Static Magnetic Fields

Introduction to Electromagnetism with Practice

Theory & Applications

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Magnetization & Permeability

Remind: Theory of Dielectric Matter: Polarization

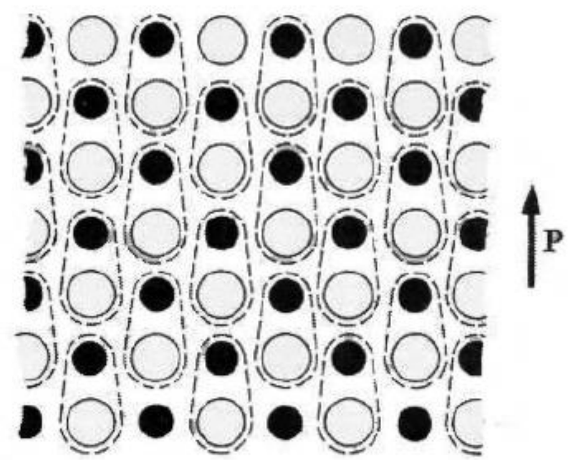
Dielectric

Polarization Charge
(Bound charge)

$$\rho_{\mathbf{P}}(\mathbf{r}) \Rightarrow \mathbf{E}_{\text{int}}(\mathbf{r})$$

Free Charge

$$\rho_{f}(\mathbf{r}) \Rightarrow \mathbf{E}_{\text{ext}}(\mathbf{r})$$

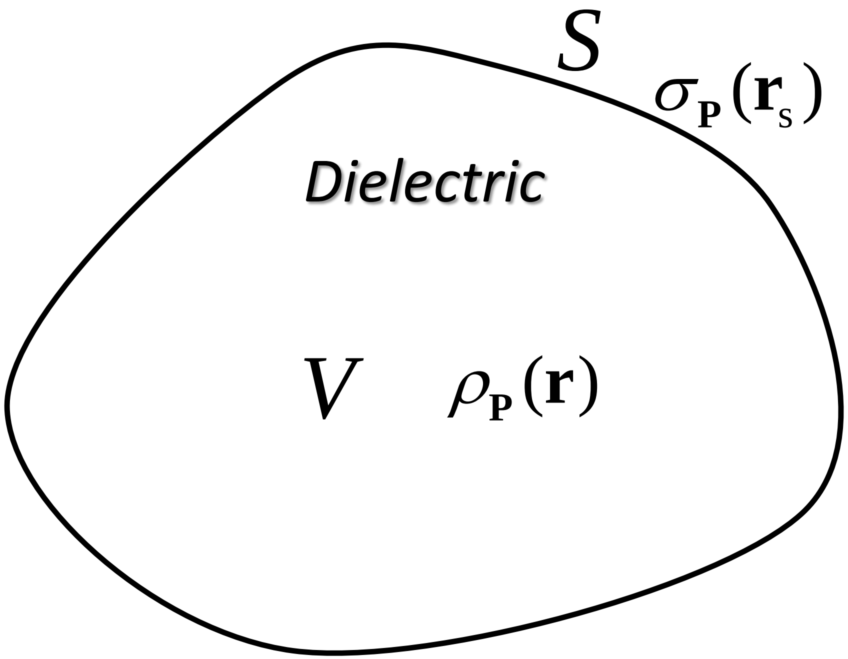


Chemical Bonding, Q.M., ...

$$\rho_{\mathbf{P}}(\mathbf{r})\mathbf{E}_{\text{tot}}(\mathbf{r}) + \boxed{\mathbf{F}_{\text{others}}(\mathbf{r})} = \mathbf{O}$$

Remind: Theory of Dielectric Matter: Polarization

Separating volume & surface parts in polarization charge



Neutral Dielectric

$$\int_V \rho_{\mathbf{P}}(\mathbf{r})dv + \oint_S \sigma_{\mathbf{P}}(\mathbf{r}_s)ds = 0$$

Remembering Gauss's Theorem!
: Introducing

$$\boxed{\rho_{\mathbf{P}}(\mathbf{r}) = -\nabla \cdot \mathbf{P}(\mathbf{r})}$$

$$-\int_V \nabla \cdot \mathbf{P}(\mathbf{r})dv + \oint_S \sigma_{\mathbf{P}}(\mathbf{r})ds$$

$$= \oint_S \mathbf{P}(\mathbf{r}) \cdot d\mathbf{s} + \oint_S \sigma_{\mathbf{P}}(\mathbf{r})ds = 0$$

$$\boxed{\sigma_{\mathbf{P}}(\mathbf{r}_s) = \mathbf{P}(\mathbf{r}_s) \cdot \mathbf{e}_{\mathbf{n}}}$$

Remind: Theory of Dielectric Matter: Polarization

Polarization

The *rearrangement* of internal charge that occurs when matter is exposed to an external field – characterized by **P**(**r**)

$$\boxed{\rho_{\mathbf{P}}(\mathbf{r}) = -\nabla \cdot \mathbf{P}(\mathbf{r})}$$

$$\boxed{\sigma_{\mathbf{P}}(\mathbf{r}_S) = \mathbf{P}(\mathbf{r}_S) \cdot \mathbf{e}_{\mathbf{n}}}$$

P(**r**) is not uniquely determined by the volume & surface charge distributions

➔ Helmholtz Theorem

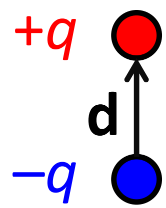
Remind: Theory of Dielectric Matter: Electric Potential

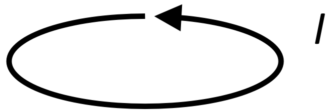
$$\boxed{V = \frac{1}{4\pi\varepsilon_0}\int_V \frac{1}{|\mathbf{x}-\mathbf{x}'|}\rho(\mathbf{x}')d^3x'}$$

$$\rho_{\mathbf{P}}(\mathbf{r}) = -\nabla\cdot\mathbf{P}(\mathbf{r}) \qquad \sigma_{\mathbf{P}}(\mathbf{r}_s) = \mathbf{P}(\mathbf{r}_s)\cdot\mathbf{e}_{\mathbf{n}}$$

$$\boxed{V = \frac{1}{4\pi\varepsilon_0}\int_V \frac{-\nabla\cdot\mathbf{P}(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|}dv' + \frac{1}{4\pi\varepsilon_0}\oint_S \frac{\mathbf{P}(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|}\cdot d\mathbf{s}'}$$

Remind: Electric Dipole VS Magnetic Dipole





$$\mathbf{p} = q\mathbf{d}$$

$$\mathbf{m} = \mathbf{e}_z I \pi b^2$$

$$\boxed{V = \frac{1}{4\pi\varepsilon_0}\frac{\mathbf{p}\cdot\mathbf{e}_r}{R^2}}$$

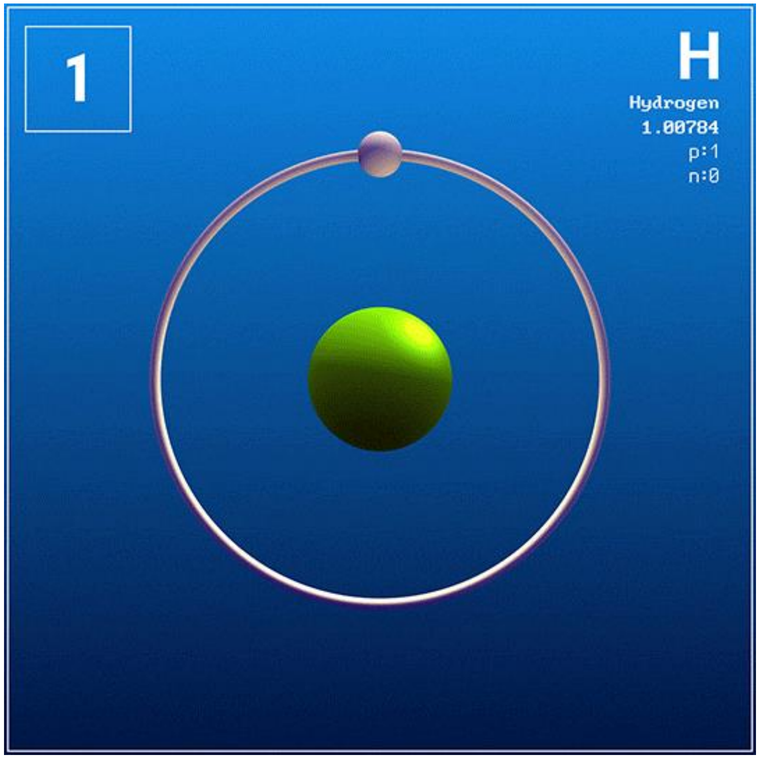
$$\boxed{\mathbf{A} = \frac{\mu_0}{4\pi}\frac{\mathbf{m}\times\mathbf{e}_r}{R^2}}$$

$$\boxed{\mathbf{E} = \frac{p}{4\pi\varepsilon_0 R^3}\left(\mathbf{e}_r 2\cos\theta + \mathbf{e}_\theta \sin\theta\right)}$$

$$\boxed{\mathbf{B} = \frac{\mu_0 m}{4\pi R^3}\left(\mathbf{e}_r 2\cos\theta + \mathbf{e}_\theta \sin\theta\right)}$$

Magnetic Dipole ~ Current Loop: Orbital Angular Moment

Two-Body Problem



The stereotypical "solar-system" model for hydrogen (WIKI)

Proton Mass:
$1.6726219236 \times 10^{-27}$ kg

Electron Mass:
$9.1093837015 \times 10^{-31}$ kg

1.8×10^{3}

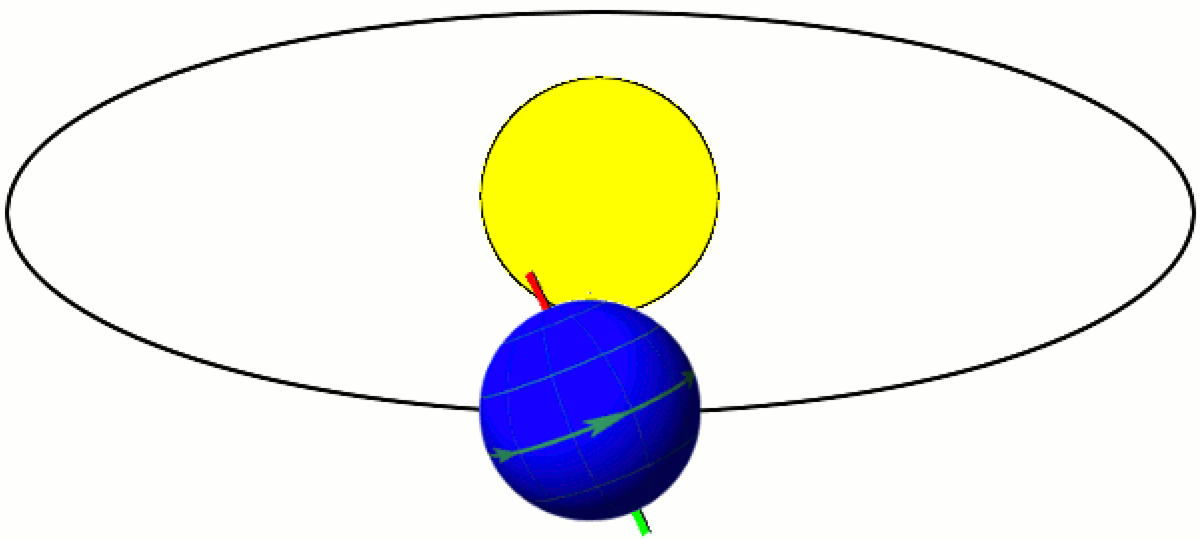
Sun Mass:
1.989×10^{30} kg

Earth Mass:
5.972×10^{24} kg

3.3×10^{5}

Magnetic Dipole ~ Current Loop: Spin Angular Moment

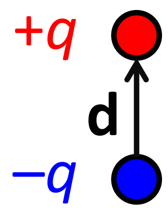
A classical spin in a macroscopic environment is actually an orbital angular momentum!

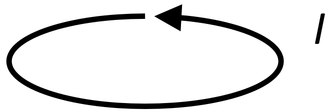


What happens when a particle becomes a point?

Spin* $\in$ *Internal Degree of Freedom for a Particle

Polarization vs Magnetization





$$\mathbf{p} = q\mathbf{d}$$

$$\mathbf{m} = \mathbf{e}_z I \pi b^2$$

$$\mathbf{P} = \lim_{\Delta v \to 0} \frac{\sum_{k=1}^{n} \mathbf{p}_k}{\Delta v}$$

$$\mathbf{M} = \lim_{\Delta v \to 0} \frac{\sum_{k=1}^{n} \mathbf{m}_k}{\Delta v}$$

Magnetization & Current Density

$$\mathbf{A} = \frac{\mu_0}{4\pi}\frac{\mathbf{m}\times\mathbf{e}_r}{R^2} \xrightarrow[d\mathbf{m} = \mathbf{M}dv']{} d\mathbf{A} = \frac{\mu_0}{4\pi}\frac{\mathbf{M}\times\mathbf{e}_r}{R^2}dv'$$

$$d\mathbf{A} = \frac{\mu_0}{4\pi}\frac{\mathbf{M}\times\mathbf{R}}{R^3}dv' = \frac{\mu_0}{4\pi}\frac{\mathbf{M}\times(\mathbf{x}-\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|^3}dv'$$

$$\nabla\frac{1}{|\mathbf{x}-\mathbf{x}'|} = -\nabla'\frac{1}{|\mathbf{x}-\mathbf{x}'|} = -\frac{\mathbf{x}-\mathbf{x}'}{|\mathbf{x}-\mathbf{x}'|^3}$$

$$d\mathbf{A} = \frac{\mu_0}{4\pi}\frac{\mathbf{M}\times\mathbf{R}}{R^3}dv' = \frac{\mu_0}{4\pi}\mathbf{M}\times\left(\nabla'\frac{1}{|\mathbf{x}-\mathbf{x}'|}\right)dv'$$

Magnetization & Current Density

$$d\mathbf{A} = \frac{\mu_0}{4\pi}\frac{\mathbf{M}\times\mathbf{R}}{R^3}dv' = \frac{\mu_0}{4\pi}\mathbf{M}\times\left(\nabla'\frac{1}{|\mathbf{x}-\mathbf{x}'|}\right)dv'$$

$$\mathbf{A} = \frac{\mu_0}{4\pi}\int_{V'}\mathbf{M}\times\left(\nabla'\frac{1}{|\mathbf{x}-\mathbf{x}'|}\right)dv'$$

$$\nabla\times(f\mathbf{A}) = f(\nabla\times\mathbf{A}) - \mathbf{A}\times(\nabla f) \Longrightarrow \mathbf{A}\times(\nabla f) = f(\nabla\times\mathbf{A}) - \nabla\times(f\mathbf{A})$$

$$\mathbf{A} = \frac{\mu_0}{4\pi}\left[\int_{V'}\frac{\nabla'\times\mathbf{M}}{|\mathbf{x}-\mathbf{x}'|}dv' - \int_{V'}\nabla'\times\frac{\mathbf{M}}{|\mathbf{x}-\mathbf{x}'|}dv'\right]$$

$$= \frac{\mu_0}{4\pi}\left[\int_{V'}\frac{\nabla'\times\mathbf{M}}{|\mathbf{x}-\mathbf{x}'|}dv' + \oint_{S'}\frac{\mathbf{M}\times\mathbf{e}_n'}{|\mathbf{x}-\mathbf{x}'|}ds'\right]$$

P. 6-20

Magnetization & Current Density

$$V = \frac{1}{4\pi\varepsilon_0}\int_V \frac{-\nabla\cdot\mathbf{P}(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|}dv' + \frac{1}{4\pi\varepsilon_0}\oint_S \frac{\mathbf{P}(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|}\cdot ds'$$

$$\mathbf{A} = \frac{\mu_0}{4\pi}\int_{V'} \frac{\nabla'\times\mathbf{M}}{|\mathbf{x}-\mathbf{x}'|}dv' + \frac{\mu_0}{4\pi}\oint_{S'} \frac{\mathbf{M}\times\mathbf{e}_n{}'}{|\mathbf{x}-\mathbf{x}'|}ds'$$

$$V = \frac{1}{4\pi\varepsilon_0}\int_V \frac{1}{|\mathbf{x}-\mathbf{x}'|}\rho(\mathbf{x}')d^3x'$$

$$\mathbf{A} = \frac{\mu_0}{4\pi}\int_V \frac{1}{|\mathbf{x}-\mathbf{x}'|}\mathbf{J}(\mathbf{x}')d^3x'$$

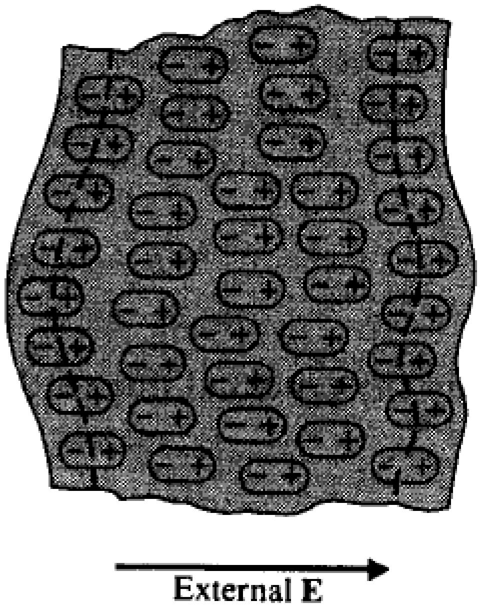
$$\rho_{\mathbf{P}}(\mathbf{r}) = -\nabla\cdot\mathbf{P}(\mathbf{r})$$

$$\mathbf{J}_m(\mathbf{r}) = \nabla\times\mathbf{M}$$

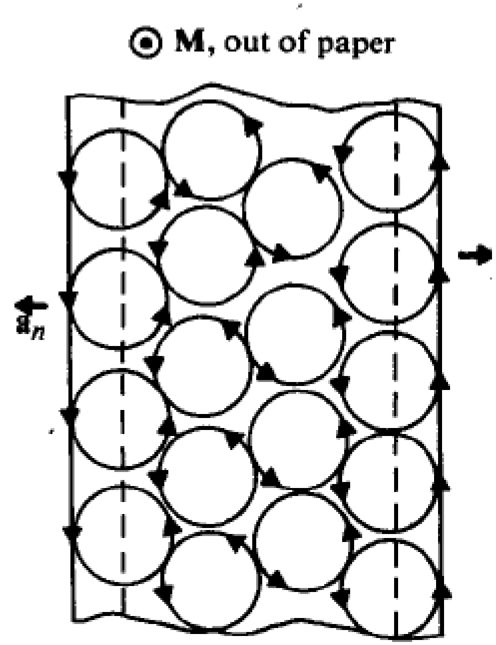
$$\sigma_{\mathbf{P}}(\mathbf{r}_s) = \mathbf{P}(\mathbf{r}_s)\cdot\mathbf{e}_\mathbf{n}$$

$$\mathbf{J}_{ms}(\mathbf{r}_s) = \mathbf{M}\times\mathbf{e}_n$$

Interpretation of Magnetization



Sec. 6-5.1 & 6.1
When J = 0:
The use of scalar potential is available also for B: Same formulation with that of Electrostatics!



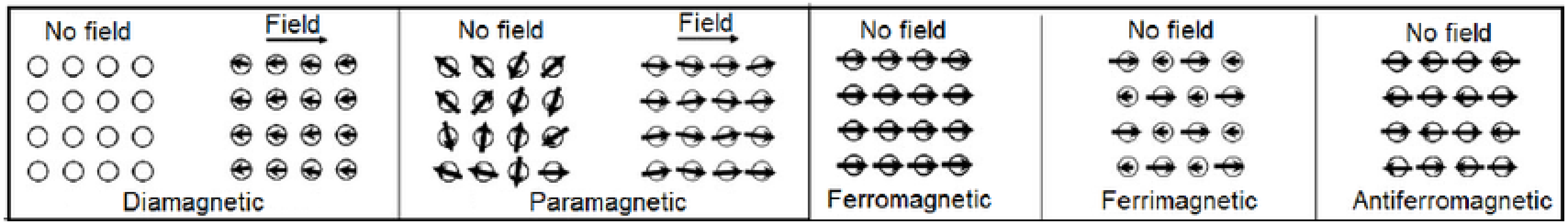
$$\boxed{\rho_{\mathbf{P}}(\mathbf{r}) = -\nabla \cdot \mathbf{P}(\mathbf{r})}$$

$$\boxed{\mathbf{J}_{\mathrm{m}}(\mathbf{r}) = \nabla \times \mathbf{M}}$$

$$\boxed{\sigma_{\mathbf{P}}(\mathbf{r}_{\mathrm{s}}) = \mathbf{P}(\mathbf{r}_{\mathrm{s}}) \cdot \mathbf{e}_{\mathbf{n}}}$$

$$\boxed{\mathbf{J}_{\mathrm{ms}}(\mathbf{r}_{\mathrm{s}}) = \mathbf{M} \times \mathbf{e}_{n}}$$

Permeability

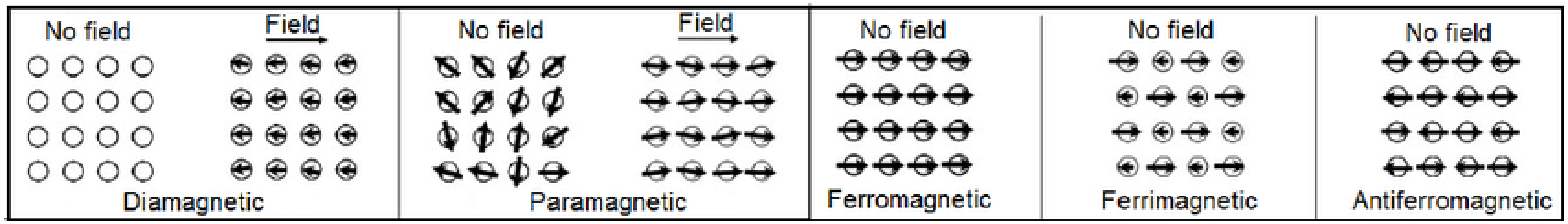


Kolhatkar, Arati G., et al. "Tuning the magnetic properties of nanoparticles." *International journal of molecular sciences* 14.8 (2013): 15977-16009.

$$\frac{1}{\mu_0}\nabla\times\mathbf{B}=\mathbf{J} \quad\Longrightarrow\quad \frac{1}{\mu_0}\nabla\times\mathbf{B}=\mathbf{J}+\mathbf{J}_{\mathrm{m}}$$

$$\frac{1}{\mu_0}\nabla\times\mathbf{B}=\mathbf{J}+\mathbf{J}_{\mathrm{m}}=\mathbf{J}+\nabla\times\mathbf{M}$$

Permeability



Kolhatkar, Arati G., et al. "Tuning the magnetic properties of nanoparticles." *International journal of molecular sciences* 14.8 (2013): 15977-16009.

$$\nabla \times \left(\frac{\mathbf{B}}{\mu_0} - \mathbf{M} \right) = \mathbf{J}$$

$$\mathbf{H} = \frac{\mathbf{B}}{\mu_0} - \mathbf{M}$$

In a simple medium…

$$\mathbf{M} = \chi_{\mathrm{m}} \mathbf{H}, \quad \mathbf{H} = \frac{\mathbf{B}}{\mu_0} - \chi_{\mathrm{m}} \mathbf{H}$$

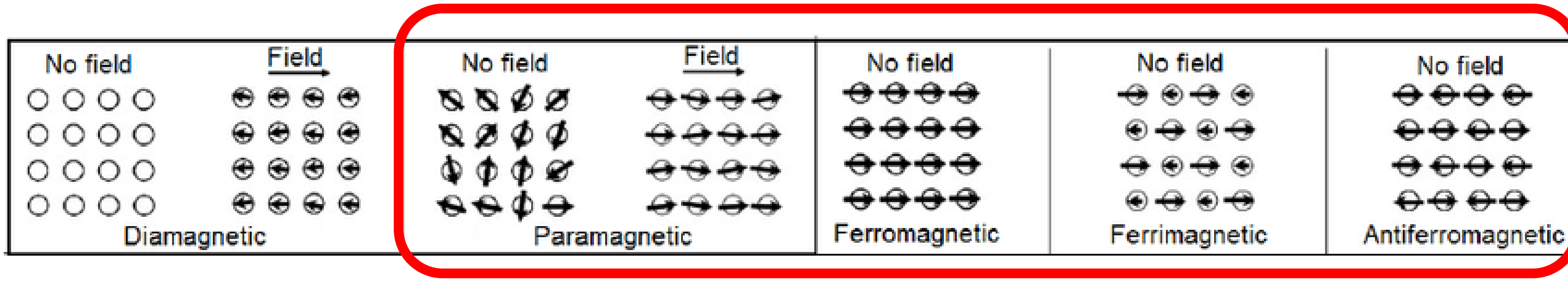
$$\boxed{\mathbf{B} = \mu_0 \left(1 + \chi_{\mathrm{m}}\right) \mathbf{H} = \mu_0 \mu_r \mathbf{H}}$$

Maxwell's Equations for Magnetostatics

$$\boxed{\begin{gathered}\nabla \times \mathbf{H} = \mathbf{J} \\ \nabla \cdot \mathbf{B} = 0 \\ \mathbf{B} = \mu_0 \mu_\mathrm{r} \mathbf{H}\end{gathered}}$$

$$\mu_r = 1 + \chi_\mathrm{m}$$

Classification of Magnetic Materials



$\mu_r \le 1$ $\mu_r \ge 1$ $\mu_r >> 1$ $\mu_r \ge 1$ $\mu_r \ge 1$

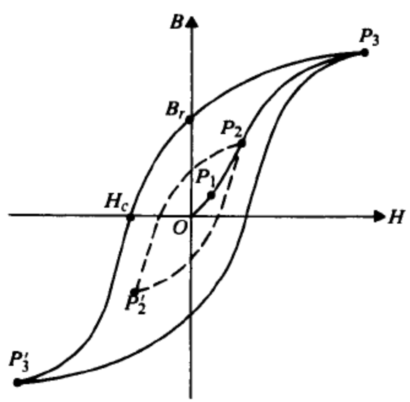
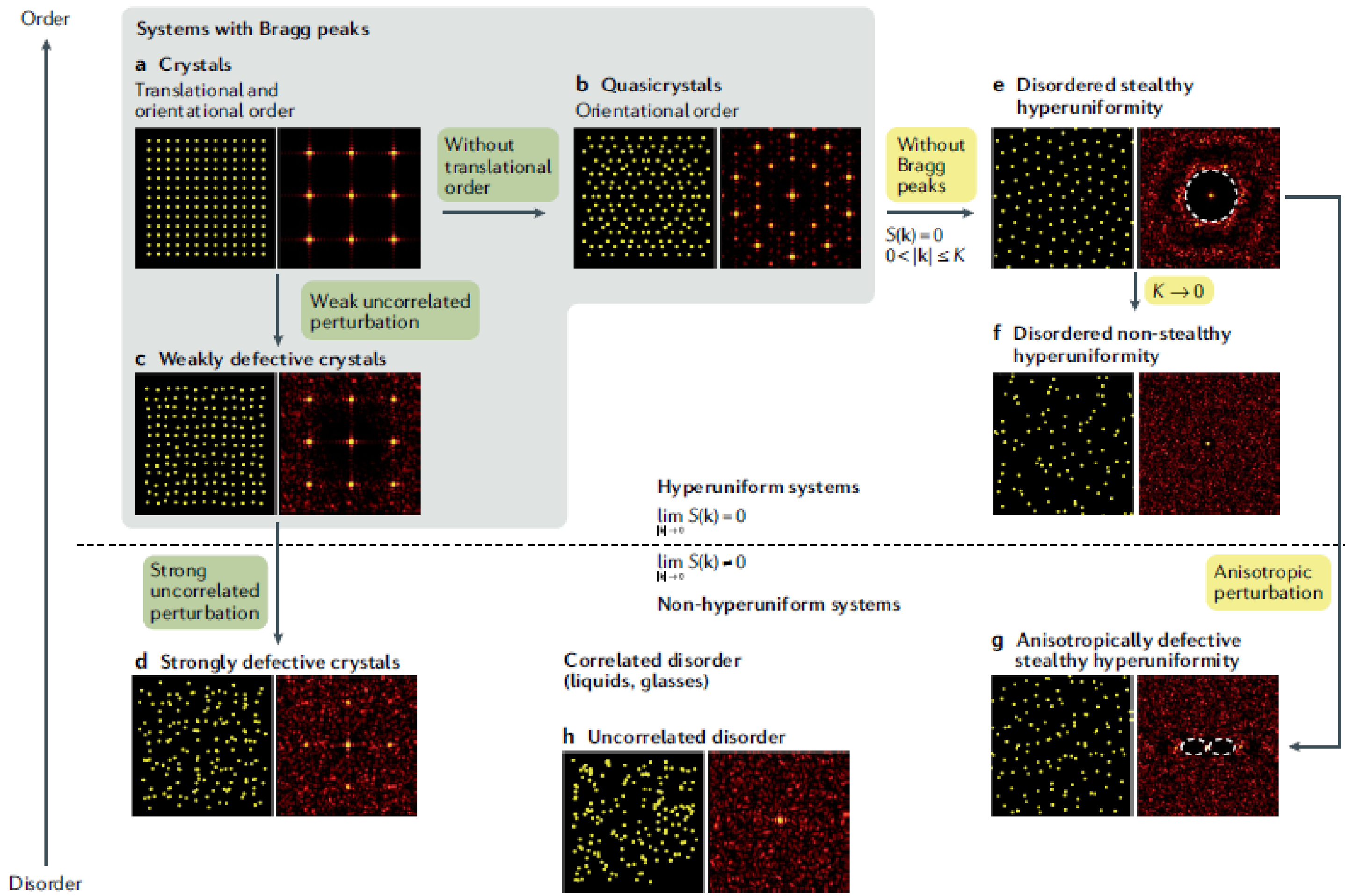


FIGURE 6–17
Hysteresis loops in the B–H plane for ferromagnetic material.

Why Paramagnetic ≠ Ferromagnetic? – Order vs Disorder



S. Yu†, C.-W. Qiu†, Y. Chong, S. Torquato* & N. Park*. Engineered Disorder in Photonics. Nature Reviews Materials 6, 226 (2021)

Paramagnetic VS Ferromagnetic Materials

Vapor, Uncorrelated Disorder	*Liquid, Liquid-like Materials*	*Crystal, Quasi-Crystal, …*
Fully Random (Uncorrelated)	Short-range Order	Short-range Order Long-range Order
	Paramagnetic	Ferromagnetic Ferrimagnetic Anti-Ferromagnetic

Magnetic Circuits

Remind: Electromotive Force (emf)

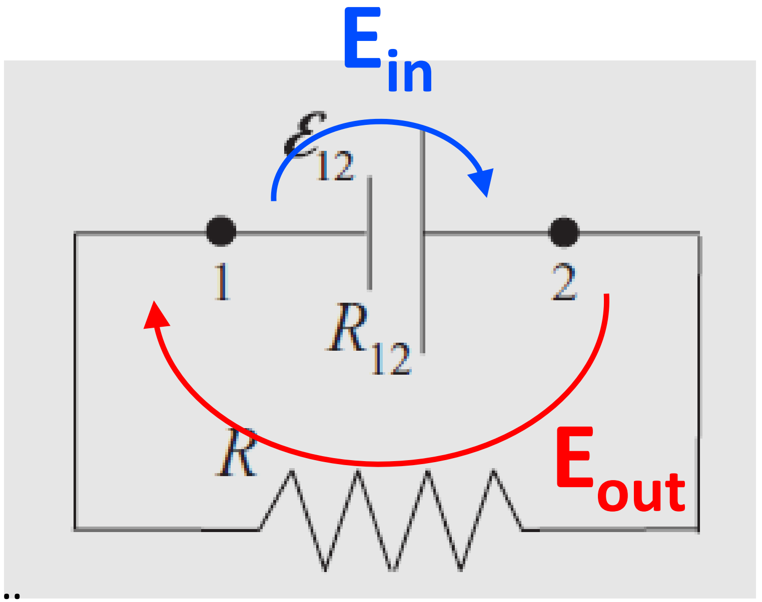
$$\oint_C \mathbf{E}\cdot d\mathbf{l} = \int_1^2 \mathbf{E}_{\text{in}}\cdot d\mathbf{l} + \int_2^1 \mathbf{E}_{\text{out}}\cdot d\mathbf{l} = 0$$

$$-\int_1^2 \mathbf{E}_{\text{in}}\cdot d\mathbf{l} = +\int_2^1 \mathbf{E}_{\text{out}}\cdot d\mathbf{l}$$

Electromotive Force (emf) ➔ Voltage Source

$$\boxed{\gamma = -\int_1^2 \mathbf{E}_{\text{in}}\cdot d\mathbf{l} = +\int_2^1 \mathbf{E}_{\text{out}}\cdot d\mathbf{l}}$$

Allowing potential drop from (+) to (–) electrode



Electric Battery, Electric Generator, Solar Cells, …

Remind: Kirchhoff's Voltage Law

$$\mathbf{J} = \sigma \mathbf{E}$$

$$\gamma = -\int_1^2 \mathbf{E}_{\text{in}} \cdot d\mathbf{l} = +\int_2^1 \mathbf{E}_{\text{out}} \cdot d\mathbf{l} = +\int_2^1 \frac{\mathbf{J}}{\sigma} \cdot d\mathbf{l} = RI$$

Kirchhoff's Voltage Law

$$\boxed{\sum_j \gamma_j = \sum_k R_k I_k}$$

It is just …

$$\oint_C \mathbf{E} \cdot d\mathbf{l} = 0$$

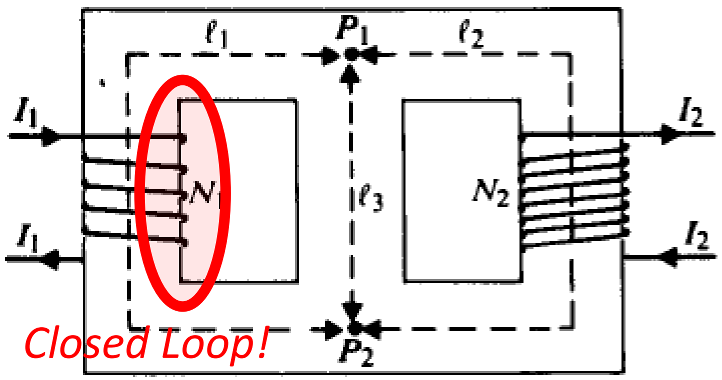
Equation (5–41) is an expression of ***Kirchhoff's voltage law***. It states that, ***around a closed path in an electric circuit, the algebraic sum of the emf's (voltage rises) is equal to the algebraic sum of the voltage drops across the resistances***. It applies to *any closed path* in a network. The direction of tracing the path can be arbitrarily assigned, and the currents in the different resistances need not be the same. Kirchhoff's voltage law is the basis for loop analysis in circuit theory.

Magnetomotive Forces (MMF)

$$\nabla \times \mathbf{H} = \mathbf{J} \quad \Rightarrow \quad \oint_C \mathbf{H} \cdot d\mathbf{l} = \int_S \mathbf{J} \cdot d\mathbf{s} = NI = \Upsilon_{\mathrm{m}}$$

$$\nabla \cdot \mathbf{B} = 0$$

$$\mathbf{B} = \mu_0 \mu_{\mathrm{r}} \mathbf{H}$$



Example 026

EXAMPLE 6–10 Assume that N turns of wire are wound around a toroidal core of a ferromagnetic material with permeability μ. The core has a mean radius r_o, a circular cross section of radius a ($a \ll r_o$), and a narrow air gap of length ℓ_g, as shown in Fig. 6–13. A steady current I_o flows in the wire. Determine (a) the magnetic flux density, $\mathbf{B}_f$, in the ferromagnetic core; (b) the magnetic field intensity, $\mathbf{H}_f$, in the core; and (c) the magnetic field intensity, $\mathbf{H}_g$, in the air gap.

Solution

a) Applying Ampère's circuital law, Eq. (6–84), around the circular contour C in Fig. 6–13, which has a mean radius r_o, we have

$$\oint_C \mathbf{H} \cdot d\boldsymbol{\ell} = NI_o. \tag{6–85}$$

Closed Loop!

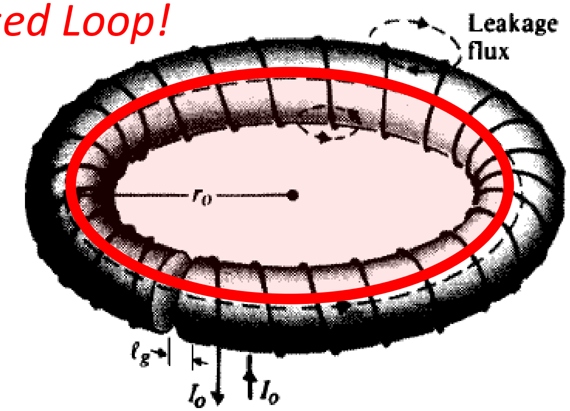


FIGURE 6–13
Coil on ferromagnetic toroid with air gap (Example 6–10).

Example 026

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If flux leakage is neglected, the same total flux will flow in both the ferromagnetic core and in the air gap. If the fringing effect of the flux in the air gap is also neglected, the magnetic flux density $\mathbf{B}$ in both the core and the air gap will also be the same. However, because of the different permeabilities, the magnetic field intensities in both parts will be different. We have

$$\mathbf{B}_f = \mathbf{B}_g = \mathbf{a}_\phi B_f, \tag{6–86}$$

where the subscripts f and g denote ferromagnetic and gap, respectively. In the ferromagnetic core,

$$\mathbf{H}_f = \mathbf{a}_\phi \frac{B_f}{\mu}; \tag{6–87}$$

and, in the air gap,

$$\mathbf{H}_g = \mathbf{a}_\phi \frac{B_f}{\mu_0}. \tag{6–88}$$

Substituting Eqs. (6–87) and (6–88) in Eq. (6–85), we obtain

$$\frac{B_f}{\mu}(2\pi r_o - \ell_g) + \frac{B_f}{\mu_0}\ell_g = NI_o$$

Example 026

EXAMPLE 6–10 Assume that N turns of wire are wound around a toroidal core of a ferromagnetic material with permeability μ. The core has a mean radius r_o, a circular cross section of radius a ($a \ll r_o$), and a narrow air gap of length ℓ_g, as shown in Fig. 6–13. A steady current I_o flows in the wire. Determine (a) the magnetic flux density, $\mathbf{B}_f$, in the ferromagnetic core; (b) the magnetic field intensity, $\mathbf{H}_f$, in the core; and (c) the magnetic field intensity, $\mathbf{H}_g$, in the air gap.

$$\mathbf{B}_f = \mathbf{a}_\phi \frac{\mu_0 \mu N I_o}{\mu_0(2\pi r_o - \ell_g) + \mu \ell_g}.$$

b) From Eqs. (6–87) and (6–89) we get

$$\mathbf{H}_f = \mathbf{a}_\phi \frac{\mu_0 N I_o}{\mu_0(2\pi r_o - \ell_g) + \mu \ell_g}. \qquad (6\text{–}90)$$

c) Similarly, from Eqs. (6–88) and (6–89) we have

$$\mathbf{H}_g = \mathbf{a}_\phi \frac{\mu N I_o}{\mu_0(2\pi r_o - \ell_g) + \mu \ell_g}. \qquad (6\text{–}91)$$

Example 026

EXAMPLE 6–10 Assume that N turns of wire are wound around a toroidal core of a ferromagnetic material with permeability μ. The core has a mean radius r_o, a circular cross section of radius a ($a \ll r_o$), and a narrow air gap of length ℓ_g, as shown in Fig. 6–13. A steady current I_o flows in the wire. Determine (a) the magnetic flux density, $\mathbf{B}_f$, in the ferromagnetic core; (b) the magnetic field intensity, $\mathbf{H}_f$, in the core; and (c) the magnetic field intensity, $\mathbf{H}_g$, in the air gap.

If the radius of the cross section of the core is much smaller than the mean radius of the toroid, the magnetic flux density $\mathbf{B}$ in the core is approximately constant, and the magnetic flux in the circuit is

$$\Phi \cong BS, \tag{6–92}$$

where S is the cross-sectional area of the core. Combination of Eqs. (6–92) and (6–89) yields

$$\Phi = \frac{NI_o}{(2\pi r_o - \ell_g)/\mu S + \ell_g/\mu_0 S}. \tag{6–93}$$

Equation (6–93) can be rewritten as

$$\Phi = \frac{\mathscr{V}_m}{\mathscr{R}_f + \mathscr{R}_g}, \tag{6–94}$$

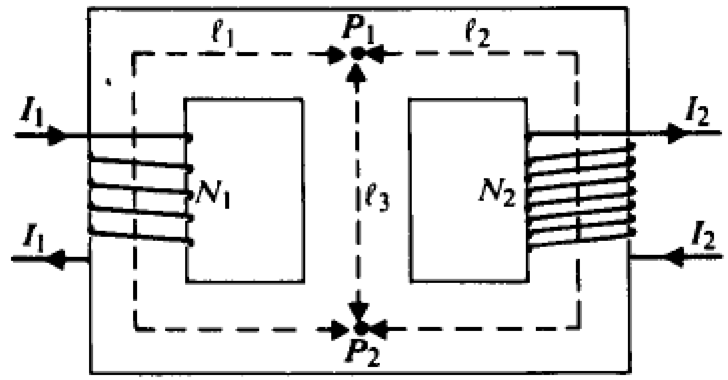
Reluctance

Magnetomotive Forces (MMF)

$$\nabla \times \mathbf{H} = \mathbf{J} \quad \Rightarrow \quad \oint_C \mathbf{H} \cdot d\mathbf{l} = \int_S \mathbf{J} \cdot d\mathbf{s} = NI = \Upsilon_{\mathrm{m}}$$

$$\nabla \cdot \mathbf{B} = 0$$

$$\mathbf{B} = \mu_0 \mu_r \mathbf{H}$$



Magnetic Circuits	Electric Circuits
mmf, $\mathscr{V}_m$ $(= NI)$	emf, $\mathscr{V}$
magnetic flux, Φ	electric current, I
reluctance, $\mathscr{R}$	resistance, R
permeability, μ	conductivity, σ

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Magnetostatics – Boundary Conditions

Remind: Maxwell's Equations: Electrostatics

$$\nabla \times \mathbf{E} = \mathbf{O}$$

$$\nabla \cdot \mathbf{D} = \rho$$

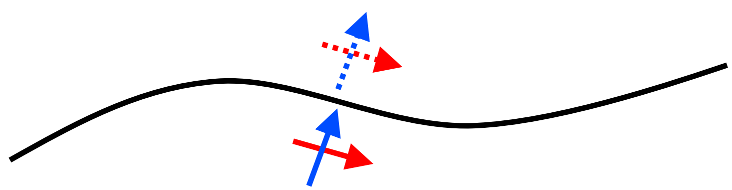
D will be discussed later…

Gauss & Stokes

$$\oint_C \mathbf{E} \cdot d\mathbf{l} = 0$$

$$\oint_S \mathbf{D} \cdot d\mathbf{S} = Q$$

Boundary Conditions: Connecting "Fields" across the boundary



Tangential & *Normal* Fields!

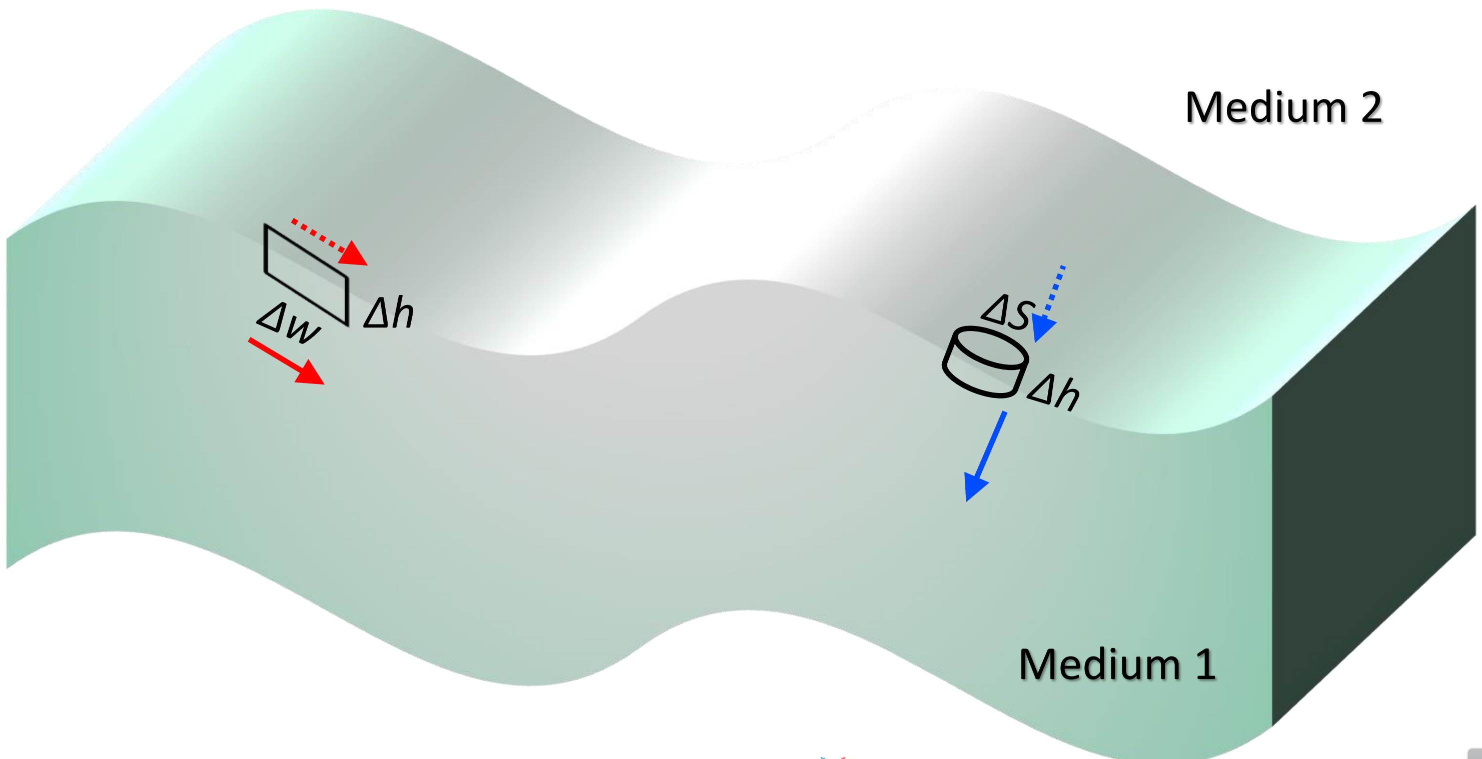
Remind: Strategy for Boundary Conditions

I. Boundary includes "different" materials ➔ Integral forms are proper

II. Stokes ➔ "Closed Loop" across materials
Gauss ➔ "Closed Surface" across materials

$$\oint_C \mathbf{E}\cdot d\mathbf{l} = 0 \qquad \oint_S \mathbf{D}\cdot d\mathbf{S} = Q$$

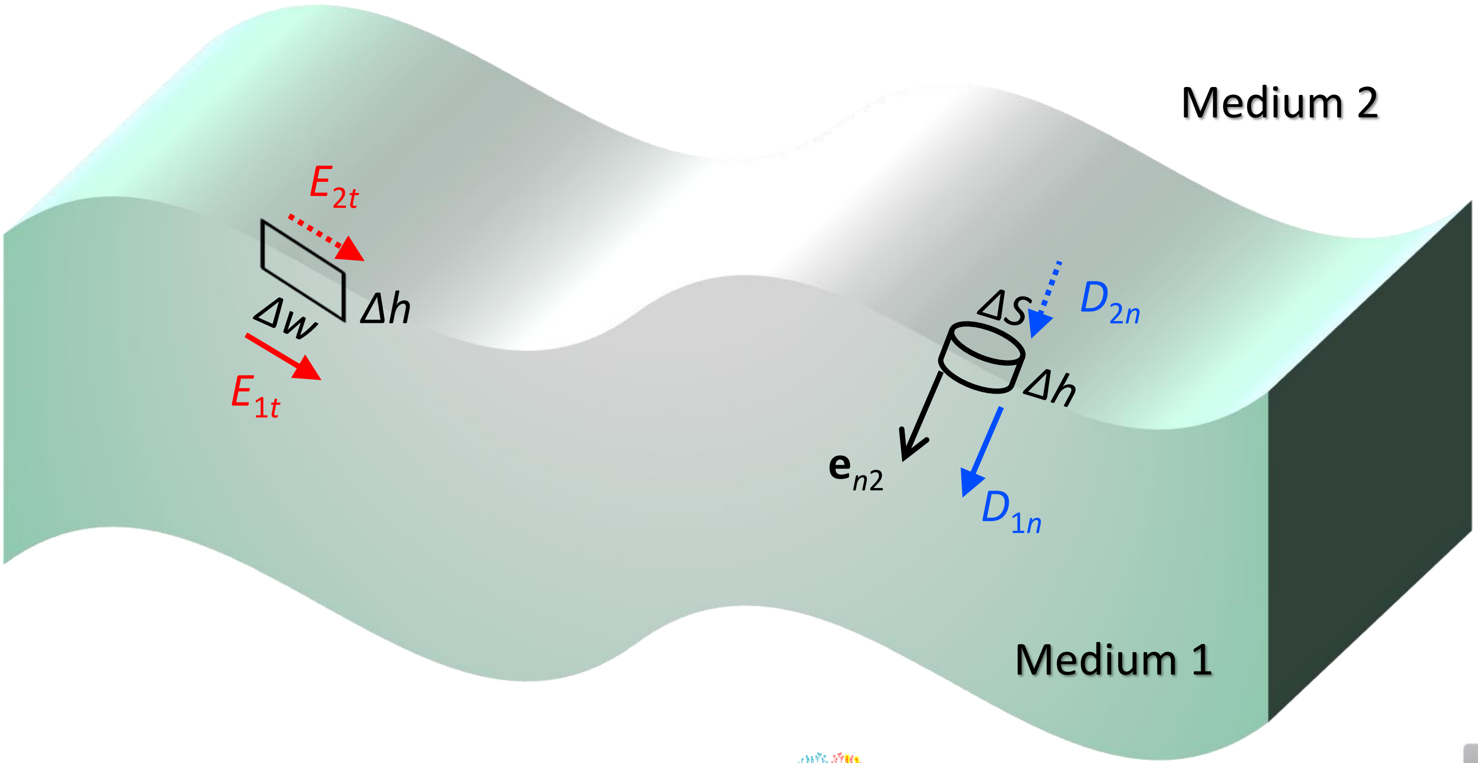
III. Loop measures tangential fields & Surface measures normal fields



Remind: Analyzing Boundary Conditions

$\Delta h \to 0$ to characterize the "boundary"

$$\oint_C \mathbf{E}\cdot d\mathbf{l}\Big|_{\Delta h=0} = E_{1t}\Delta w - E_{2t}\Delta w = 0 \qquad \oint_S \mathbf{D}\cdot d\mathbf{S} = \mathbf{e}_{n2}\cdot\left(\mathbf{D}_1 - \mathbf{D}_2\right)\Delta S = \rho_s \Delta S$$



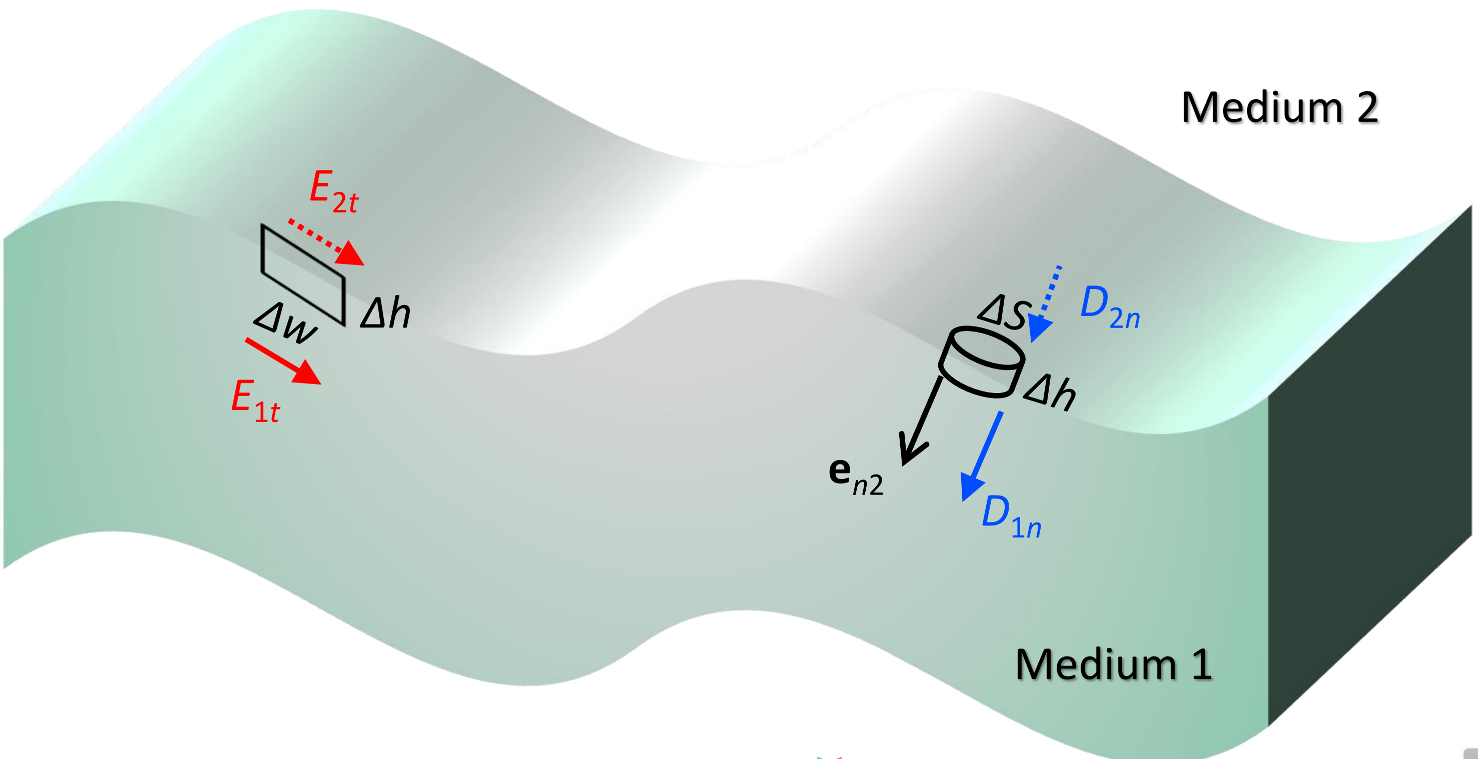
Remind: Boundary Conditions: Electrostatics

Tangential Fields

$$E_{1t} = E_{2t}$$

Normal Fields

$$\mathbf{e}_{n2} \cdot \left(\mathbf{D}_1 - \mathbf{D}_2\right) = \rho_s$$



Remind: Governing Equations for Steady Current Density

Irrotational Electric Field

$$\nabla \times \frac{\mathbf{J}}{\sigma} = \nabla \times \mathbf{E} = \mathbf{O}$$

$$\oint_C \mathbf{E} \cdot d\mathbf{l} = 0$$

$$\oint_C \frac{\mathbf{J}}{\sigma} \cdot d\mathbf{l} = 0$$

Equation of Continuity

$$\nabla \cdot \mathbf{J} = -\frac{d\rho}{dt}$$

$$\oint_S \mathbf{J} \cdot d\mathbf{s} = -\int_V \frac{d\rho}{dt} dv$$

Remind: Boundary Conditions for Current Density

Tangential Fields

$$\frac{J_{1t}}{\sigma_1} = \frac{J_{2t}}{\sigma_2}$$

Normal Fields

$$\mathbf{e}_{n2} \cdot \left(\mathbf{J}_1 - \mathbf{J}_2 \right) = -\frac{d\rho}{dt}$$

If there is no external source…

$$J_{1n} = J_{2n}$$

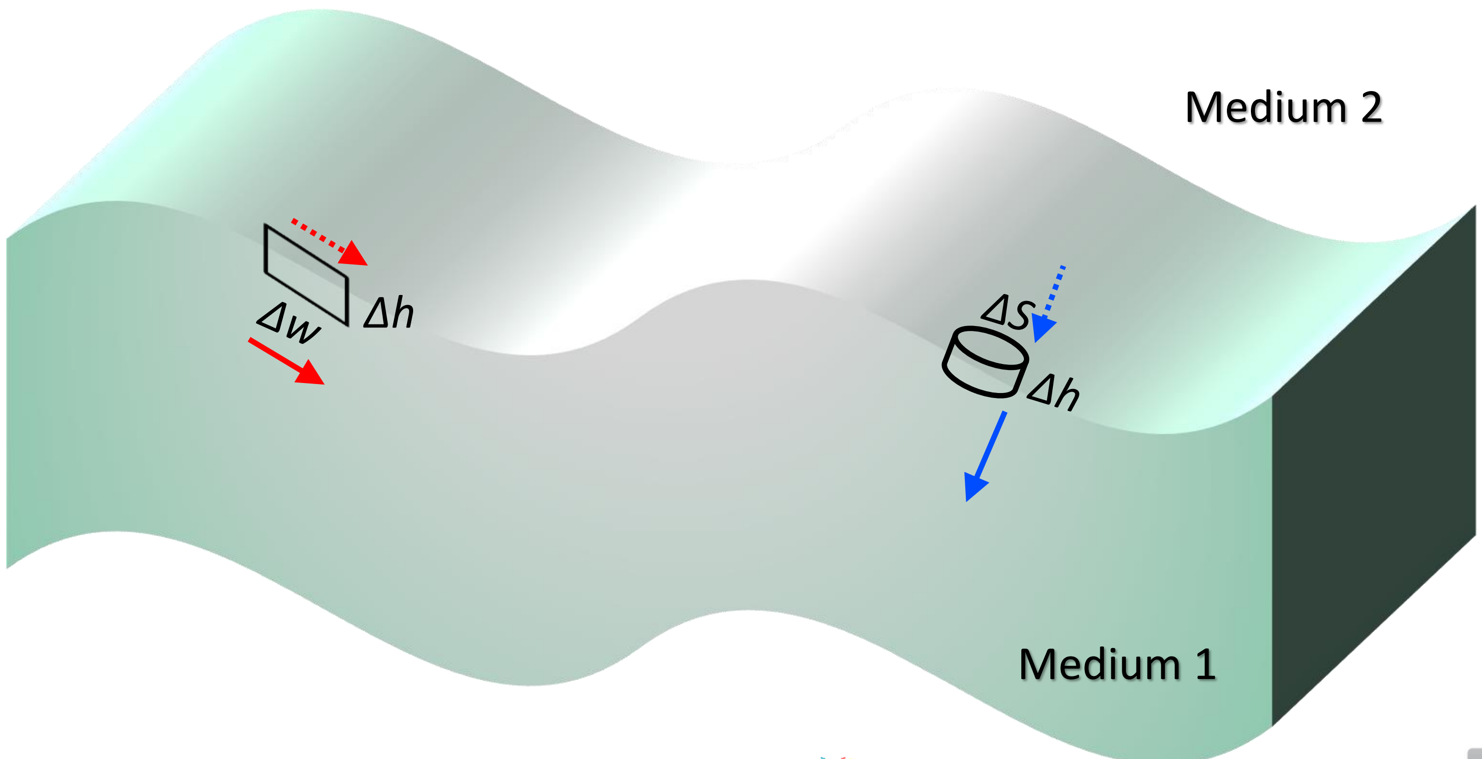
Strategy for Boundary Conditions

I. Boundary includes "different" materials ➔ Integral forms are proper

II. Stokes ➔ "Closed Loop" across materials
Gauss ➔ "Closed Surface" across materials

$$\oint_C \mathbf{H}\cdot d\mathbf{l} = \int_S \mathbf{J}\cdot d\mathbf{s}, \quad \oint_S \mathbf{B}\cdot d\mathbf{S} = 0$$

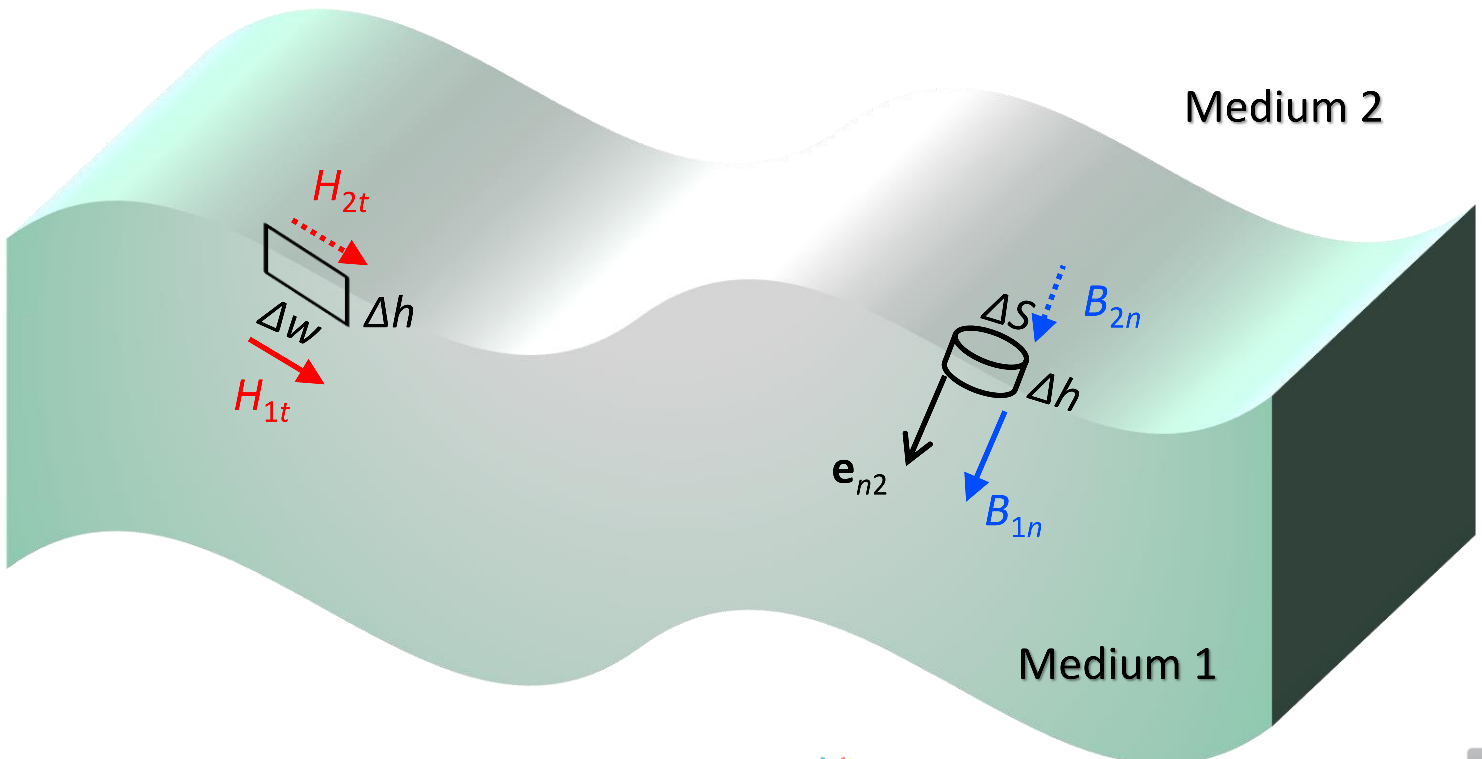
III. Loop measures tangential fields & Surface measures normal fields



Analyzing Boundary Conditions

$\Delta h \rightarrow 0$ to characterize the "boundary"

$$\oint_C \mathbf{H}\cdot d\mathbf{l}\Big|_{\Delta h=0} = H_{1t}\Delta w - H_{2t}\Delta w = J_{\mathrm{sn}}\Delta w, \qquad \oint_S \mathbf{B}\cdot d\mathbf{S} = \mathbf{e}_{n2}\cdot\left(\mathbf{B}_1 - \mathbf{B}_2\right)\Delta S = 0$$



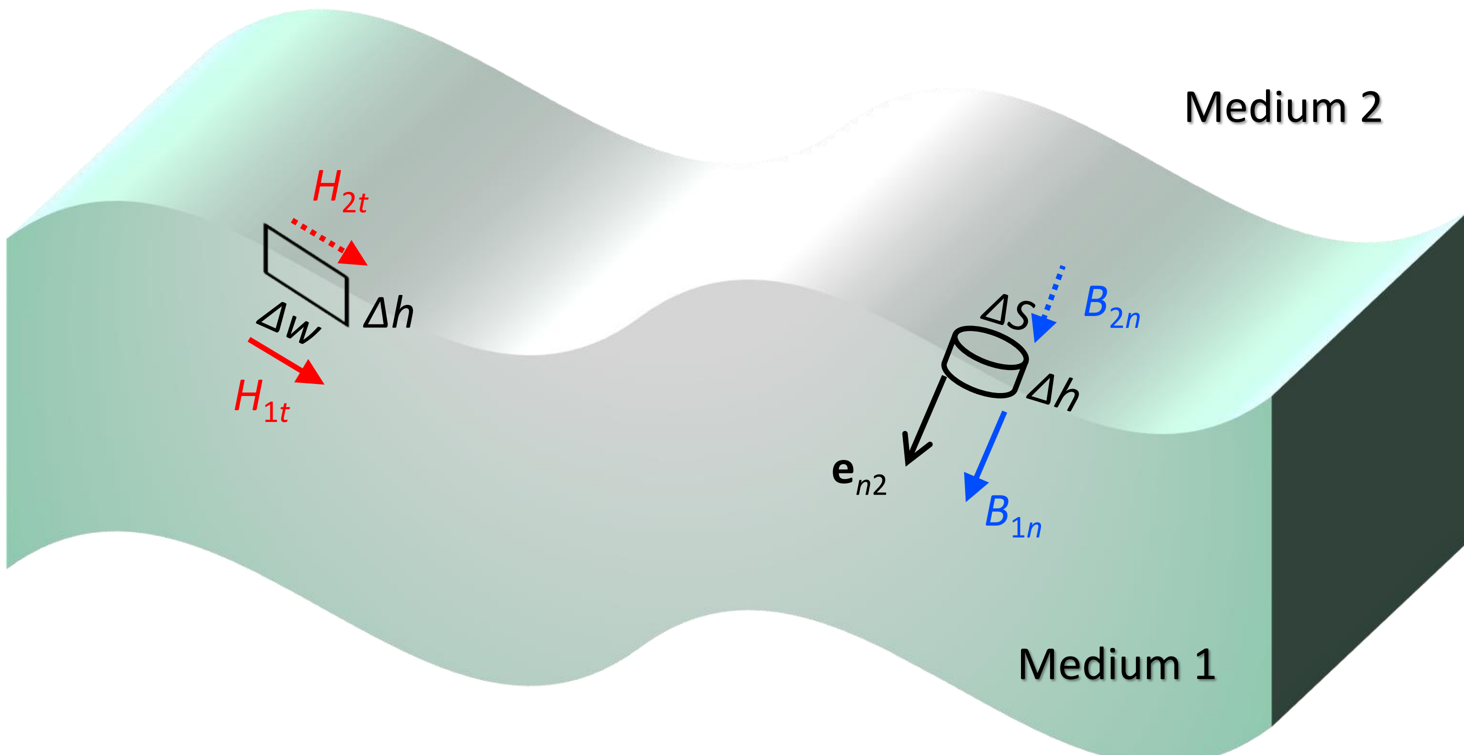
Boundary Conditions: Magnetostatics

Tangential Fields

$$\mathbf{e}_{n2} \times \left(\mathbf{H}_1 - \mathbf{H}_2 \right) = \mathbf{J}_s$$

Normal Fields

$$B_{1n} = B_{2n}$$



Boundary Conditions: Electro-/Magneto-Statics + Current

Tangential Fields

$$E_{1t} = E_{2t}$$

Normal Fields

$$\mathbf{e}_{n2} \cdot (\mathbf{D}_1 - \mathbf{D}_2) = \rho_s$$

Tangential Fields

$$\mathbf{e}_{n2} \times (\mathbf{H}_1 - \mathbf{H}_2) = \mathbf{J}_s$$

Normal Fields

$$B_{1n} = B_{2n}$$

Tangential Fields

$$\frac{J_{1t}}{\sigma_1} = \frac{J_{2t}}{\sigma_2}$$

Normal Fields

$$\mathbf{e}_{n2} \cdot (\mathbf{J}_1 - \mathbf{J}_2) = -\frac{d\rho}{dt}$$

Remember the Relations between Maxwell & B.C.

$$\nabla \times \mathbf{E} = \mathbf{O}$$

$$\nabla \cdot \mathbf{D} = \rho$$

$$\nabla \times \mathbf{H} = \mathbf{J}$$

$$\nabla \cdot \mathbf{B} = 0$$

Tangential Fields

$$E_{1t} = E_{2t}$$

Normal Fields

$$\mathbf{e}_{n2} \cdot \left(\mathbf{D}_1 - \mathbf{D}_2\right) = \rho_s$$

Tangential Fields

$$\mathbf{e}_{n2} \times \left(\mathbf{H}_1 - \mathbf{H}_2\right) = \mathbf{J}_s$$

Normal Fields

$$B_{1n} = B_{2n}$$

Example 027

EXAMPLE 6–12 Two magnetic media with permeabilities μ_1 and μ_2 have a common boundary, as shown in Fig. 6–20. The magnetic field intensity in medium 1 at the

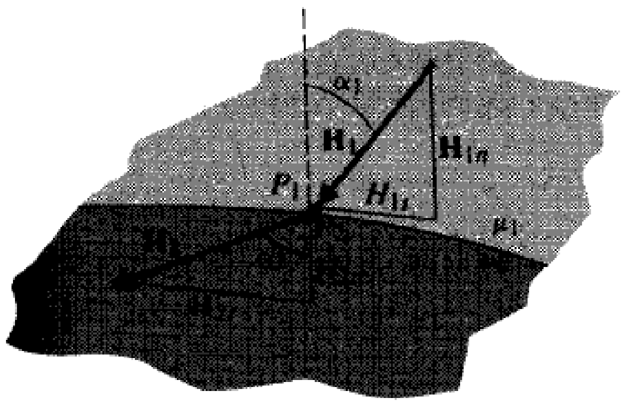


FIGURE 6–20
Boundary conditions for magnetostatic field at an interface (Example 6–12).

point P_1 has a magnitude H_1 and makes an angle α_1 with the normal. Determine the magnitude and the direction of the magnetic field intensity at point P_2 in medium 2.

Solution The desired unknown quantities are H_2 and α_2. Continuity of the normal component of **B** field requires, from Eq. (6–108),

$$\mu_2 H_2 \cos\alpha_2 = \mu_1 H_1 \cos\alpha_1. \tag{6–112}$$

Since neither of the media is a perfect conductor, the tangential component of **H** field is continuous. We have

$$H_2 \sin\alpha_2 = H_1 \sin\alpha_1. \tag{6–113}$$

Example 027

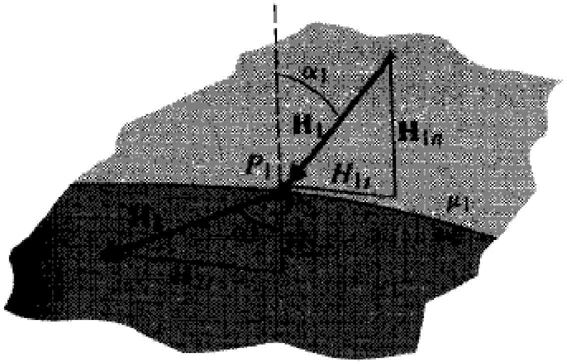


FIGURE 6–20
Boundary conditions for magnetostatic field at an interface (Example 6–12).

Division of Eq. (6–113) by Eq. (6–112) gives

$$\boxed{\frac{\tan \alpha_2}{\tan \alpha_1} = \frac{\mu_2}{\mu_1}} \tag{6–114}$$

or

$$\alpha_2 = \tan^{-1}\left(\frac{\mu_2}{\mu_1}\tan \alpha_1\right), \tag{6–115}$$

which describes the refraction property of the magnetic field. The magnitude of $\mathbf{H}_2$ is

$$H_2 = \sqrt{H_{2t}^2 + H_{2n}^2} = \sqrt{(H_2 \sin \alpha_2)^2 + (H_2 \cos \alpha_2)^2}.$$

From Eqs. (6–112) and (6–113) we obtain

$$\boxed{H_2 = H_1\left[\sin^2 \alpha_1 + \left(\frac{\mu_1}{\mu_2}\cos \alpha_1\right)^2\right]^{1/2}.} \tag{6–116}$$

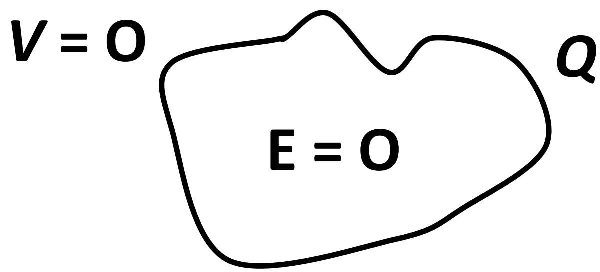
∎

Inductance

Remind: Self-Capacitance

Perfect Conductor

Equipotential Body!



Volume

$$V = \frac{1}{4\pi\varepsilon_0}\int_V \frac{1}{|\mathbf{x}-\mathbf{x}'|}\rho(\mathbf{x}')d^3x'$$

Surface

$$\boxed{V = \frac{1}{4\pi\varepsilon_0}\int_S \frac{1}{|\mathbf{x}-\mathbf{x}'|}\rho_s(\mathbf{x}')d^2x'}$$

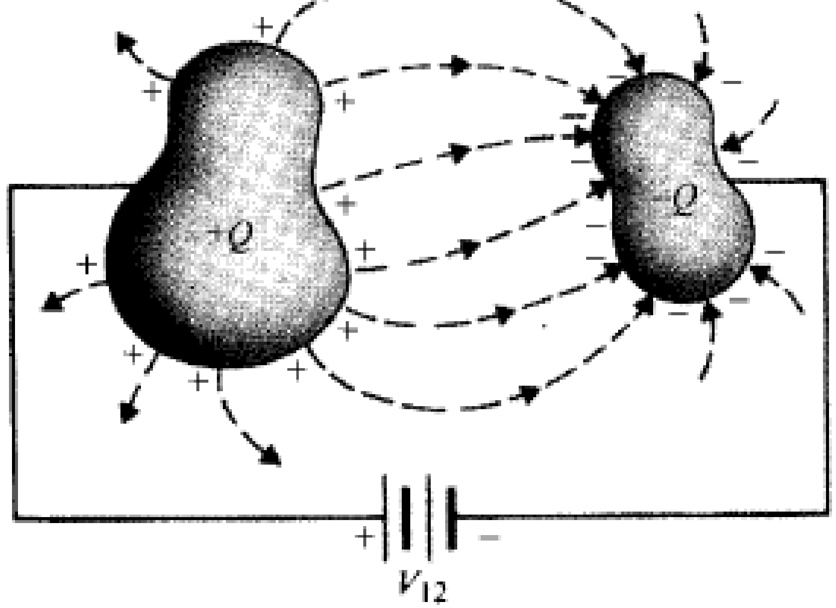
$$\boxed{V_2 - V_1 = -\int_{P_1}^{P_2} \mathbf{E}\cdot d\mathbf{l} = 0}$$

$$Q = \int_S \rho_s(\mathbf{x}')d^2x'$$

$$\rho_s(\mathbf{x}') \to \alpha\rho_s(\mathbf{x}'): \quad Q \to \alpha Q, \quad V \to \alpha V$$

$$\boxed{Q = CV}$$

Remind: Capacitor



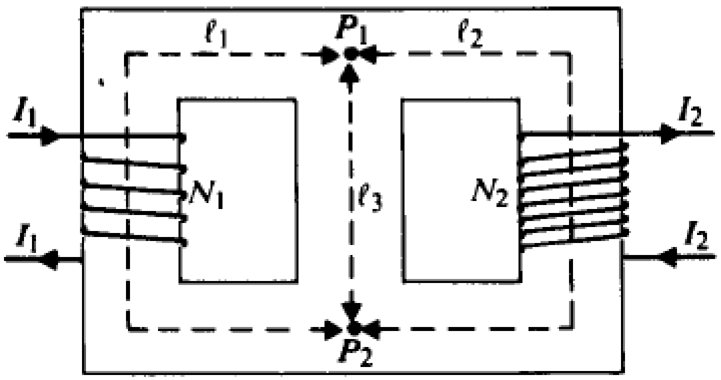
$$\boxed{Q = CV_{12}}$$

FIGURE 3–27
A two-conductor capacitor.

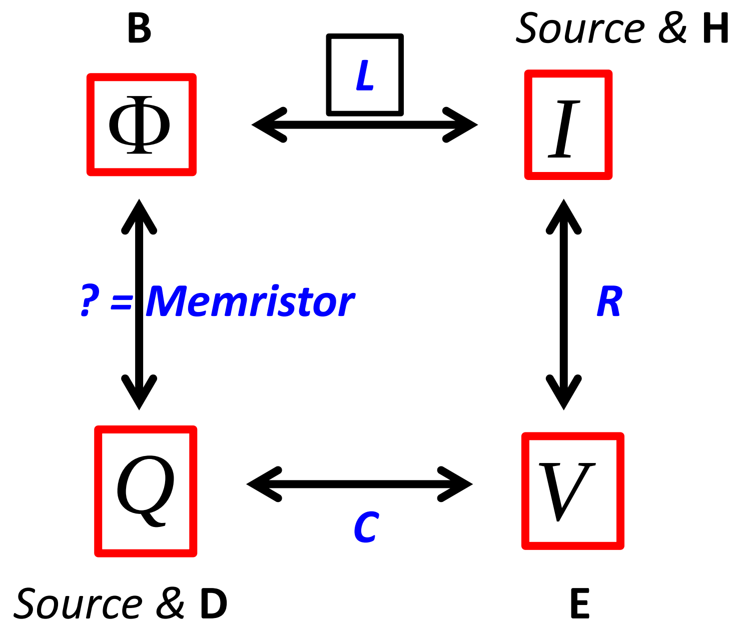
Inductance: Linking Magnetic Flux & Current

$$Q = CV_{12}$$

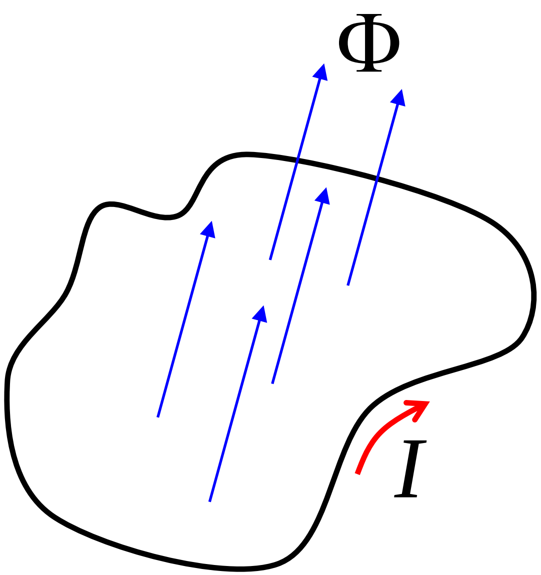
Magnetic Circuit



Magnetic Circuits	Electric Circuits
mmf, $\mathcal{V}_m$ $(=NI)$	emf, $\mathcal{V}$
magnetic flux, Φ	electric current, I
reluctance, $\mathcal{R}$	resistance, R
permeability, μ	conductivity, σ



Self-Inductance



$$\Phi = \int_{S'} \mathbf{B}(\mathbf{x}')d^2\mathbf{s}'$$

$$\nabla \times \mathbf{H} = \mathbf{J}$$

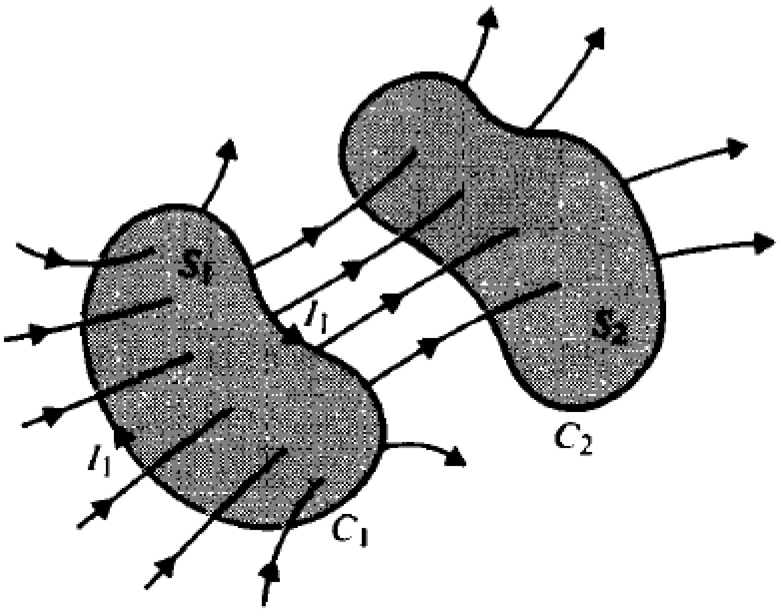
$$I = \int_{S''} \left(\nabla \times \mathbf{H}\right) d^2\mathbf{s}''$$

$$= \int_{S''} \left(\nabla \times \frac{\mathbf{B}}{\mu}\right) d^2\mathbf{s}''$$

$$\mathbf{B}(\mathbf{x}') \rightarrow \alpha\mathbf{B}(\mathbf{x}'): \quad I \rightarrow \alpha I, \quad \Phi \rightarrow \alpha\Phi$$

$$\boxed{\Phi = LI}$$

Mutual Inductance



$$\Phi_{21} = \int_{S_2} \mathbf{B}_1(\mathbf{x}')d^2\mathbf{s}_2'$$

Induced Source

$$\mathbf{B}_1 = \frac{\mu_0 I_1}{4\pi}\oint_{C_1}\frac{d\mathbf{x}_l \times (\mathbf{x}-\mathbf{x}_l)}{|\mathbf{x}-\mathbf{x}_l|^3}$$

$$\Phi_{21} = L_{21}I_1$$

When C_2 has N turns

$$\Lambda_{21} = N\Phi_{21}$$

$$\Lambda_{21} = L_{21}^{\text{Total}}I_1 = NL_{21}I_1$$

$$\boxed{L_{21}^{\text{Total}} = \frac{\Lambda_{21}}{I_1}}$$

Generalization of Circuit Elements: Nonlinear Responses

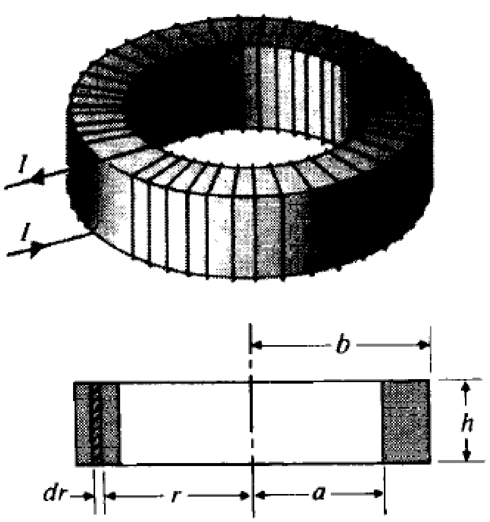
$$\boxed{Q = CV} \qquad \boxed{\Phi = LI}$$

$$dQ = CdV \qquad d\Phi = LdI$$

$$\boxed{C = \frac{dQ}{dV}} \qquad \boxed{L = \frac{d\Phi}{dI}}$$

Example 028

EXAMPLE 6–14 Assume that N turns of wire are tightly wound on a toroidal frame of a rectangular cross section with dimensions as shown in Fig. 6–23. Then, assuming the permeability of the medium to be μ_0, find the self-inductance of the toroidal coil.



Solution It is clear that the cylindrical coordinate system is appropriate for this problem because the toroid is symmetrical about its axis. Assuming a current I in the conducting wire, we find, by applying Eq. (6–10) to a circular path with radius r $(a < r < b)$:

$$\mathbf{B} = \mathbf{a}_\phi B_\phi,$$
$$d\boldsymbol{\ell} = \mathbf{a}_\phi r d\phi,$$
$$\oint_C \mathbf{B} \cdot d\boldsymbol{\ell} = \int_0^{2\pi} B_\phi r \, d\phi = 2\pi r B_\phi.$$

This result is obtained because both B_ϕ and r are constant around the circular path C. Since the path encircles a total current NI, we have

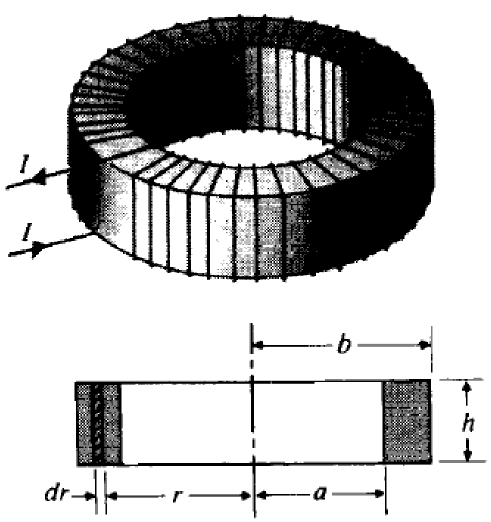
$$2\pi r B_\phi = \mu_0 N I$$

and

$$\boxed{B_\phi = \frac{\mu_0 N I}{2\pi r}.}$$

Example 028

EXAMPLE 6–14 Assume that N turns of wire are tightly wound on a toroidal frame of a rectangular cross section with dimensions as shown in Fig. 6–23. Then, assuming the permeability of the medium to be μ_0, find the self-inductance of the toroidal coil.



Next we find

$$\Phi = \int_S \mathbf{B} \cdot d\mathbf{s} = \int_S \left(\mathbf{a}_\phi \frac{\mu_0 NI}{2\pi r}\right) \cdot (\mathbf{a}_\phi h\, dr)$$

$$= \frac{\mu_0 NIh}{2\pi} \int_a^b \frac{dr}{r} = \frac{\mu_0 NIh}{2\pi} \ln \frac{b}{a}.$$

The flux linkage Λ is $N\Phi$ or

$$\Lambda = \frac{\mu_0 N^2 Ih}{2\pi} \ln \frac{b}{a}.$$

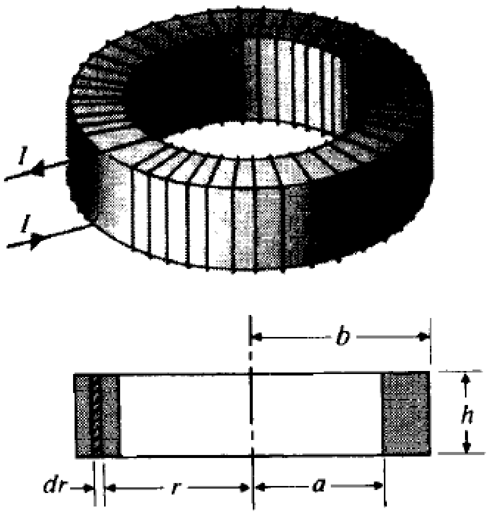
Finally, we obtain

$$L = \frac{\Lambda}{I} = \boxed{\frac{\mu_0 N^2 h}{2\pi} \ln \frac{b}{a}} \quad \text{(H)}. \tag{6–132}$$

We note that the self-inductance is not a function of I (for a constant medium permeability). The qualification that the coil be closely wound on the toroid is to minimize the linkage flux around the individual turns of the wire. ▬

Example 028

EXAMPLE 6–14 Assume that N turns of wire are tightly wound on a toroidal frame of a rectangular cross section with dimensions as shown in Fig. 6–23. Then, assuming the permeability of the medium to be μ_0, find the self-inductance of the toroidal coil.



$$L = \frac{\Lambda}{I} = \boxed{\frac{\mu_0 N^2 h}{2\pi} \ln \frac{b}{a}}$$

Material Parameter + Structural Parameters

Example 029

Prove the reciprocity:

$$\boxed{L_{21} = L_{12}}$$

We may vaguely and intuitively expect that the answer is in the affirmative "because of reciprocity." But how do we prove it? We may proceed as follows. Combining Eqs. (6–123), (6–125) and (6–127), we obtain

$$L_{12} = \frac{N_2}{I_1} \int_{S_2} \mathbf{B}_1 \cdot d\mathbf{s}_2. \tag{6–147}$$

But in view of Eq. (6–15), $\mathbf{B}_1$ can be written as the curl of a vector magnetic potential $\mathbf{A}_1$, $\mathbf{B}_1 = \nabla \times \mathbf{A}_1$. We have

$$\begin{aligned} L_{12} &= \frac{N_2}{I_1} \int_{S_2} (\nabla \times \mathbf{A}_1) \cdot d\mathbf{s}_2 \\ &= \frac{N_2}{I_1} \oint_{C_2} \mathbf{A}_1 \cdot d\boldsymbol{\ell}_2. \end{aligned} \tag{6–148}$$

Now, from Eq. (6–27),

$$\mathbf{A}_1 = \frac{\mu_0 N_1 I_1}{4\pi} \oint_{C_1} \frac{d\boldsymbol{\ell}_1}{R}. \tag{6–149}$$

In Eqs. (6–148) and (6–149) the contour integrals are evaluated only *once* over the periphery of the *loops* C_2 and C_1, respectively—the effects of multiple turns having been taken care of separately by the factors N_2 and N_1. Substitution of Eq. (6–149) in Eq. (6–148) yields

$$L_{12} = \frac{\mu_0 N_1 N_2}{4\pi} \oint_{C_1} \oint_{C_2} \frac{d\boldsymbol{\ell}_1 \cdot d\boldsymbol{\ell}_2}{R}, \tag{6–150a}$$

where R is the distance between the differential lengths $d\boldsymbol{\ell}_1$ and $d\boldsymbol{\ell}_2$. It is customary to write Eq. (6–150a) as

$$\boxed{L_{12} = \frac{\mu_0}{4\pi} \oint_{C_1} \oint_{C_2} \frac{d\boldsymbol{\ell}_1 \cdot d\boldsymbol{\ell}_2}{R} \quad \text{(H)},} \tag{6–150b}$$

Magnetic Energy

Remind: Electrostatic Energy: Materials

$$\boxed{U_E = \frac{1}{2}\int \mathbf{D}\cdot\mathbf{E}d^3\mathbf{x}}$$

Homogeneous (position-independent) Materials

$$\mathbf{D} = \varepsilon_0\varepsilon_r\mathbf{E}$$

$$\boxed{U_E = \frac{1}{2}\varepsilon_0\varepsilon_r\int|\mathbf{E}|^2\, d^3\mathbf{x}}$$

In the vacuum

$$\boxed{U_E = \frac{1}{2}\varepsilon_0\int|\mathbf{E}|^2\, d^3\mathbf{x} \geq 0}$$

Remind: Electrostatic Energy: Materials

$$\boxed{U_E = \frac{1}{2}\int \mathbf{D}\cdot\mathbf{E}\,d^3\mathbf{x}}$$

Anisotropic Materials

$$\mathbf{D} = \varepsilon_0 \boldsymbol{\varepsilon}_r \mathbf{E} = \varepsilon_0 \begin{bmatrix} \varepsilon_{xx} & \varepsilon_{xy} & \varepsilon_{xz} \\ \varepsilon_{yx} & \varepsilon_{yy} & \varepsilon_{yz} \\ \varepsilon_{zx} & \varepsilon_{zy} & \varepsilon_{zz} \end{bmatrix} \mathbf{E}$$

For Example:

$$\mathbf{D} = \varepsilon_0 \boldsymbol{\varepsilon}_r \mathbf{E} = \varepsilon_0 \begin{bmatrix} \varepsilon_x & 0 & 0 \\ 0 & \varepsilon_y & 0 \\ 0 & 0 & \varepsilon_z \end{bmatrix} \mathbf{E}$$

$$\boxed{U_E = \frac{1}{2}\varepsilon_0 \int \left(\varepsilon_x E_x^{\,2} + \varepsilon_y E_y^{\,2} + \varepsilon_z E_z^{\,2}\right) d^3\mathbf{x}}$$

Magnetostatic Energy: Materials

$$\boxed{U_E = \frac{1}{2}\int \mathbf{D}\cdot\mathbf{E}\,d^3\mathbf{x}}$$

$$\boxed{U_H = \frac{1}{2}\int \mathbf{B}\cdot\mathbf{H}\,d^3\mathbf{x}}$$

$$\mathbf{D} = \varepsilon_0\varepsilon_r\mathbf{E}$$

$$\mathbf{B} = \mu_0\mu_r\mathbf{H}$$

$$\boxed{U_E = \frac{1}{2}\varepsilon_0\varepsilon_r\int |\mathbf{E}|^2\,d^3\mathbf{x}}$$

$$\boxed{U_H = \frac{1}{2}\mu_0\mu_r\int |\mathbf{H}|^2\,d^3\mathbf{x}}$$

In the vacuum

$$\boxed{U_E = \frac{1}{2}\varepsilon_0\int |\mathbf{E}|^2\,d^3\mathbf{x} \geq 0}$$

In the vacuum

$$\boxed{U_H = \frac{1}{2}\mu_0\int |\mathbf{H}|^2\,d^3\mathbf{x} \geq 0}$$

Magnetostatic Energy: Materials

$$\boxed{U_H = \frac{1}{2}\int \mathbf{B}\cdot\mathbf{H}\, d^3\mathbf{x}}$$

Anisotropic Materials $\mathbf{B} = \mu_0 \boldsymbol{\mu}_r \mathbf{H} = \mu_0 \begin{bmatrix} \mu_{xx} & \mu_{xy} & \mu_{xz} \\ \mu_{yx} & \mu_{yy} & \mu_{yz} \\ \mu_{zx} & \mu_{zy} & \mu_{zz} \end{bmatrix} \mathbf{H}$

For Example: $\mathbf{B} = \mu_0 \boldsymbol{\mu}_r \mathbf{H} = \mu_0 \begin{bmatrix} \mu_x & 0 & 0 \\ 0 & \mu_y & 0 \\ 0 & 0 & \mu_z \end{bmatrix} \mathbf{H}$

$$\boxed{U_H = \frac{1}{2}\mu_0 \int \left(\mu_x {H_x}^2 + \mu_y {H_y}^2 + \mu_z {H_z}^2 \right) d^3\mathbf{x}}$$