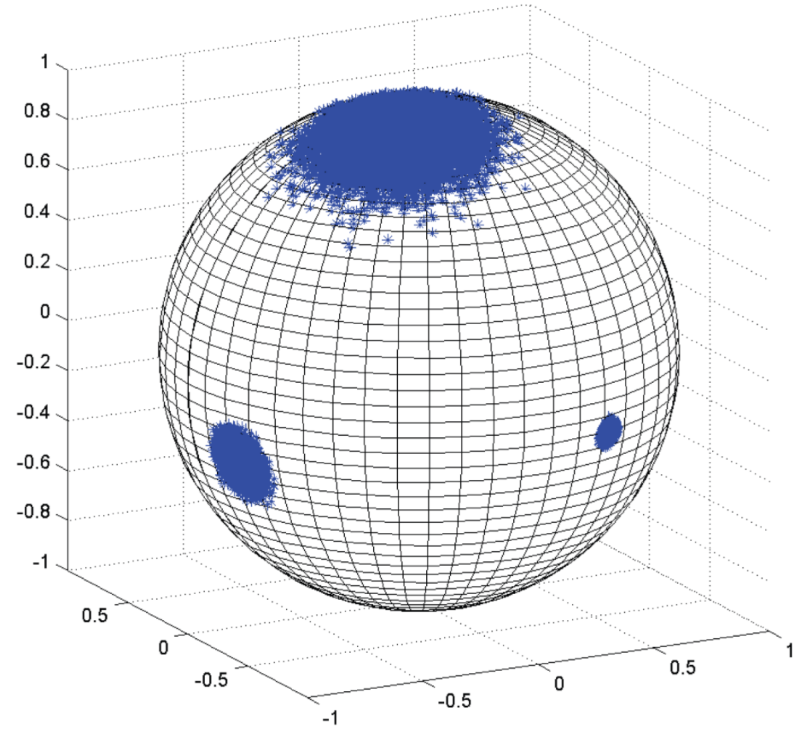
- **Sampling by "von Mises-Fisher Mixture" model**

$$f_{\text{vMFM}}(\bar{\mathbf{u}}; \mathbf{v}) = \sum_{k=1}^{K} \alpha_k f_{\text{vMF}}(\bar{\mathbf{u}}; \mathbf{v}_k)$$

where $\sum_{k=1}^{K} \alpha_k = 1$, $\alpha_k > 0$ for $\forall k$

$$f_{\text{vMF}}(\bar{\mathbf{u}}) = c_d(\kappa) e^{\kappa \boldsymbol{\mu}^T \bar{\mathbf{u}}}$$

- κ: concentration parameter
- $\boldsymbol{\mu}$: mean direction
- α_k: weight for the k-th vMF



Wang, Z., and J. Song (2016). Cross-entropy-based adaptive importance sampling using von Mises–Fisher mixture for high dimensional reliability analysis. *Structural Safety*. 59: 42-52.

Updating rules derived for CE-AIS-vMFM

$$\alpha_k = \frac{\sum_{i=1}^{N} I_R(\bar{\mathbf{u}}_i) W(\bar{\mathbf{u}}_i; \mathbf{w}) \Upsilon_i(z_k)}{\sum_{i=1}^{N} I_R(\bar{\mathbf{u}}_i) W(\bar{\mathbf{u}}_i; \mathbf{w})}$$

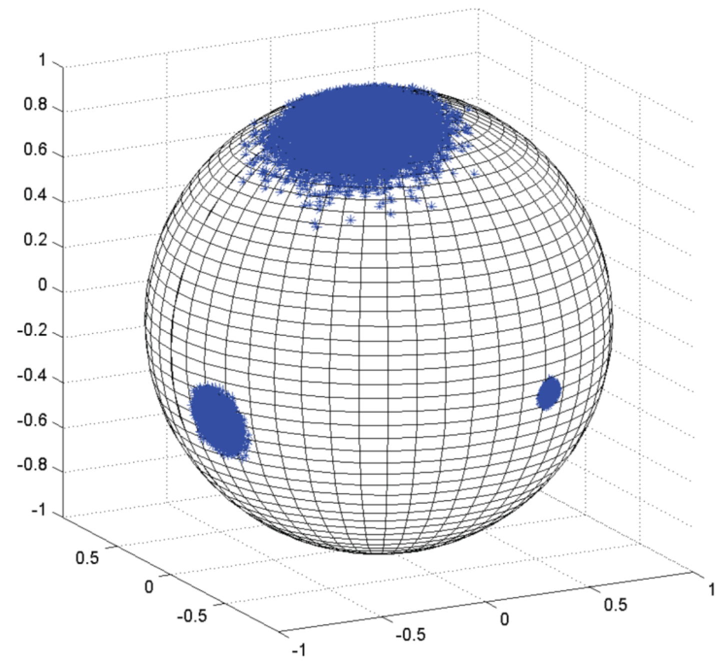
$$\boldsymbol{\mu}_k = \frac{\sum_{i=1}^{N} I_R(\bar{\mathbf{u}}_i) W(\bar{\mathbf{u}}_i; \mathbf{w}) \Upsilon_i(z_k) \bar{\mathbf{u}}_i}{\left\| \sum_{i=1}^{N} I_R(\bar{\mathbf{u}}_i) W(\bar{\mathbf{u}}_i; \mathbf{w}) \Upsilon_i(z_k) \bar{\mathbf{u}}_i \right\|}$$

$$\kappa_k \cong \frac{\xi n - \xi^3}{1 - \xi^2}$$

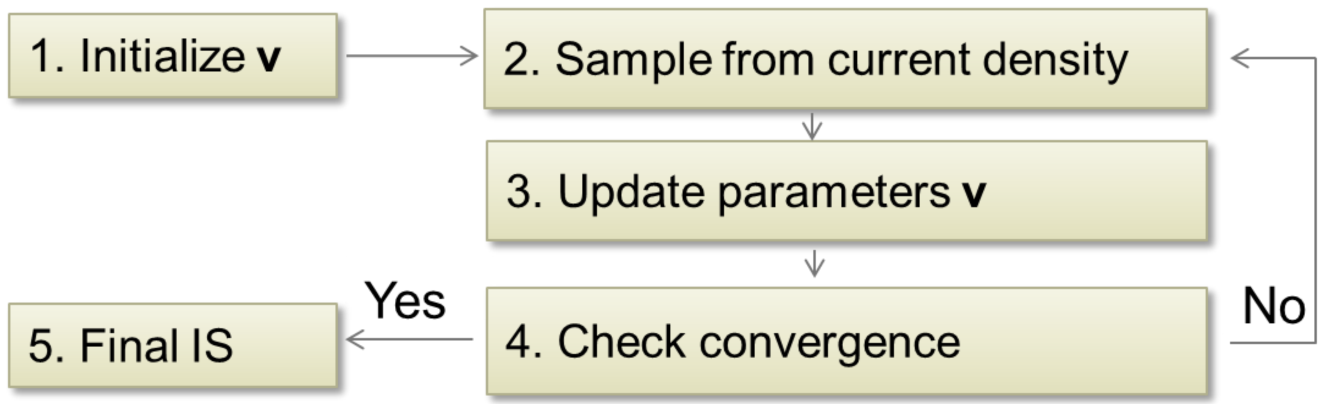
where

$$\xi = \frac{\left\| \sum_{i=1}^{N} I_R(\bar{\mathbf{u}}_i) W(\bar{\mathbf{u}}_i; \mathbf{w}) \Upsilon_i(z_k) \bar{\mathbf{u}}_i \right\|}{\sum_{i=1}^{N} I_R(\bar{\mathbf{u}}_i) W(\bar{\mathbf{u}}_i; \mathbf{w}) \Upsilon_i(z_k)}$$

$$\Upsilon_i(z_k) = \frac{\alpha_k f_{\mathrm{vMF}}(\bar{\mathbf{u}}_i; \mathbf{v}_k)}{\sum_{k=1}^{K} \alpha_k f_{\mathrm{vMF}}(\bar{\mathbf{u}}_i; \mathbf{v}_k)}$$

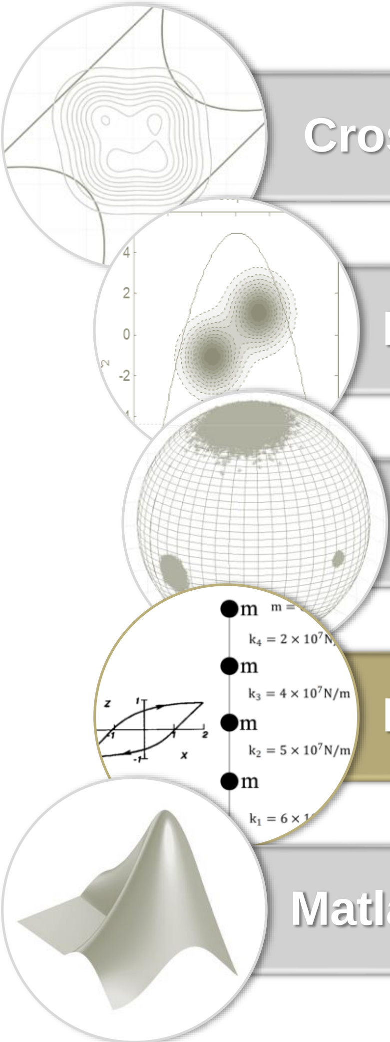


Procedures of CE-AIS-vMFM



1. Pre-sampling to obtain near-optimal (i.e. minimum CE) vMFM sampling density using updating rules
2. Perform the final IS on the hyper-sphere with radius $\sqrt{n}$ (**CE-AIS-vMFM-1**)
3. Alternatively, perform the final IS on hyper-spheres with radius drawn from the χ-distribution
(**CE-AIS-vMFM-2**)

Contents



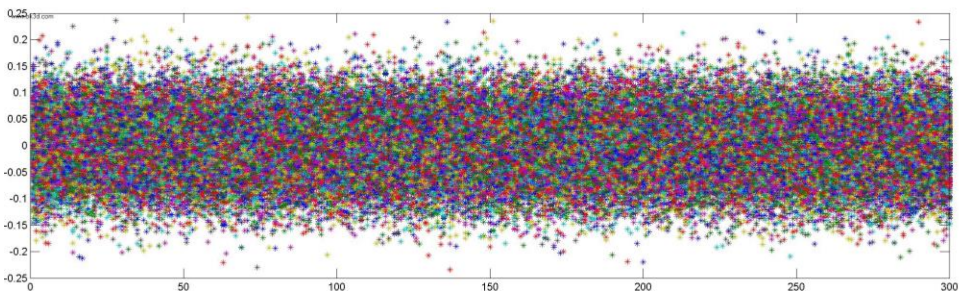
Example 1: Series system reliability in high-dimension

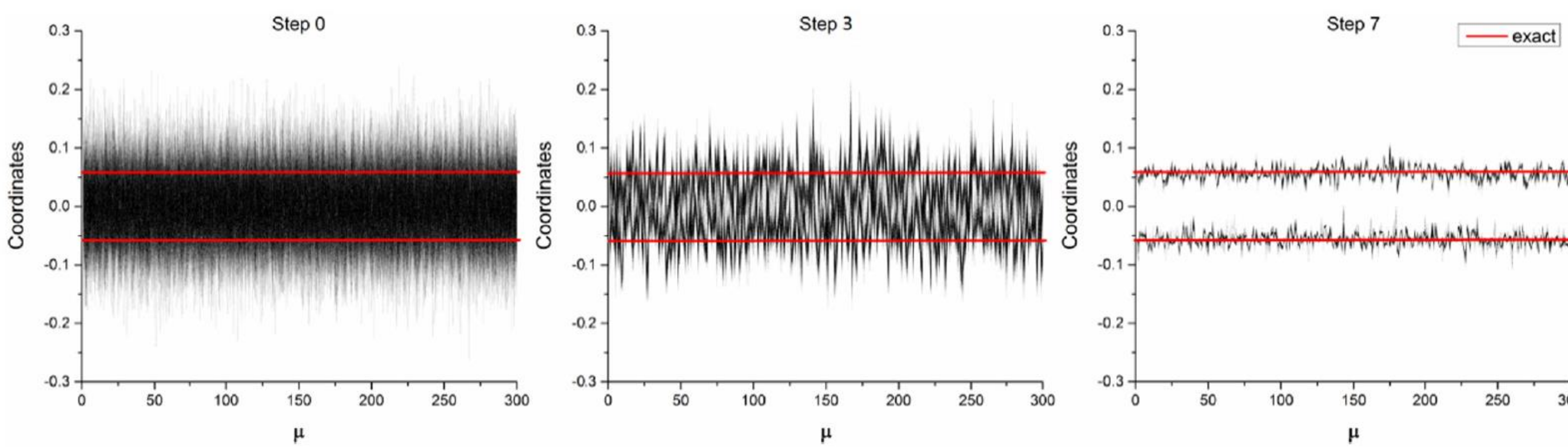
$$G_1(\mathbf{u}) = \beta_1\sqrt{n} - \sum_{i=1}^{n} \mathbf{u}_i \,, G_2(\mathbf{u}) = \beta_2\sqrt{n} + \sum_{i=1}^{n} \mathbf{u}_i$$

System failure domain: $G_1(\mathbf{u}) \leq \mathbf{0} \cup G_2(\mathbf{u}) \leq \mathbf{0}$

$\beta_1 = \beta_2 = 3.5$, $n = 300$

Updating of mean directions:





Example 2: Nonlinear random vibration analysis of MDOF system

- **Discrete representation of stochastic process representing ground acceleration (in frequency domain)**

$$\ddot{U}_g(t) = \sum_{j=1}^{n/2} \sigma_j [\mathrm{u}_j \cos(\omega_j t) + \hat{\mathrm{u}}_j \sin(\omega_j t)]$$

where

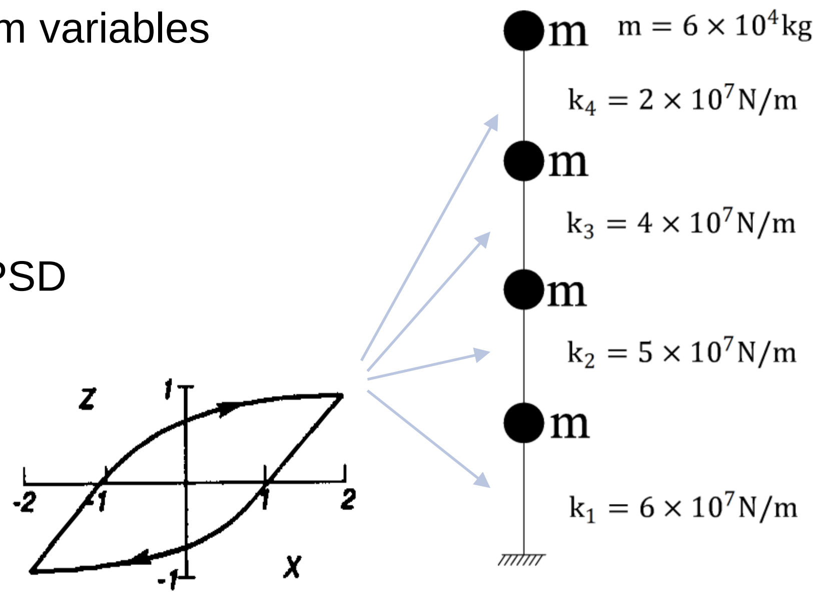
u_j, $\hat{\mathrm{u}}_j$: independent standard normal random variables

ω_j: discretized frequency points

$\sigma_j = \sqrt{2S(\omega_j)\Delta\omega}$

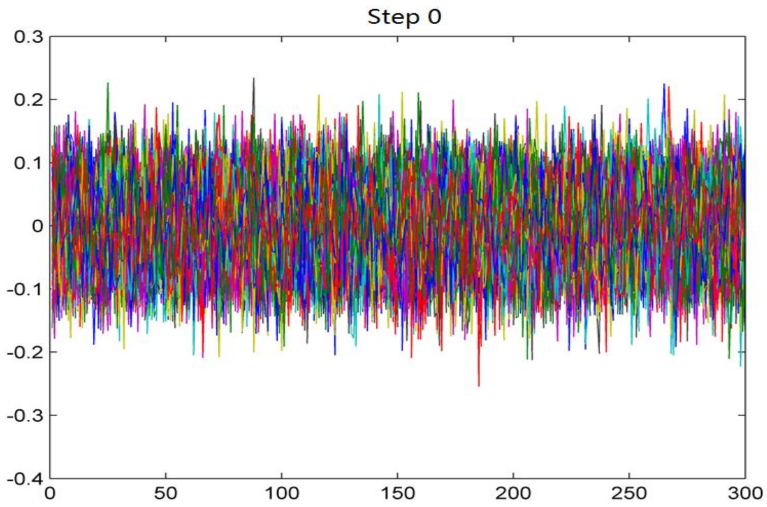
$S(\omega_j)$: two-sided power spectrum density/PSD

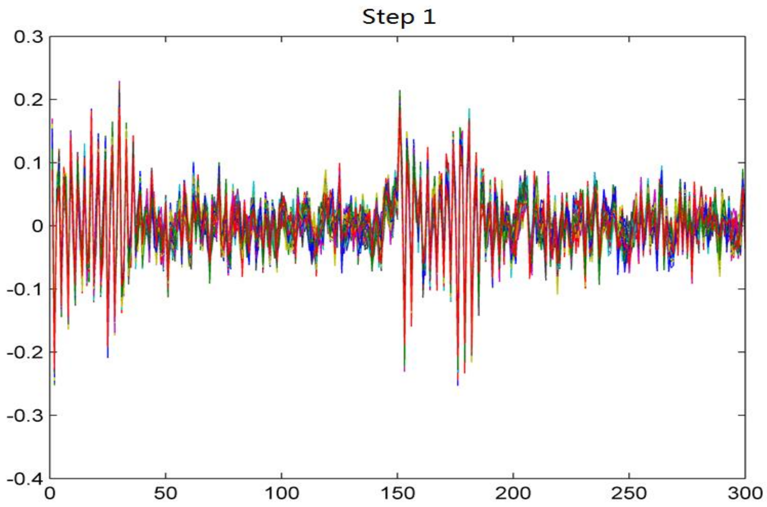
$\Delta\omega$: frequency step size



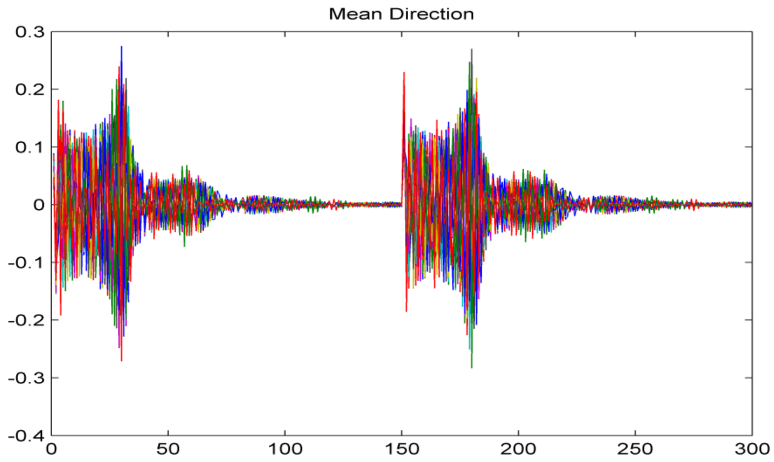
Example 2: Updating of vMFM

- **Instantaneous failure**





- **First-passage failure (series system)**



Example 2: Accuracy and efficiency of CE-AIS-vMFM

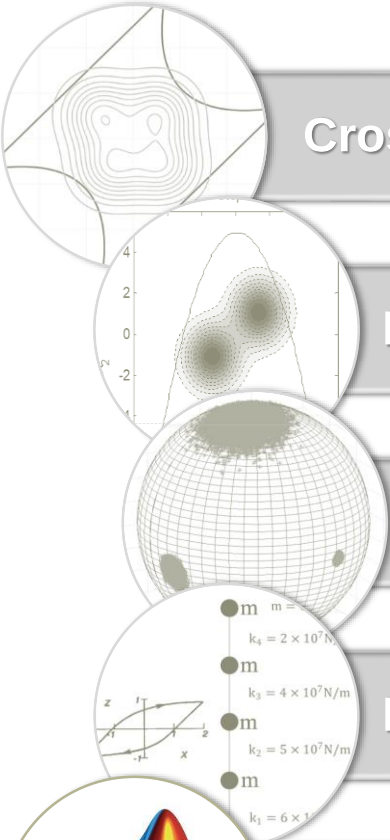
- **Instantaneous failure**

Displ. Threshold	CE-AIS-vMFM-1		CE-AIS-vMFM-2		MCS	
	$N_{tot}(\times 10^4)$	$P_f(10^{-3})$	$N_{tot}(\times 10^4)$	$P_f(10^{-3})$	$N_{tot}(\times 10^4)$	$P_f(10^{-3})$
0.005	0.36	183.3	0.36	196.9		193.7
0.01	0.21	49.6	0.21	52.6		55.4
0.02	1.13	2.3	1.55	3.1		2.9
0.03	0.82	0.03	1.24	0.04	275.93	0.04

- **First passage failure (series system)**

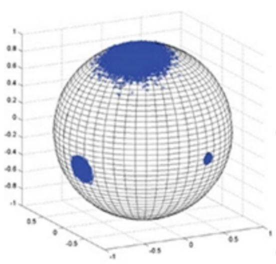
Displ. Threshold	CE-AIS-vMFM-1		CE-AIS-vMFM-2		MCS	
	$N_{tot}(\times 10^4)$	$P_f(10^{-3})$	$N_{tot}(\times 10^4)$	$P_f(10^{-3})$	$N_{tot}(\times 10^4)$	$P_f(10^{-3})$
0.02	0.89	475	0.96	483		459
0.03	0.81	10.1	1.01	11.3		13.2
0.04	0.94	0.20	1.17	0.27	322.52	0.31

Contents



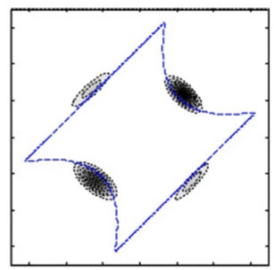
Matlab® codes of CE-AIS

http://systemreliability.wordpress.com/software/



Cross-entropy-based Adaptive Importance Sampling Using von Mises-Fisher Mixture for High Dimensional Reliability Analysis

- Developer: Ziqi Wang (currently at Earthquake Engineering Research & Test Center, Guangzhou University, Guanzhou, China)
- What it does: Perform adaptive importance sampling for component and system reliability problems using the "Cross-entropy-based Adaptive Importance Sampling Using von Mises-Fisher Mixture for High Dimensional reliability analysis" (Wang and Song
- Purpose of development: International collaborative res reliability problem
- Reference: Wang, Z., and J. Song, Cross-entropy-based sampling using von Mises-Fisher mixture for high dime *Structural Safety*. Vol. 59, 42-52, 2016.
- Download: HERE
- How to use: Unzip the file into a local folder and set patl "TestExample.m" to reproduce the results of 5.2 in the Read "ReadMe.txt" for more details.



Cross-entropy-based Adaptive Importance Sampling Using Gaussian Mixture

- Developer: Ryan H.Y. Wong (currently working at AECOM in Hong Kong)
- What it does: Perform adaptive importance sampling for component and system reliability problems using the "Cross-entropy-based Adaptive Importance Sampling Using Gaussian Mixture" (Kurtz and Song 2013).
- Purpose of development: Undergraduate research to develop free computer codes for state-of-the-art algorithms (sponsored by the Department of Civil and Environmental Engineering at UIUC).
- Reference: N. Kurtz and J. Song, "Cross-entropy-based Adaptive Importance Sampling Using Gaussian Mixture," Structural Safety, Vol. 42, pp. 35-44, 2013.
- Download: HERE (The "accepted author manuscript" of the paper is included)
- How to use: Unzip the file into a local folder and set paths. Run an input file (see the examples and input file template). Then, run "ceaisgm.m". Read "CEAISGM.txt" for more details.

Organizing Association CERRA

Co-Organized by KIBSE Korean Institute of Bridge & Structural Engineers · Institute of Construction and Environmental Engineering Seoul National University · KBRC

ICASP13

13th International Conference on Applications of Statistics and Probability in Civil Engineering

May 26-30, 2019
Seoul National University Seoul, South Korea