457.646 Topics in Structural Reliability

In-Class Material: Class 23

◎ Latin Hypercube Sampling (Mckay et al. 1979)

Extension of "Latin Square" – appearing exactly once in each row and exactly once in each column)

(←) 7x7 Latin Square stained glass honoring R.A. Fisher's work on DOE

Evenly distribute sampling points to promote early convergence

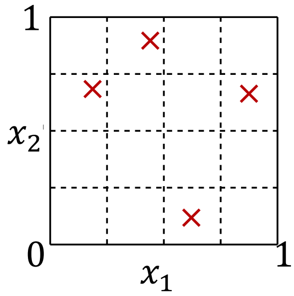
e.g. $\mathbf{X} = \{X_1, X_2\}$ uniform (0,1), s.i

⇒ 4 samples

- Brute force MCS:

  Samples are generated independently

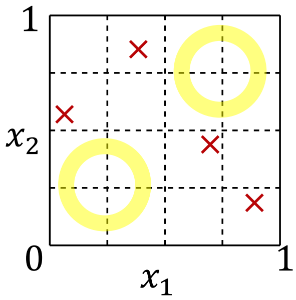
  No memory



- Latin Hypercube Sampling:

  There is only one sample in each row and column

  (w/ memory)



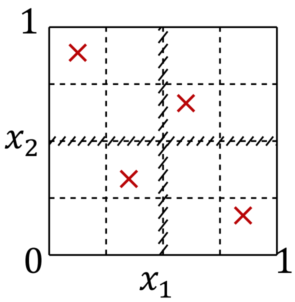
Possible LHS combination?

$$\prod_{n=0}^{M-1}(M-n)^{N-1} \quad \begin{array}{l} M = 4,\ N = 2 \\ \therefore 24 \text{ cases} \end{array}$$

- Orthogonal Sampling:

  LHS + subspace sampled

  w/ same frequency



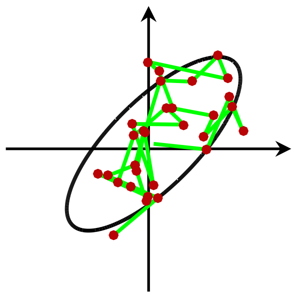
choose LHS combination that satisfy orthogonal sampling conditions

Example: Y.S. Kim et al. (2009)

→ Seismic Performance Assessment of Interdependent Lifeline Systems

⇒ Generated random samples of post-disaster conditions of network components using LHS

◎ Markov Chain Monte Carlo (MCMC) Simulation



→ MCS method generating random samples as a Markov chain according to transitional probabilities $p(\mathbf{z}^{(m+1)}|\mathbf{z}^{(m)})$

→ Good for high-dimensional problems

① Metropolis-Hastings algorithm

Generate a random sample using the proposal distribution $q(\mathbf{z}|\mathbf{z}^{(\tau)})$ and then accept or reject with the pre-determined probability.

<u>Metropolis algorithm</u> (Metropolis et al. 1953)

Works for the symmetric proposal distribution, i.e. $q(\mathbf{z}_A|\mathbf{z}_B) = q(\mathbf{z}_B|\mathbf{z}_A)$

The candidate sample $\boldsymbol{z}^*$ proposed by $q(\mathbf{z}|\mathbf{z}^{(\tau)})$ is accepted with the probability

$$A(\boldsymbol{z}^*, \boldsymbol{z}^{(\tau)}) = \min\left(1, \frac{\tilde{p}(\boldsymbol{z}^*)}{\tilde{p}(\boldsymbol{z}^{(\tau)})}\right)$$

<u>Metropolis-Hasting algorithm</u> (Hastings, 1970)

Works even if the proposal distribution is not symmetric → generalized version

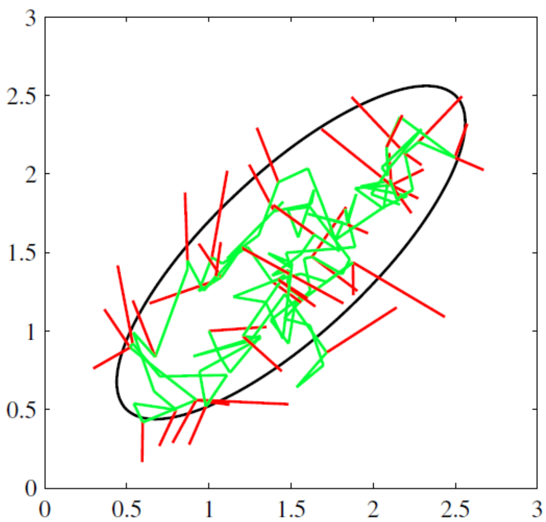
The candidate sample $\boldsymbol{z}^*$ proposed by $q(\mathbf{z}|\mathbf{z}^{(\tau)})$ is accepted with the probability

$$A(\boldsymbol{z}^*, \boldsymbol{z}^{(\tau)}) = \min\left(1, \frac{\tilde{p}(\boldsymbol{z}^*)q(\boldsymbol{z}^{(\tau)}|\boldsymbol{z}^*)}{\tilde{p}(\boldsymbol{z}^{(\tau)})q(\boldsymbol{z}^*|\boldsymbol{z}^{(\tau)})}\right)$$

Note: The evaluation of the acceptance criterion does NOT require knowledge of the normalizing constant Z_p in the probability distribution $p(\mathbf{z}) = \tilde{p}(\mathbf{z})/Z_p$ → Good for Bayesian updating $f(\boldsymbol{\theta}) = cL(\boldsymbol{\theta})p(\boldsymbol{\theta})$

Example: Generating samples of a bi-variate Gaussian distribution using Metropolis algorithm (Bishop 2006)

A simple illustration using Metropolis algorithm to sample from a Gaussian distribution whose one standard-deviation contour is shown by the ellipse. The proposal distribution is an isotropic Gaussian distribution whose standard deviation is 0.2. Steps that are accepted are shown as green lines, and rejected steps are shown in red. A total of 150 candidate samples are generated, of which 43 are rejected.



② Gibbs sampling (Geman & Geman 1984)

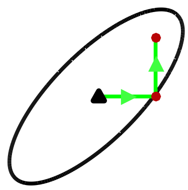
Sample "one" element each time based on conditional distribution given the outcomes of the other elements

e.g. $P(Z_1, Z_2, Z_3)$

sample $Z_1^{\tau+1}$ by $P(Z_1 | Z_2^{\tau}, Z_3^{\tau})$

$Z_2^{\tau+1}$ by $P(Z_2 | Z_1^{\tau}, Z_3^{\tau})$

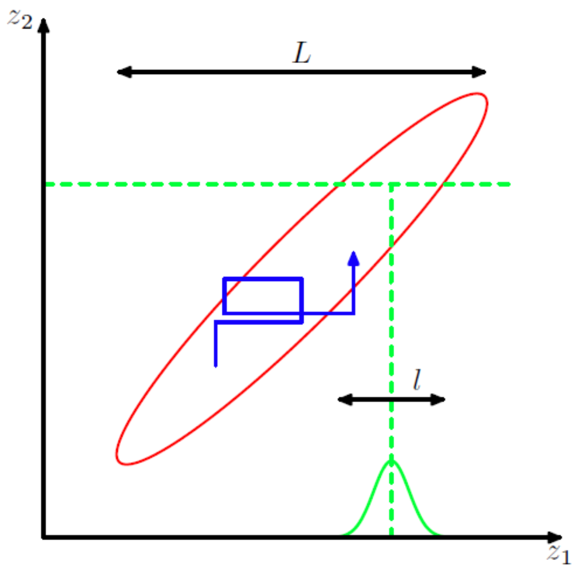
$Z_3^{\tau+1}$ by $P(Z_3 | Z_1^{\tau}, Z_2^{\tau})$



<u>Can show this is a special case of Metropolis-Hasting algorithm</u>

Example: Generating samples of a bi-variate Gaussian distribution using Gibbs sampling

Illustration of Gibbs sampling by alternate updates of two variables whose distribution is a correlated Gaussian. The step size is governed by the standard deviation of the conditional distribution (green curve), and is $O(l)$, leading to slow progress in the direction of elongation of the joint distribution (red ellipse). The number of steps needed to obtain an independent sample from the distribution is $O((L/l)^2)$.



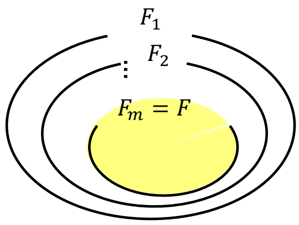
◎ Subset Simulation (Au & Beck, 2001)

$F_1 \supset F_2 \supset \cdots \supset F_m = F$ event of interest

$P(F) = P(F_m)$ too low

e.g. $F_i = \{D > C_i\}$

$C_1 < C_2 < \cdots < C_m = C$

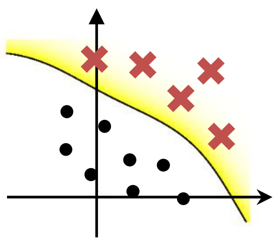


$$P(F) = P(F_m) = P(\bigcap_{i=1}^{m} F_i)$$

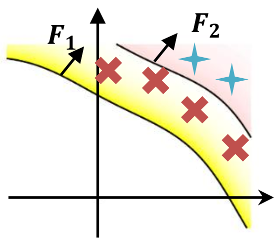
$$= P(F_m | \bigcap_{i=1}^{m-1} F_i) \cdot P(\bigcap_{i=1}^{m-1} F_i)$$

$$= P(F_m | f_{m-1}) \cdot P(\bigcap_{i=1}^{m-1} F_i)$$

$$= \underbrace{P(F_1) \times P(F_2 | F_1)}_{\text{Each larger than } P(F)} \times \cdots \times P(F_m | F_1 \cdots F_{m-1})$$



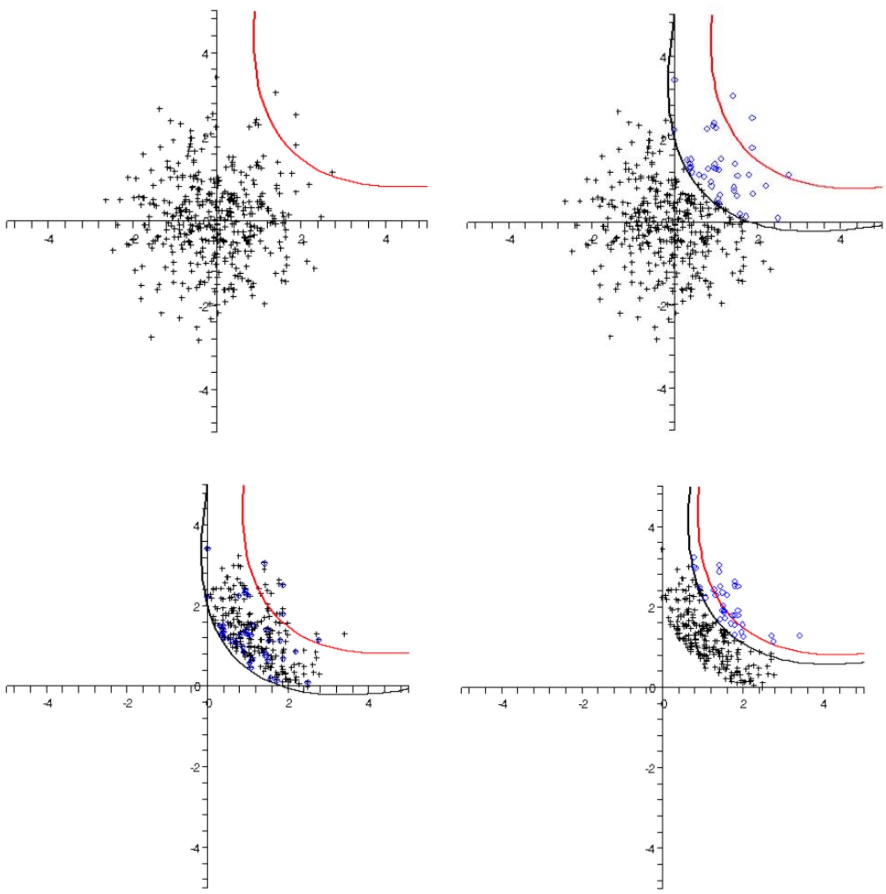
$P(F_1)$



$P(F_2|F_1)$

Identify intermediate failure domains adaptively for a given probability $p_0 = P(F_{i+1}|F_i)$

(figure credit: Dr. Iason Papaioannou)



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Example: Hamiltonian Monte Carlo methods for Subset Simulation (Wang, Broccardo and Song, 2019)

The location of the next sample is determined by a moving particle described by Hamiltonian mechanics → more efficient

$$H(\boldsymbol{q}, \boldsymbol{p}) = V(\boldsymbol{q}) + K(\boldsymbol{p})$$

The potential energy $V(\boldsymbol{q})$ is defined by the probability density function while the kinetic energy $K(\boldsymbol{p})$ represents the proposal distribution

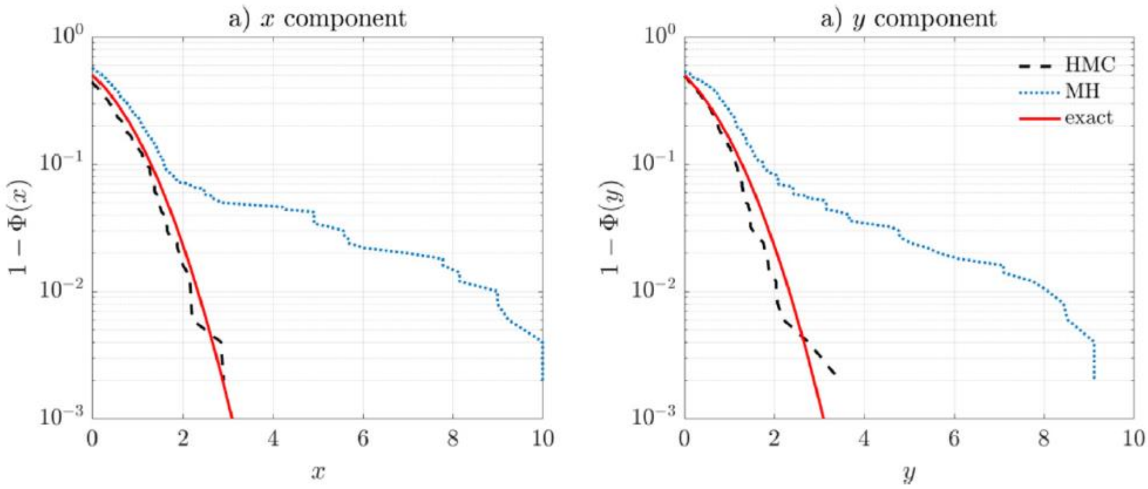

Fig. 3. Marginal complementary CDFs obtained from CW-MH and HMC.

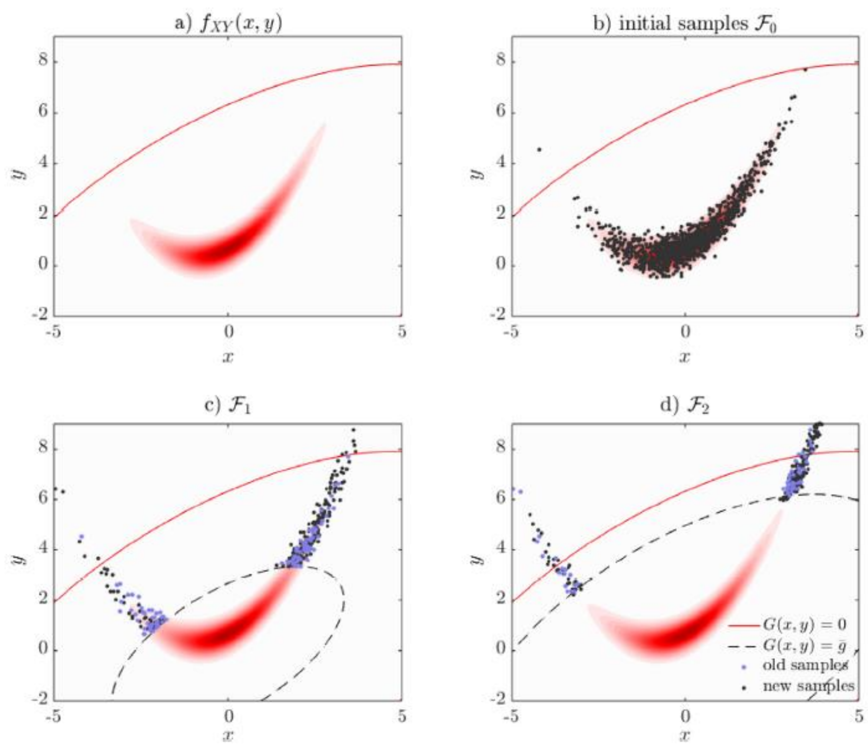

Fig. 11. a) The elliptical limit-state function in the space of the banana-shaped distribution; b) initial sampling; c) first subset; d) second subset.

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◎ Extrapolation-based MCS (Naess et al. 2009)

$g(\lambda) = g - \mu_g(1-\lambda)$ $\qquad$ $\lambda = 0:$ $\quad g(\lambda) = g - \mu_g$ $\quad P_f \approx 0.5$

$0 \le \lambda \le 1$ $\qquad$ $\lambda = 1:$ $\quad g(\lambda) = g$ $\quad P_f \approx 1.0$

Generate samples $\{g_1, \cdots, g_n\}$ and use to estimate

$\tilde{P}_f(\lambda) = \dfrac{N_f(\lambda)}{N}$ while varying λ

Fitted to $\underset{\lambda\to 1}{\cong} q(\lambda)\cdot\exp\{-a(\lambda-b)^c\}$ (can assume constant q), i.e.

$$\tilde{P}_f(\lambda) \underset{\lambda\to 1}{\cong} q^*\cdot\exp\{-a^*(\lambda-b^*)^{c^*}\}$$

Find a, b, c, q by fitting and extrapolate as $\tilde{P}_f(\lambda)$ as $\lambda \to 1$

⇒ Has been applied to component/system (Naess et al. 2009)

and large-size system problems (Naess et al. 2010)

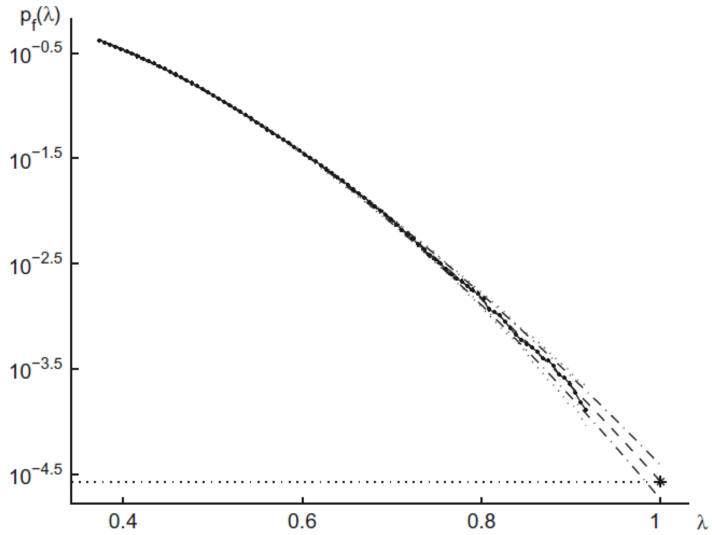

Fig. 9. Plot of $\log \hat{p}_f(\lambda_j)$ for Example 4: Monte Carlo (·); fitted optimal curve (– –); reanchored empirical confidence band (···); fitted confidence band (–·). $\log q = -0.303$, $a = 16.231$, $b = 0.252$, $c = 1.591$.

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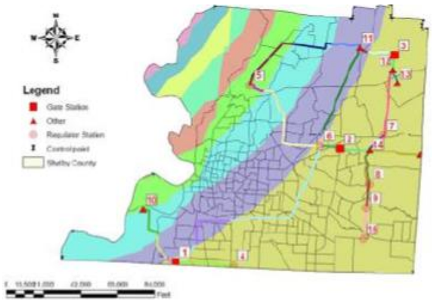
In-Class Material: Class 24

VII. Random fields

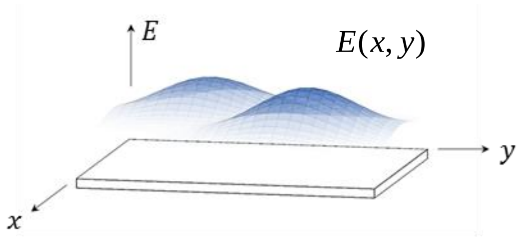
~ Random quantity distributed over ______________ field (space or time)

Ex1) Spatial Distribution of Random Ground Motion Intensity)



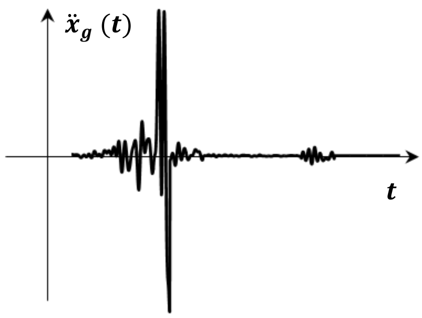
(Song & Ok, 2010)

Ex2) Spatial distribution of material property (Young's Modulus)



Ex3) Ground acceleration time history $\ddot{x}_g(t)$

e.g.



random process, stochastic process

⇒ ( ) # of random variables

⇒ ( ) representation is required

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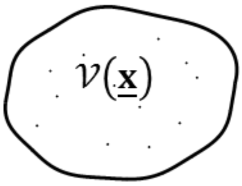
◎ Discretization of Random field → Random vector



→ $\mathbf{v} = \{v_1, \cdots, v_n\}^T$

Random n-vector

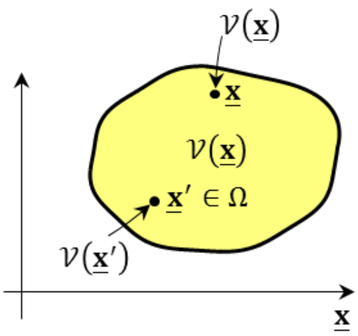
$$
? \begin{cases}
\mathbf{M_v} = E[\mathbf{v}] = \{\mu_{v_i}\} \\
\mathbf{\Sigma_{vv}} = E[(\mathbf{v}-\mathbf{M}_v)(\mathbf{v}-\mathbf{M}_v)^T] \\
\quad = \mathbf{D_v R_{vv} D_v} \qquad \text{covariance matrix} \\
\text{where} \quad D_v = \text{diag}[\sigma_{v_i}] \\
\qquad\qquad R_{vv} = [\rho_{v_i v_j}] \\
f_\mathbf{v}(\mathbf{v}) \rightarrow \text{joint PDF of } \mathbf{v}
\end{cases}
$$

◎ Theoretical Representation of R.F

$v(\mathbf{x}),\ \mathbf{x} \in \Omega$ random field in domain Ω

Partial descriptors:

$$
\begin{cases}
\mu(\mathbf{x}) : \text{mean function} \quad E[v(\mathbf{x})] \\
\sigma^2(\mathbf{x}) : \text{variance function} \quad E[v^2(\mathbf{x})] - \mu^2(x) \\
\rho(\mathbf{x}, \mathbf{x}') : \text{correlation coefficient function} \quad \rho_{v(\mathbf{x})v(\mathbf{x}')}
\end{cases}
$$



For Gaussian R.F. the above gives a complete specification

For Nataf R.F., also specify $F_v(v; \mathbf{x})$

For general RF's, specify joint PDF of (   ) and (   )

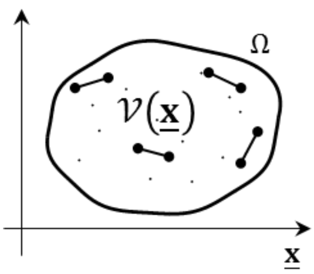
for, $x, x' \in \Omega$, $f_{vv}(v(x), v(x'))$

e.g. _____________ Random field

~ _____________ does not change over the domain Ω

$v(\mathbf{x})$, $x \in \Omega$

$$
\begin{cases}
\mu(\mathbf{x}) = \\
\sigma^2(\mathbf{x}) = \\
\rho(\mathbf{x}, \mathbf{x}') = \\
F(v; \mathbf{x}) =
\end{cases}
$$



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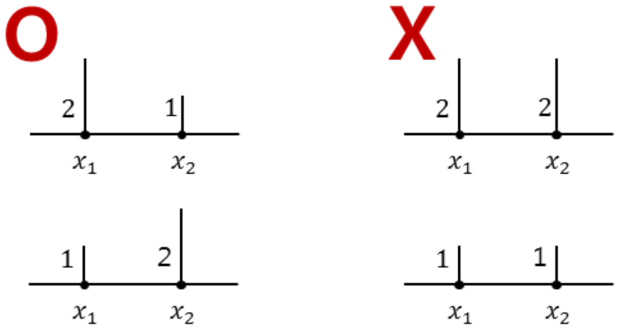
Note; This doesn't mean $v(\mathbf{x}) = v$ (not constant over the domain)

Scenario 1

Scenario 2

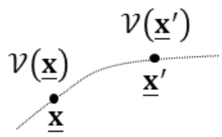


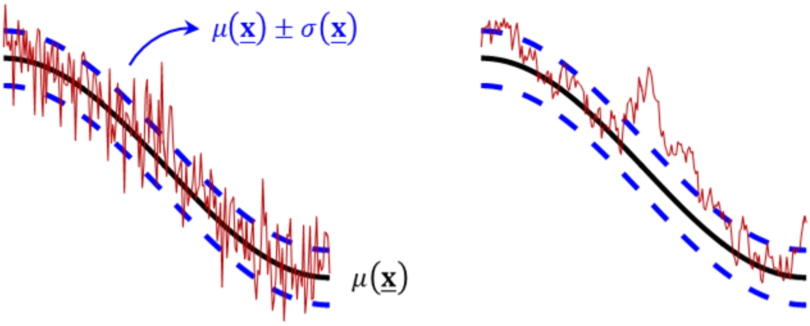
$\mu_{x_1} = 1.5 \ \mu_{x_2} = 1.5$

$\mu(\mathbf{x}) = \mu = 1.5$

$v(\mathbf{x}) = v$

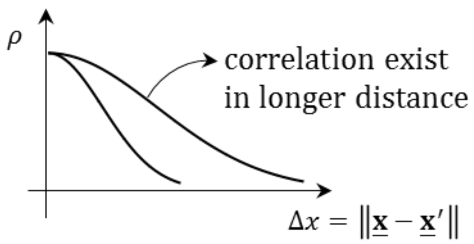
Q: Correlation Function $\rho(\mathbf{x}, \mathbf{x}')$ ←meaning?



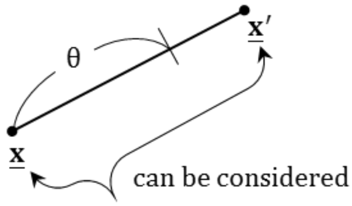


How to capture this from $\rho(\mathbf{x}, \mathbf{x}')$?

◎ Correlation length



$$\theta = \int_0^\infty \rho(\Delta x) dx$$



~ measure of the distance over which significant loss of correlation occurs

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Examples

- $\rho(\Delta x) = \exp\left(-\dfrac{\Delta x}{a}\right)$

$$\theta = \int_0^\infty \exp\left(-\frac{\Delta x}{a}\right) d\Delta x$$
$$= -a\exp\left(-\frac{\Delta x}{a}\right)\Big|_0^\infty = a$$

- $\rho(\Delta x) = \exp\left(-\dfrac{\Delta x^2}{a^2}\right)$

$$\theta = \int_0^\infty \exp\left(-\frac{\Delta x^2}{a^2}\right) d\Delta x$$
$$= \frac{1}{2}\int_{-\infty}^\infty \exp\left(-\frac{\Delta x^2}{a^2}\right) d\Delta x$$
$$= \frac{1}{2}\sqrt{\pi}a \qquad\qquad \theta \propto a$$

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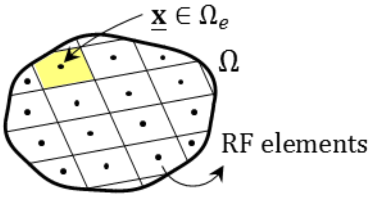
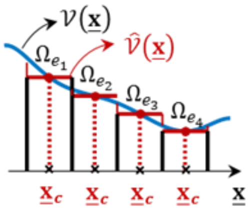
◎ Discrete Representation of RFs (Summary: Sudret & ADK 2000; 2002 PEM)

① Mid-point method

$$v(\mathbf{x}) \simeq \hat{v}(\mathbf{x})$$
$$= v(\mathbf{x}_c),\ \mathbf{x} \in \Omega_e$$

(constant in each Ω_e)





- Represented by a constant r.v.

  over each RF element

- Positive definiteness problem of $\mathbf{R}$ … if RF element size is small relative to θ

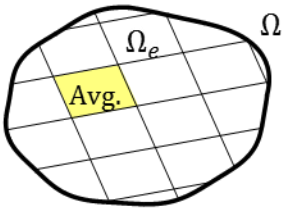
  Recommended size of RF element size

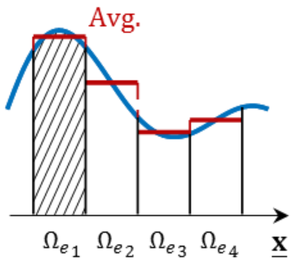
$$\frac{\theta}{10} \sim \frac{\theta}{15} \leq \text{ RF size } \leq \frac{\theta}{3} \sim \frac{\theta}{5}$$

Numerical stability (Positive definiteness)   Accurate representation

② Spatial averaging method

$$\hat{v}(\mathbf{x}) = \frac{\int_{\Omega_e} v(\mathbf{x}) d\Omega}{\int_{\Omega_e} d\Omega}, \quad \mathbf{x} \in \Omega_e$$





- Represented by a <u>single</u> r.v per Ω_e
- Variances are (   ) → _____-estimate P_f
- Positive definiteness problem

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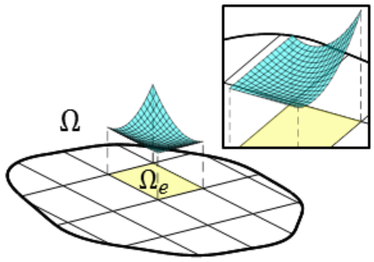
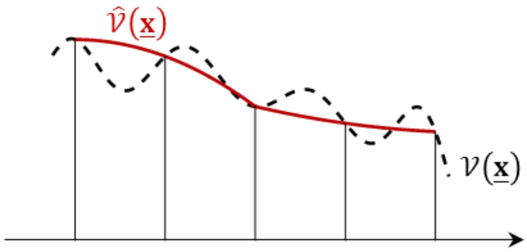
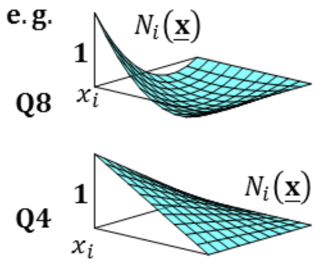
③ Shape function method (←motivated by FE people)

$$v(\mathbf{x}) \simeq \hat{v}(\mathbf{x}) = \sum_{\substack{\text{element}\\ \text{nodes}}} N_i(\mathbf{x}) v(\mathbf{x}_i)$$



- Represented by continuous function





$$N_i(\mathbf{x}_j) = \delta_{ij}$$

to guarantee $\hat{v}(\mathbf{x}_i) = v(\mathbf{x}_i)$

④ Karhunen-Loève (KL) expansion (Gaussian RFs)

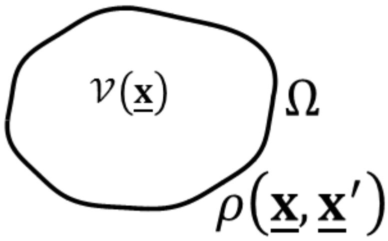
→ Describe RF in terms of finite # of shape functions

defined over ________ domain

(no geometric discretization)

→ Discretization based on

_____________ structure $\rho(\mathbf{x},\mathbf{x}')$



Goal: Want to descrive $\rho(\mathbf{x},\mathbf{x}')$ by

$$\rho(\mathbf{x},\mathbf{x}') = \sum_{i=1}^{\infty} \lambda_i \varphi_i(\mathbf{x}) \varphi_i(\mathbf{x}')$$

Orthogonal shape (base) functions

Can find λ, φ by solving an integral eigenvalue problem, i.e.

$$\int_{\Omega} \rho(\mathbf{x},\mathbf{x}')\varphi_i(\mathbf{x}')d\mathbf{x}' = \lambda_i \varphi_i(\mathbf{x})$$ (Fredholem integral eqn – 2nd kind)

Note $\rho(\mathbf{x},\mathbf{x}')$ is bounded, symmetric, (+) definite.

If so, one can find

$\varphi_i(\mathbf{x})$: orthogonal $\int \varphi_i(\mathbf{x})\varphi_j(\mathbf{x})d\mathbf{x} = \delta_{ij}$

λ_i: real & positive

Can drop λ_i's if $\lambda_r \cong 0$

Then using $\varphi_i(\mathbf{x})$, and λ_i, $i=1,\ldots,r$, one can describe Gaussian RF $v(x)$ by

KL expansion of Gaussian RF

$$\boxed{v(\mathbf{x}) \simeq \hat{v}(\mathbf{x}) = \mu(\mathbf{x}) + \sigma(\mathbf{x})\sum_{i=1}^{r}(u_i\sqrt{\lambda_i}\varphi_i(\mathbf{x}))\ , \quad x \in \Omega} \Rightarrow \ v(\mathbf{x}) \ \Rightarrow \ \{u_1,\cdots,u_r\}$$

$u_i \to N(0,1),\ u_i$ s.i

Let's check!

i. Gaussian? Yes, function of u_i's

ii. $E[\hat{v}(\mathbf{x})] = \mu(\mathbf{x})$? $E[\hat{v}(\mathbf{x})] =$

iii. $Var[\hat{v}(\mathbf{x})] = E[(\qquad)^2]$

$$= E[\qquad \sum_{i=1}^{r}\sum_{j=1}^{r} \qquad]$$

$$= \sigma^2(\mathbf{x})\sum_{i=1}^{r}\sum_{j=1}^{r} \qquad \sqrt{\lambda_i}\sqrt{\lambda_j}\varphi_i(\mathbf{x})\varphi_j(\mathbf{x})$$

$$= \sigma^2(\mathbf{x})\sum_{i=1}^{r}\lambda_i\varphi_i^2(\mathbf{x})$$

$$= \sigma^2(\mathbf{x})$$

(because $\rho(\mathbf{x},\mathbf{x}) =$ $\qquad = \qquad$)

iv. $\rho_{\hat{v}\hat{v}}(\mathbf{x},\mathbf{x}') \overset{?}{=} \rho(\mathbf{x},\mathbf{x}')$

$$= E[(\hat{v}(\mathbf{x}) - \mu(\mathbf{x}))(\hat{v}(\mathbf{x}') - \mu(\mathbf{x}'))] / \sigma(\mathbf{x})\sigma(\mathbf{x}')$$

$$= E[\sum_{i=1}^{r}\sum_{j=}^{r} u_i\sqrt{\lambda_i}\varphi_i(\mathbf{x})u_j\sqrt{\lambda_j}\varphi_j(\mathbf{x}')]$$

$$= \sum_{i=1}^{r}\sum_{j=1}^{r} E[\qquad]\sqrt{\lambda_i}\sqrt{\lambda_j}\varphi_i(\mathbf{x})\varphi_j(\mathbf{x}')$$

$$= \sum_{i=1}^{r}\lambda_i\varphi_i(\mathbf{x})\varphi_i(\mathbf{x}')$$

$$= \varphi(\mathbf{x},\mathbf{x}')$$

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- # of RV's:
- Represented by function
- No necessary
- Most efficient (in terms of # of)
- Requires solution of an integral eigenvalue problem.

⑤ Orthogonal expansion (eigen-expansion, but correlated rv's)

⑥ Optimal linear estimation (OLE)~ linear regression

⑦ Expansion OLE

⋮ See Sudret & ADK (2000)

◎ Nataf RF

$v(\mathbf{x}) \Rightarrow F(v,\mathbf{x})$, $\rho_{ZZ}(\mathbf{x},\mathbf{x}')$

$v(\mathbf{x}) = F_v^{-1}\{\Phi(\hat{Z}(\mathbf{x}))\}$, $Z(\mathbf{x}) \sim N(\mathbf{0}, \rho_{ZZ}(\mathbf{x},\mathbf{x}'))$ ($Z(\mathbf{x}) \rightarrow$ Gaussian RF)

⇒ Construct $Z(\mathbf{x})$ and discrete to $\hat{Z}(\mathbf{x})$

⇒ $v(\mathbf{x}) = F^{-1}\{\Phi(\hat{Z}(\mathbf{x}))\}$