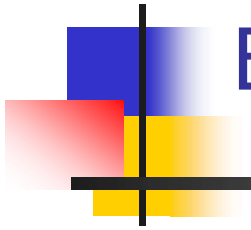
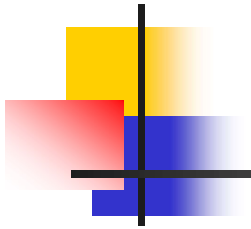


# Bellman-Ford Algorithm



# Bellman-Ford Algorithm

```
BELLMAN-FORD(G, w, s)
1.      INITIALIZE-SINGLE-SOURCE(G,s)
2.      for i=1 to |G.V|-1
3.          for each edge (u,v) ∈ G.E
4.              RELAX(u, v, w)
5.      for each edge (u,v) ∈ G.E
6.          if  v.d > u.d + w(u,v)
7.              return FALSE
8.      return TRUE
```

# Bellman-Ford Algorithm

```
BELLMAN-FORD(G, w, s)
1.      INITIALIZE-SINGLE-SOURCE(G,s)
2.      for i=1 to |G.V|-1
3.          for each edge (u,v) ∈ G.E
4.              RELAX(u, v, w)
5.      for each edge (u,v) ∈ G.E
6.          if  v.d > u.d + w(u,v)
7.              return FALSE
8.      return TRUE
```

Relaxation: Make |V|-1 passes, relaxing each edge

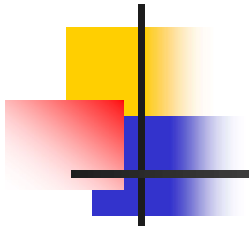
Test whether negative -weight cycle exists

- **Edge weights may be negative**

# Bellman-Ford Algorithm

```
BELLMAN-FORD(G, w, s)
1.      INITIALIZE-SINGLE-SOURCE(G,s)        ← O(|V|)
2.      for i=1 to |G.V|-1                   ← O(|V||E|)
3.          for each edge (u,v) ∈ G.E
4.              RELAX(u, v, w)
5.      for each edge (u,v) ∈ G.E            ← O(|E|)
6.          if  v.d > u.d + w(u,v)
7.              return FALSE
8.      return TRUE
```

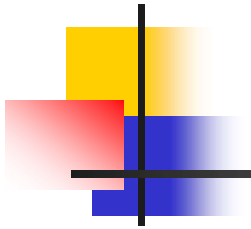
- **Running time O(|V||E|)**



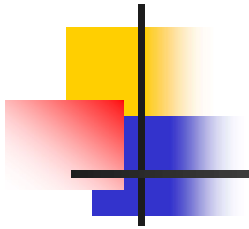
# Lemma 24.2

- Let G=(V,E) be a weighted, directed graph with source s and weight function w:E→R.
- Assume that G contains no negative-weight cycles that are reachable from s.
- Then, after the |V|-1 iterations of the **for** loop of lines 2 - 4 of BELLMAN-FORD, we have v.d=δ(s,v) for all vertices that are reachable from s.

# Lemma 24.2 (Proof)



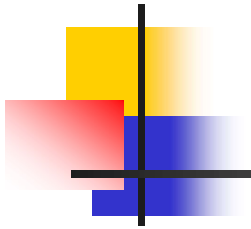
- We prove the lemma by appealing to the path-relaxation property.
- Consider any vertex $v$ that is reachable from $s$, and let $p=\langle v_0,v_1,\ldots,v_k\rangle$, where $v_0= s$ and $v_k = v$, be any shortest path from $s$ to $v$.
- Because shortest paths are simple, $p$ has at most $|V|-1$ edges, and so $k\leq|V|-1$. Each of the $|V|-1$ iterations of the **for** loop of lines 2–4 relaxes all $|E|$ edges.
- The edge $(v_{i-1}, v_i)$ is one among the edges relaxed in the $i$-th iteration, for $i = 1, 2, \ldots, k$.
- By the path-relaxation property, therefore, $v.d=v_k.d=\delta(s,v_k)=\delta(s,v)$.



# Corollary 24.3

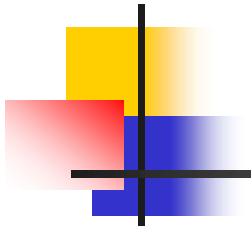
- Let G=(V,E) be a weighted, directed graph with source vertex s and weight function w:E→R.
- Assume that G contains no negative-weight cycles that are reachable from s.
- Then, for each vertex $v \in V$, there is a path from s to v if and only if BELLMAN-FORD terminates with $v.d < \infty$ when it is run on G.
- The proof is left as Exercise 24.1-2.

# Correctness of the Bellman-Ford Algorithm



- Let BELLMAN-FORD be run on a weighted, directed graph $G=(V,E)$ with source $s$ and weight function $w:E\rightarrow R$.
- If $G$ contains no negative-weight cycles that are reachable from $s$, then the algorithm returns TRUE, we have $v.d=\delta(s,v)$ for all vertices $v \in V$, and the predecessor subgraph $G$ is a shortest-paths tree rooted at $s$.
- It $G$ does contain a negative-weight cycle reachable from $s$, then the algorithm returns FALSE

# Correctness of the Bellman-Ford Algorithm (Proof)



- Suppose that G contains no negative-weight cycles that are reachable from s.
- We first prove the claim that at termination, $v.d=\delta(s,v)$ for all vertices $v \in V$.
  - If vertex v is reachable from s, then Lemma 24.2 proves this.
  - If v is not reachable from s, then the claim follows from the no-path property.
  - Thus, the claim is proven.
- The predecessor-subgraph property, along with the claim, implies that $G.\pi$ is a shortest-paths tree.
- Now we use the claim to show that BELLMAN-FORD returns TRUE.
- At termination, we have for all edges $(u,v) \in E$,
  - $v.d = \delta(s,v)$
    $\leq \delta(s,u) + w(u,v)$ (by the triangle inequality)
    $= u.d+w(u,v)$
- and so none of the tests in line 6 causes BELLMAN-FORD to return FALSE.
- Thus, it returns TRUE.

# Correctness of the Bellman-Ford Algorithm (Proof)

- Suppose G contains a negative-weight cycle that is reachable from s.
- Let this cycle c=<$v_0$,$v_1$,…,$v_k$>, where $v_0=v_k$ and $\sum_{i=1}^{k} w(v_{i-1}, v_i) < 0$ **(24.1)**
- Assume Bellman-Fold algorithm returns TRUE.

  $v_i.d \leq v_{i-1}.d + w(v_{i-1}, v_i)$ for $i = 1,2,\dots,k$
- Summing the inequalities around cycle c

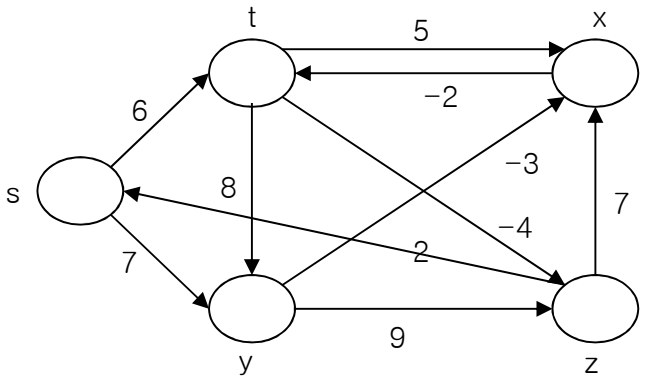
$$\sum_{i=1}^{k} v_i.d \leq \sum_{i=1}^{k} \left(v_{i-1}.d + w(v_{i-1}, v_i)\right)$$

$$= \sum_{i=1}^{k} v_{i-1}.d + \sum_{i=1}^{k} w(v_{i-1}, v_i)$$

- Since $v_0=v_k$, each vertex in c appears exactly once in each of the summations, $\sum_{i=1}^{k} v_i.d = \sum_{i=1}^{k} v_{i-1}.d$.
- Moreover, by Corollary 24.3, $v_i$.d is finite for i=1,2,…,k.
- Thus, $0 \leq \sum_{i=1}^{k} w(v_{i-1}, v_i)$ which contradicts inequality **(24.1)**

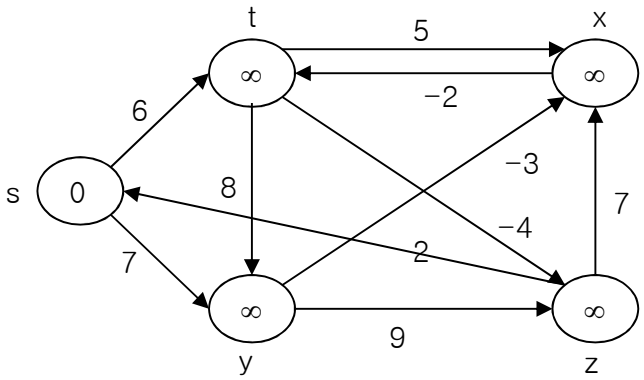
# Bellman-Ford Algorithm

```
BELLMAN-FORD(G, w, s)
1.      INITIALIZE-SINGLE-SOURCE(G,s)
2.      for i=1 to |G.V|-1
3.          for each edge (u,v) ∈ G.E
4.              RELAX(u, v, w)
5.      for each edge (u,v) ∈ G.E
6.          if  v.d > u.d + w(u,v)
7.              return FALSE
8.      return TRUE
```



# Bellman-Ford Algorithm

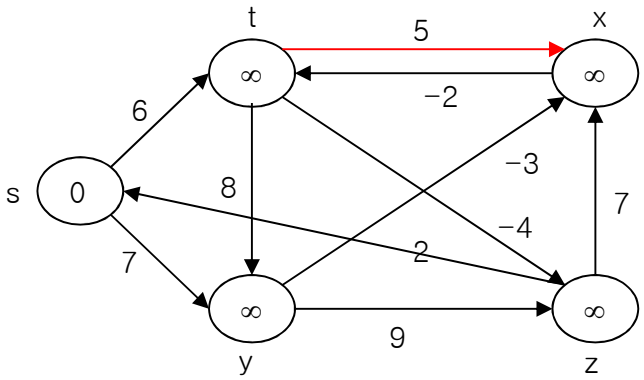
```
BELLMAN-FORD(G, w, s)
1.      INITIALIZE-SINGLE-SOURCE(G,s)
2.      for i=1 to |G.V|-1
3.          for each edge (u,v) ∈ G.E
4.              RELAX(u, v, w)
5.      for each edge (u,v) ∈ G.E
6.          if  v.d > u.d + w(u,v)
7.              return FALSE
8.      return TRUE
```



# Bellman-Ford Algorithm

```
BELLMAN-FORD(G, w, s)
1.      INITIALIZE-SINGLE-SOURCE(G,s)
2.      for i=1 to |G.V|-1
3.          for each edge (u,v) ∈ G.E
4.              RELAX(u, v, w)
5.      for each edge (u,v) ∈ G.E
6.          if  v.d > u.d + w(u,v)
7.              return FALSE
8.      return TRUE
```

i = 1

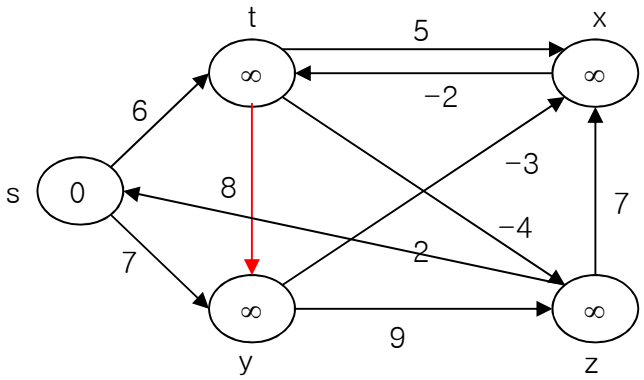



| G.E | (t,x) | (t,y) | (t,z) | (x,t) | (y,x) | (y,z) | (z,x) | (z,s) | (s,t) | (s,y) |
|---|---|---|---|---|---|---|---|---|---|---|

# Bellman-Ford Algorithm

```
BELLMAN-FORD(G, w, s)
1.      INITIALIZE-SINGLE-SOURCE(G,s)
2.      for i=1 to |G.V|-1
3.          for each edge (u,v) ∈ G.E
4.              RELAX(u, v, w)
5.      for each edge (u,v) ∈ G.E
6.          if  v.d > u.d + w(u,v)
7.              return FALSE
8.      return TRUE
```

i = 1

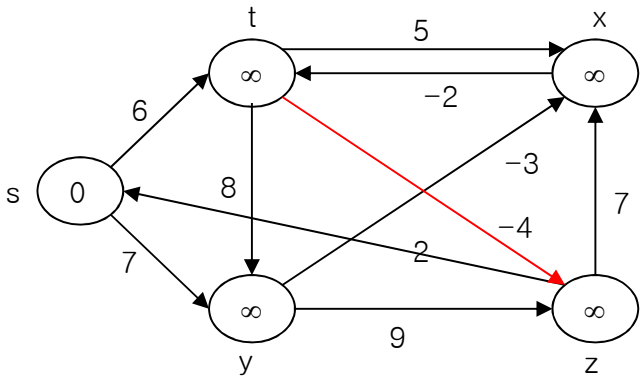



| G.E | (t,x) | (t,y) | (t,z) | (x,t) | (y,x) | (y,z) | (z,x) | (z,s) | (s,t) | (s,y) |
|---|---|---|---|---|---|---|---|---|---|---|

# Bellman-Ford Algorithm

```
BELLMAN-FORD(G, w, s)
1.      INITIALIZE-SINGLE-SOURCE(G,s)
2.      for i=1 to |G.V|-1
3.          for each edge (u,v) ∈ G.E
4.              RELAX(u, v, w)
5.      for each edge (u,v) ∈ G.E
6.          if  v.d > u.d + w(u,v)
7.              return FALSE
8.      return TRUE
```

i = 1

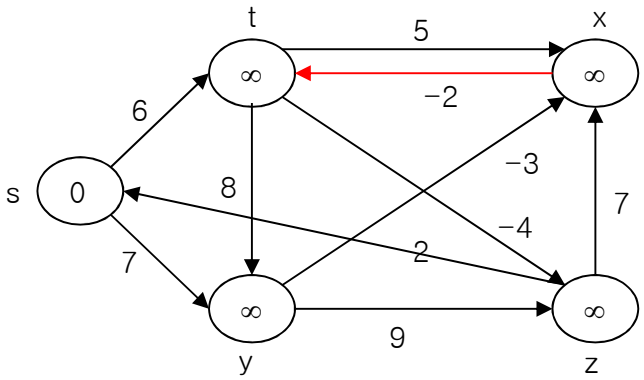



| G.E | (t,x) | (t,y) | (t,z) | (x,t) | (y,x) | (y,z) | (z,x) | (z,s) | (s,t) | (s,y) |
|---|---|---|---|---|---|---|---|---|---|---|

# Bellman-Ford Algorithm

```
BELLMAN-FORD(G, w, s)
1.      INITIALIZE-SINGLE-SOURCE(G,s)
2.      for i=1 to |G.V|-1
3.          for each edge (u,v) ∈ G.E
4.              RELAX(u, v, w)
5.      for each edge (u,v) ∈ G.E
6.          if  v.d > u.d + w(u,v)
7.              return FALSE
8.      return TRUE
```

i = 1

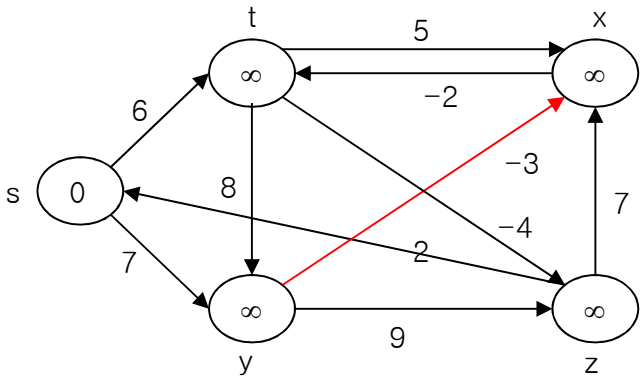



| G.E | (t,x) | (t,y) | (t,z) | (x,t) | (y,x) | (y,z) | (z,x) | (z,s) | (s,t) | (s,y) |
|---|---|---|---|---|---|---|---|---|---|---|

# Bellman-Ford Algorithm

```
BELLMAN-FORD(G, w, s)
1.      INITIALIZE-SINGLE-SOURCE(G,s)
2.      for i=1 to |G.V|-1
3.          for each edge (u,v) ∈ G.E
4.              RELAX(u, v, w)
5.      for each edge (u,v) ∈ G.E
6.          if  v.d > u.d + w(u,v)
7.              return FALSE
8.      return TRUE
```

i = 1

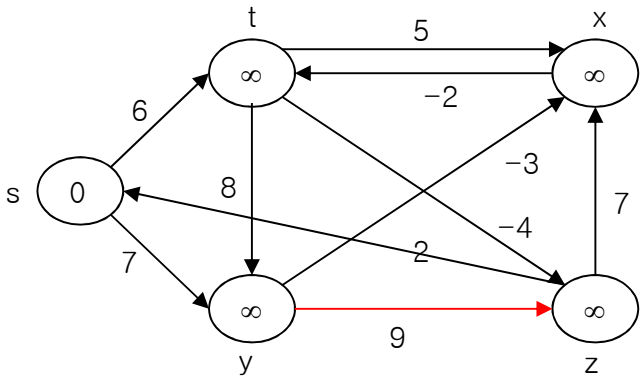



| G.E | (t,x) | (t,y) | (t,z) | (x,t) | (y,x) | (y,z) | (z,x) | (z,s) | (s,t) | (s,y) |
|---|---|---|---|---|---|---|---|---|---|---|

# Bellman-Ford Algorithm

```
BELLMAN-FORD(G, w, s)
1.      INITIALIZE-SINGLE-SOURCE(G,s)
2.      for i=1 to |G.V|-1
3.          for each edge (u,v) ∈ G.E
4.              RELAX(u, v, w)
5.      for each edge (u,v) ∈ G.E
6.          if  v.d > u.d + w(u,v)
7.              return FALSE
8.      return TRUE
```

i = 1

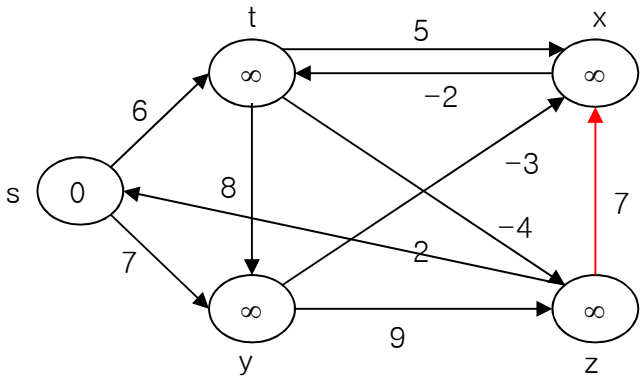



| G.E | (t,x) | (t,y) | (t,z) | (x,t) | (y,x) | (y,z) | (z,x) | (z,s) | (s,t) | (s,y) |
|---|---|---|---|---|---|---|---|---|---|---|

# Bellman-Ford Algorithm

```
BELLMAN-FORD(G, w, s)
1.      INITIALIZE-SINGLE-SOURCE(G,s)
2.      for i=1 to |G.V|-1
3.          for each edge (u,v) ∈ G.E
4.              RELAX(u, v, w)
5.      for each edge (u,v) ∈ G.E
6.          if  v.d > u.d + w(u,v)
7.              return FALSE
8.      return TRUE
```

i = 1

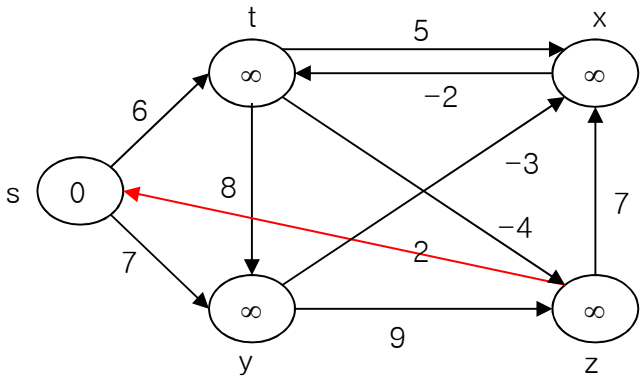



| G.E | (t,x) | (t,y) | (t,z) | (x,t) | (y,x) | (y,z) | (z,x) | (z,s) | (s,t) | (s,y) |
|---|---|---|---|---|---|---|---|---|---|---|

# Bellman-Ford Algorithm

```
BELLMAN-FORD(G, w, s)
1.      INITIALIZE-SINGLE-SOURCE(G,s)
2.      for i=1 to |G.V|-1
3.          for each edge (u,v) ∈ G.E
4.              RELAX(u, v, w)
5.      for each edge (u,v) ∈ G.E
6.          if  v.d > u.d + w(u,v)
7.              return FALSE
8.      return TRUE
```

i = 1

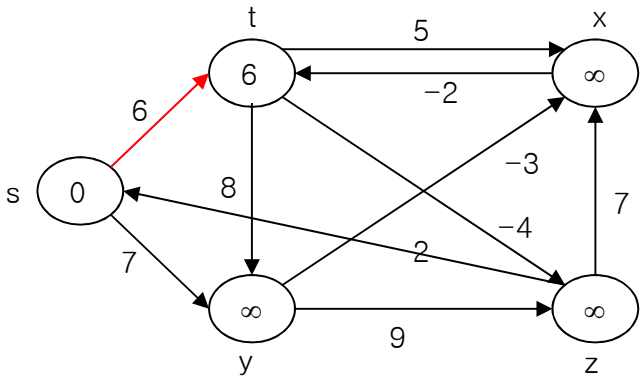



| G.E | (t,x) | (t,y) | (t,z) | (x,t) | (y,x) | (y,z) | (z,x) | (z,s) | (s,t) | (s,y) |
|---|---|---|---|---|---|---|---|---|---|---|

# Bellman-Ford Algorithm

```
BELLMAN-FORD(G, w, s)
1.      INITIALIZE-SINGLE-SOURCE(G,s)
2.      for i=1 to |G.V|-1
3.          for each edge (u,v) ∈ G.E
4.              RELAX(u, v, w)
5.      for each edge (u,v) ∈ G.E
6.          if  v.d > u.d + w(u,v)
7.              return FALSE
8.      return TRUE
```

i = 1

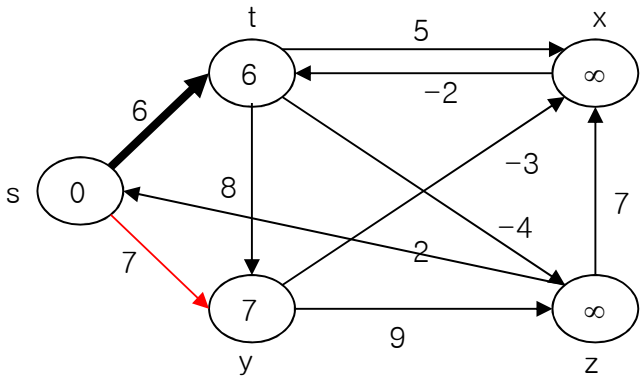



| G.E | (t,x) | (t,y) | (t,z) | (x,t) | (y,x) | (y,z) | (z,x) | (z,s) | (s,t) | (s,y) |
|---|---|---|---|---|---|---|---|---|---|---|

# Bellman-Ford Algorithm

```
BELLMAN-FORD(G, w, s)
1.      INITIALIZE-SINGLE-SOURCE(G,s)
2.      for i=1 to |G.V|-1
3.          for each edge (u,v) ∈ G.E
4.              RELAX(u, v, w)
5.      for each edge (u,v) ∈ G.E
6.          if  v.d > u.d + w(u,v)
7.              return FALSE
8.      return TRUE
```

i = 1

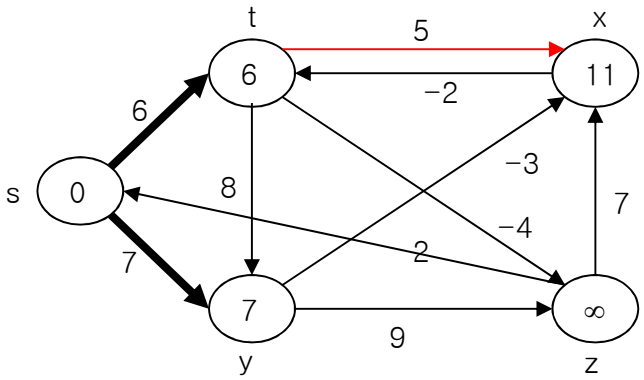



| G.E | (t,x) | (t,y) | (t,z) | (x,t) | (y,x) | (y,z) | (z,x) | (z,s) | (s,t) | (s,y) |
|---|---|---|---|---|---|---|---|---|---|---|

# Bellman-Ford Algorithm

```
BELLMAN-FORD(G, w, s)
1.      INITIALIZE-SINGLE-SOURCE(G,s)
2.      for i=1 to |G.V|-1
3.          for each edge (u,v) ∈ G.E
4.              RELAX(u, v, w)
5.      for each edge (u,v) ∈ G.E
6.          if  v.d > u.d + w(u,v)
7.              return FALSE
8.      return TRUE
```

i = 2

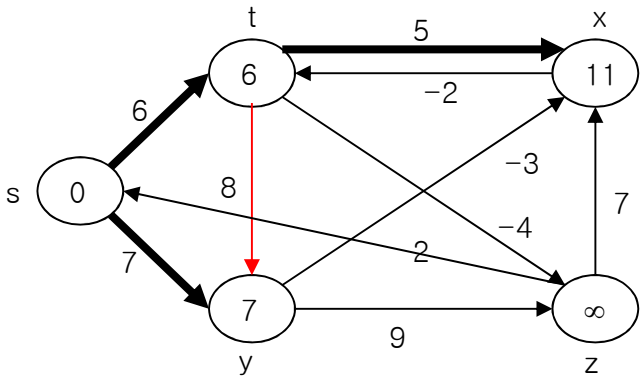



| G.E | (t,x) | (t,y) | (t,z) | (x,t) | (y,x) | (y,z) | (z,x) | (z,s) | (s,t) | (s,y) |
|---|---|---|---|---|---|---|---|---|---|---|

# Bellman-Ford Algorithm

```
BELLMAN-FORD(G, w, s)
1.      INITIALIZE-SINGLE-SOURCE(G,s)
2.      for i=1 to |G.V|-1
3.          for each edge (u,v) ∈ G.E
4.              RELAX(u, v, w)
5.      for each edge (u,v) ∈ G.E
6.          if  v.d > u.d + w(u,v)
7.              return FALSE
8.      return TRUE
```

i = 2

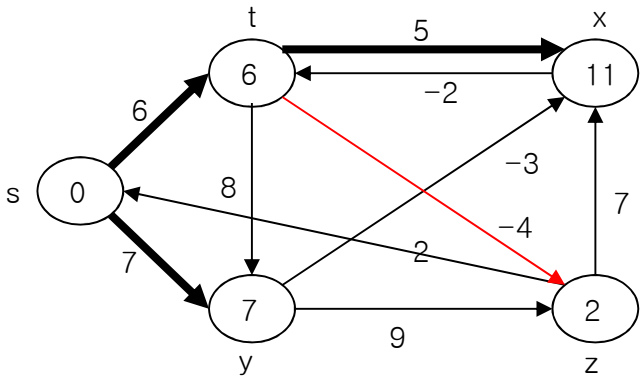



| G.E | (t,x) | (t,y) | (t,z) | (x,t) | (y,x) | (y,z) | (z,x) | (z,s) | (s,t) | (s,y) |
|---|---|---|---|---|---|---|---|---|---|---|

# Bellman-Ford Algorithm

```
BELLMAN-FORD(G, w, s)
1.      INITIALIZE-SINGLE-SOURCE(G,s)
2.      for i=1 to |G.V|-1
3.          for each edge (u,v) ∈ G.E
4.              RELAX(u, v, w)
5.      for each edge (u,v) ∈ G.E
6.          if  v.d > u.d + w(u,v)
7.              return FALSE
8.      return TRUE
```

i = 2

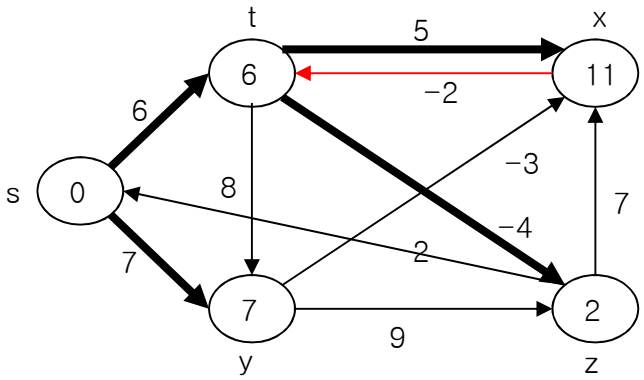



| G.E | (t,x) | (t,y) | (t,z) | (x,t) | (y,x) | (y,z) | (z,x) | (z,s) | (s,t) | (s,y) |
|---|---|---|---|---|---|---|---|---|---|---|

# Bellman-Ford Algorithm

```
BELLMAN-FORD(G, w, s)
1.      INITIALIZE-SINGLE-SOURCE(G,s)
2.      for i=1 to |G.V|-1
3.          for each edge (u,v) ∈ G.E
4.              RELAX(u, v, w)
5.      for each edge (u,v) ∈ G.E
6.          if  v.d > u.d + w(u,v)
7.              return FALSE
8.      return TRUE
```

i = 2

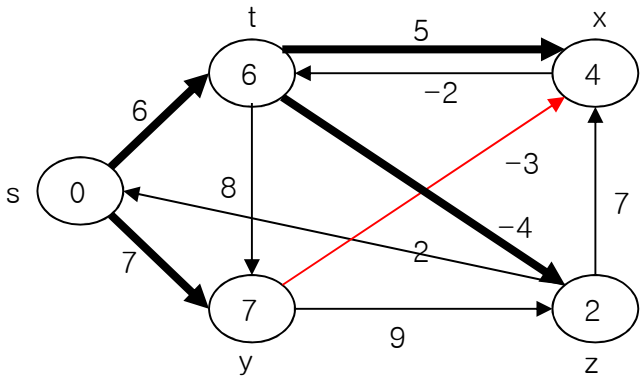



| G.E | (t,x) | (t,y) | (t,z) | (x,t) | (y,x) | (y,z) | (z,x) | (z,s) | (s,t) | (s,y) |
|---|---|---|---|---|---|---|---|---|---|---|

# Bellman-Ford Algorithm

```
BELLMAN-FORD(G, w, s)
1.      INITIALIZE-SINGLE-SOURCE(G,s)
2.      for i=1 to |G.V|-1
3.          for each edge (u,v) ∈ G.E
4.              RELAX(u, v, w)
5.      for each edge (u,v) ∈ G.E
6.          if  v.d > u.d + w(u,v)
7.              return FALSE
8.      return TRUE
```

i = 2

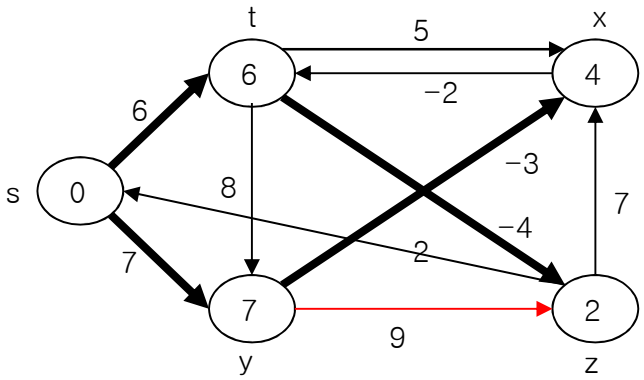



| G.E | (t,x) | (t,y) | (t,z) | (x,t) | (y,x) | (y,z) | (z,x) | (z,s) | (s,t) | (s,y) |
|---|---|---|---|---|---|---|---|---|---|---|

# Bellman-Ford Algorithm

```
BELLMAN-FORD(G, w, s)
1.      INITIALIZE-SINGLE-SOURCE(G,s)
2.      for i=1 to |G.V|-1
3.          for each edge (u,v) ∈ G.E
4.              RELAX(u, v, w)
5.      for each edge (u,v) ∈ G.E
6.          if  v.d > u.d + w(u,v)
7.              return FALSE
8.      return TRUE
```

i = 2

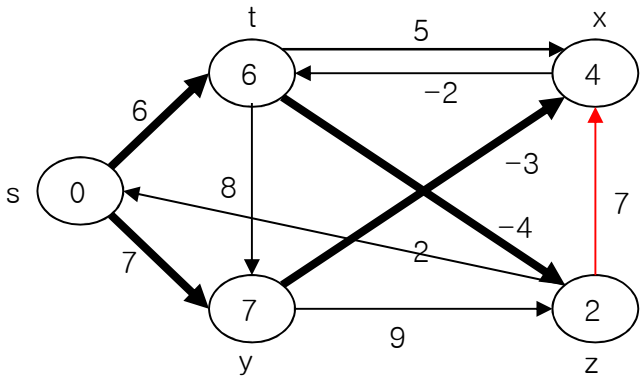



| G.E | (t,x) | (t,y) | (t,z) | (x,t) | (y,x) | (y,z) | (z,x) | (z,s) | (s,t) | (s,y) |
|---|---|---|---|---|---|---|---|---|---|---|

# Bellman-Ford Algorithm

```
BELLMAN-FORD(G, w, s)
1.      INITIALIZE-SINGLE-SOURCE(G,s)
2.      for i=1 to |G.V|-1
3.          for each edge (u,v) ∈ G.E
4.              RELAX(u, v, w)
5.      for each edge (u,v) ∈ G.E
6.          if  v.d > u.d + w(u,v)
7.              return FALSE
8.      return TRUE
```

i = 2

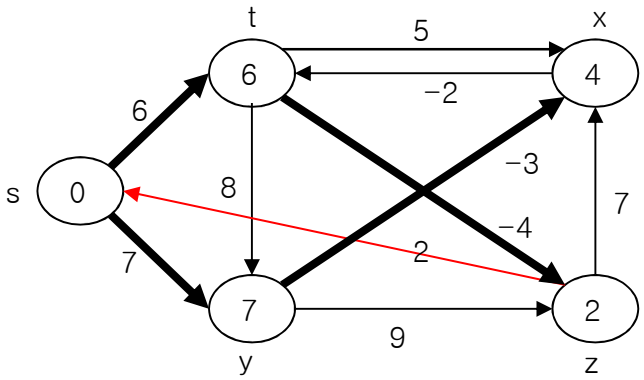



| G.E | (t,x) | (t,y) | (t,z) | (x,t) | (y,x) | (y,z) | (z,x) | (z,s) | (s,t) | (s,y) |
|---|---|---|---|---|---|---|---|---|---|---|

# Bellman-Ford Algorithm

```
BELLMAN-FORD(G, w, s)
1.      INITIALIZE-SINGLE-SOURCE(G,s)
2.      for i=1 to |G.V|-1
3.          for each edge (u,v) ∈ G.E
4.              RELAX(u, v, w)
5.      for each edge (u,v) ∈ G.E
6.          if  v.d > u.d + w(u,v)
7.              return FALSE
8.      return TRUE
```

i = 2

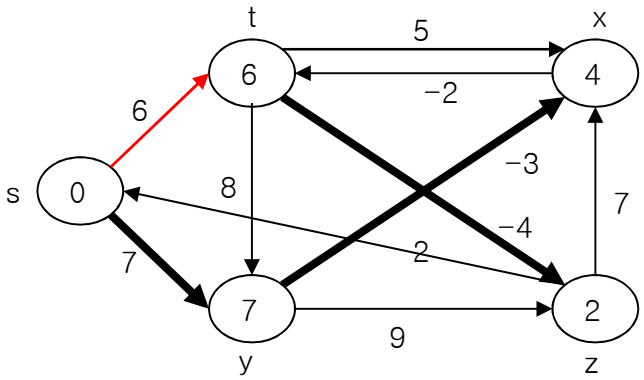



| G.E | (t,x) | (t,y) | (t,z) | (x,t) | (y,x) | (y,z) | (z,x) | (z,s) | (s,t) | (s,y) |
|---|---|---|---|---|---|---|---|---|---|---|

# Bellman-Ford Algorithm

```
BELLMAN-FORD(G, w, s)
1.      INITIALIZE-SINGLE-SOURCE(G,s)
2.      for i=1 to |G.V|-1
3.          for each edge (u,v) ∈ G.E
4.              RELAX(u, v, w)
5.      for each edge (u,v) ∈ G.E
6.          if  v.d > u.d + w(u,v)
7.              return FALSE
8.      return TRUE
```

i = 2

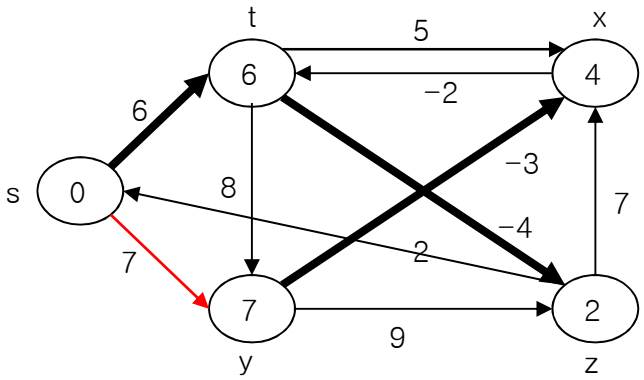



| G.E | (t,x) | (t,y) | (t,z) | (x,t) | (y,x) | (y,z) | (z,x) | (z,s) | (s,t) | (s,y) |
|---|---|---|---|---|---|---|---|---|---|---|

# Bellman-Ford Algorithm

```
BELLMAN-FORD(G, w, s)
1.      INITIALIZE-SINGLE-SOURCE(G,s)
2.      for i=1 to |G.V|-1
3.          for each edge (u,v) ∈ G.E
4.              RELAX(u, v, w)
5.      for each edge (u,v) ∈ G.E
6.          if  v.d > u.d + w(u,v)
7.              return FALSE
8.      return TRUE
```

i = 2

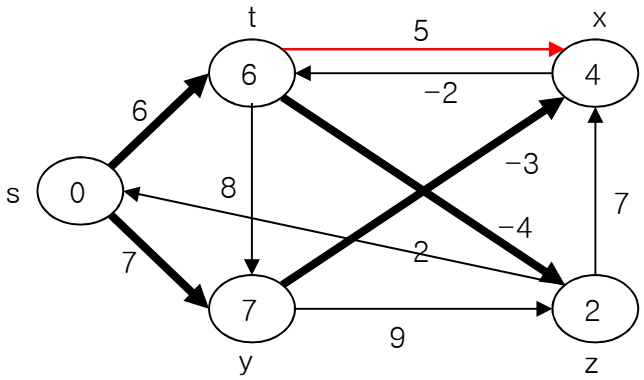



| G.E | (t,x) | (t,y) | (t,z) | (x,t) | (y,x) | (y,z) | (z,x) | (z,s) | (s,t) | (s,y) |
|---|---|---|---|---|---|---|---|---|---|---|

# Bellman-Ford Algorithm

```
BELLMAN-FORD(G, w, s)
1.      INITIALIZE-SINGLE-SOURCE(G,s)
2.      for i=1 to |G.V|-1
3.          for each edge (u,v) ∈ G.E
4.              RELAX(u, v, w)
5.      for each edge (u,v) ∈ G.E
6.          if  v.d > u.d + w(u,v)
7.              return FALSE
8.      return TRUE
```

i = 3

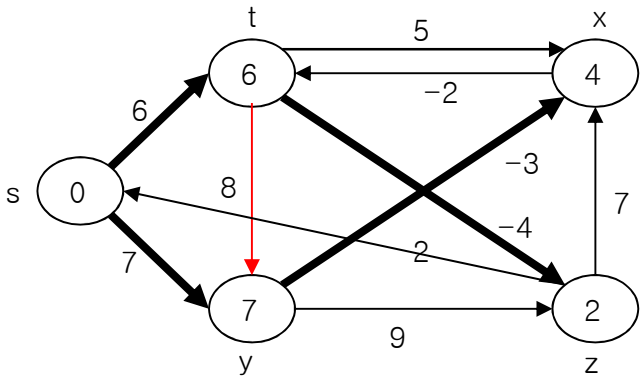



| G.E | (t,x) | (t,y) | (t,z) | (x,t) | (y,x) | (y,z) | (z,x) | (z,s) | (s,t) | (s,y) |
|---|---|---|---|---|---|---|---|---|---|---|

# Bellman-Ford Algorithm

```
BELLMAN-FORD(G, w, s)
1.      INITIALIZE-SINGLE-SOURCE(G,s)
2.      for i=1 to |G.V|-1
3.          for each edge (u,v) ∈ G.E
4.              RELAX(u, v, w)
5.      for each edge (u,v) ∈ G.E
6.          if  v.d > u.d + w(u,v)
7.              return FALSE
8.      return TRUE
```

i = 3

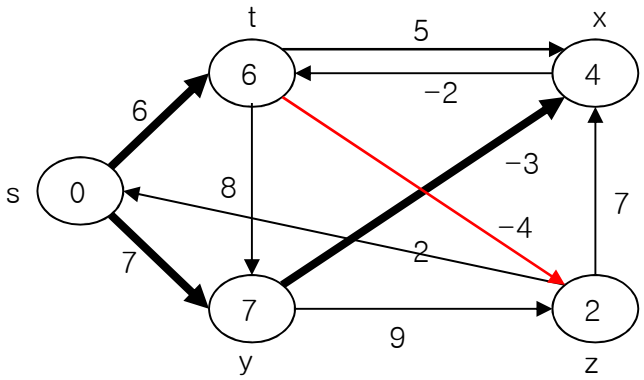



| G.E | (t,x) | (t,y) | (t,z) | (x,t) | (y,x) | (y,z) | (z,x) | (z,s) | (s,t) | (s,y) |
|---|---|---|---|---|---|---|---|---|---|---|

# Bellman-Ford Algorithm

```
BELLMAN-FORD(G, w, s)
1.      INITIALIZE-SINGLE-SOURCE(G,s)
2.      for i=1 to |G.V|-1
3.          for each edge (u,v) ∈ G.E
4.              RELAX(u, v, w)
5.      for each edge (u,v) ∈ G.E
6.          if  v.d > u.d + w(u,v)
7.              return FALSE
8.      return TRUE
```

i = 3

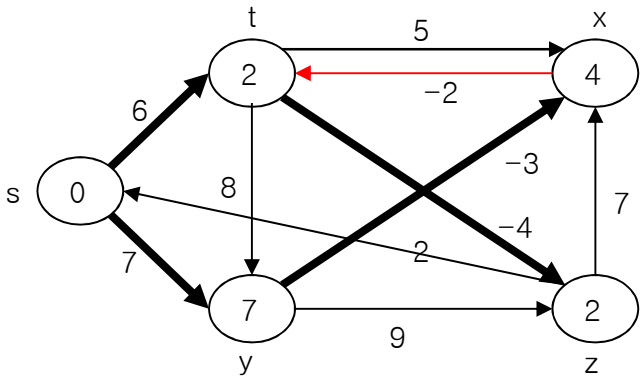



| G.E | (t,x) | (t,y) | (t,z) | (x,t) | (y,x) | (y,z) | (z,x) | (z,s) | (s,t) | (s,y) |
|---|---|---|---|---|---|---|---|---|---|---|

# Bellman-Ford Algorithm

```
BELLMAN-FORD(G, w, s)
1.      INITIALIZE-SINGLE-SOURCE(G,s)
2.      for i=1 to |G.V|-1
3.          for each edge (u,v) ∈ G.E
4.              RELAX(u, v, w)
5.      for each edge (u,v) ∈ G.E
6.          if  v.d > u.d + w(u,v)
7.              return FALSE
8.      return TRUE
```

i = 3

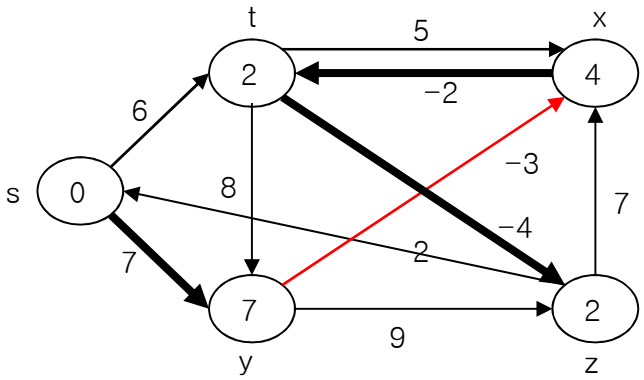



| G.E | (t,x) | (t,y) | (t,z) | (x,t) | (y,x) | (y,z) | (z,x) | (z,s) | (s,t) | (s,y) |
|---|---|---|---|---|---|---|---|---|---|---|

# Bellman-Ford Algorithm

```
BELLMAN-FORD(G, w, s)
1.      INITIALIZE-SINGLE-SOURCE(G,s)
2.      for i=1 to |G.V|-1
3.          for each edge (u,v) ∈ G.E
4.              RELAX(u, v, w)
5.      for each edge (u,v) ∈ G.E
6.          if  v.d > u.d + w(u,v)
7.              return FALSE
8.      return TRUE
```

i = 3

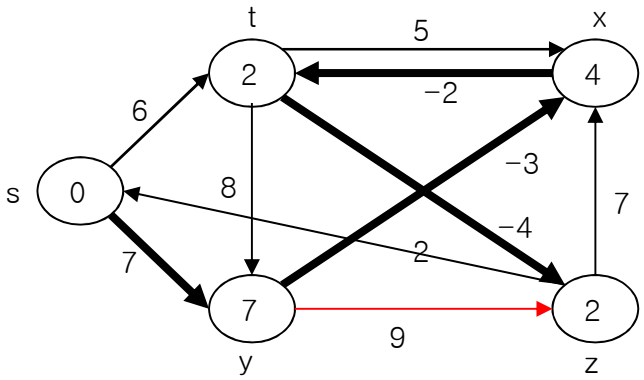



| G.E | (t,x) | (t,y) | (t,z) | (x,t) | (y,x) | (y,z) | (z,x) | (z,s) | (s,t) | (s,y) |
|---|---|---|---|---|---|---|---|---|---|---|

# Bellman-Ford Algorithm

```
BELLMAN-FORD(G, w, s)
1.      INITIALIZE-SINGLE-SOURCE(G,s)
2.      for i=1 to |G.V|-1
3.          for each edge (u,v) ∈ G.E
4.              RELAX(u, v, w)
5.      for each edge (u,v) ∈ G.E
6.          if  v.d > u.d + w(u,v)
7.              return FALSE
8.      return TRUE
```

i = 3

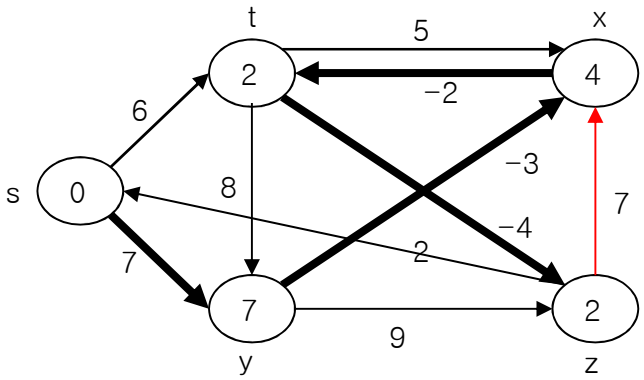



| G.E | (t,x) | (t,y) | (t,z) | (x,t) | (y,x) | (y,z) | (z,x) | (z,s) | (s,t) | (s,y) |
|---|---|---|---|---|---|---|---|---|---|---|

# Bellman-Ford Algorithm

```
BELLMAN-FORD(G, w, s)
1.      INITIALIZE-SINGLE-SOURCE(G,s)
2.      for i=1 to |G.V|-1
3.          for each edge (u,v) ∈ G.E
4.              RELAX(u, v, w)
5.      for each edge (u,v) ∈ G.E
6.          if  v.d > u.d + w(u,v)
7.              return FALSE
8.      return TRUE
```

i = 3

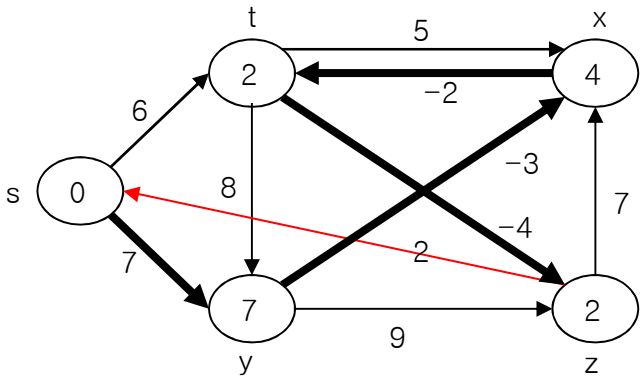



| G.E | (t,x) | (t,y) | (t,z) | (x,t) | (y,x) | (y,z) | (z,x) | (z,s) | (s,t) | (s,y) |
|---|---|---|---|---|---|---|---|---|---|---|

# Bellman-Ford Algorithm

```
BELLMAN-FORD(G, w, s)
1.      INITIALIZE-SINGLE-SOURCE(G,s)
2.      for i=1 to |G.V|-1
3.          for each edge (u,v) ∈ G.E
4.              RELAX(u, v, w)
5.      for each edge (u,v) ∈ G.E
6.          if  v.d > u.d + w(u,v)
7.              return FALSE
8.      return TRUE
```

i = 3

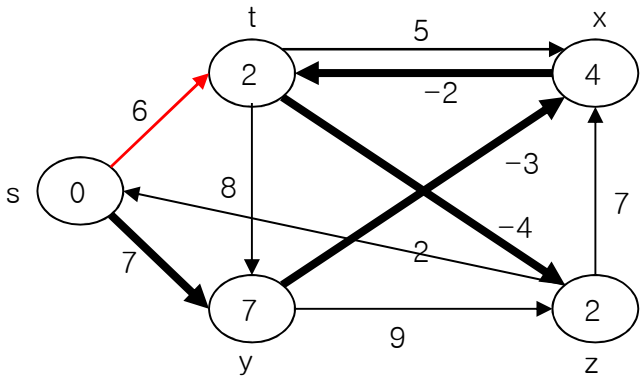



| G.E | (t,x) | (t,y) | (t,z) | (x,t) | (y,x) | (y,z) | (z,x) | (z,s) | (s,t) | (s,y) |
|---|---|---|---|---|---|---|---|---|---|---|

# Bellman-Ford Algorithm

```
BELLMAN-FORD(G, w, s)
1.      INITIALIZE-SINGLE-SOURCE(G,s)
2.      for i=1 to |G.V|-1
3.          for each edge (u,v) ∈ G.E
4.              RELAX(u, v, w)
5.      for each edge (u,v) ∈ G.E
6.          if  v.d > u.d + w(u,v)
7.              return FALSE
8.      return TRUE
```

i = 3

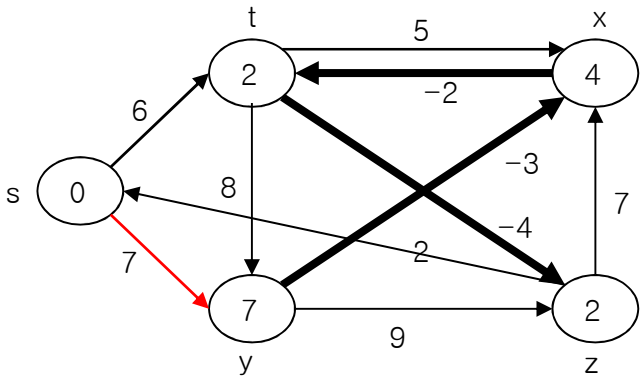



| G.E | (t,x) | (t,y) | (t,z) | (x,t) | (y,x) | (y,z) | (z,x) | (z,s) | (s,t) | (s,y) |
|---|---|---|---|---|---|---|---|---|---|---|

# Bellman-Ford Algorithm

```
BELLMAN-FORD(G, w, s)
1.      INITIALIZE-SINGLE-SOURCE(G,s)
2.      for i=1 to |G.V|-1
3.          for each edge (u,v) ∈ G.E
4.              RELAX(u, v, w)
5.      for each edge (u,v) ∈ G.E
6.          if  v.d > u.d + w(u,v)
7.              return FALSE
8.      return TRUE
```

i = 3

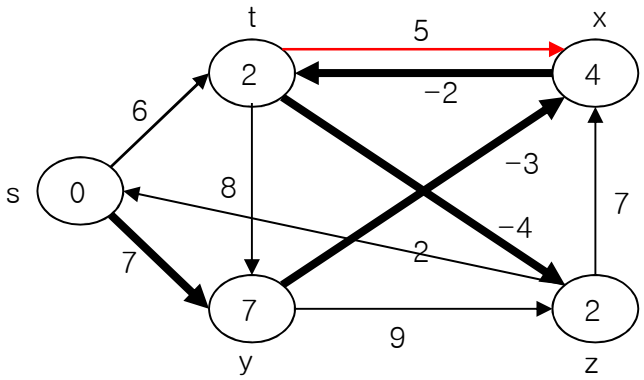



| G.E | (t,x) | (t,y) | (t,z) | (x,t) | (y,x) | (y,z) | (z,x) | (z,s) | (s,t) | (s,y) |
|---|---|---|---|---|---|---|---|---|---|---|

# Bellman-Ford Algorithm

```
BELLMAN-FORD(G, w, s)
1.      INITIALIZE-SINGLE-SOURCE(G,s)
2.      for i=1 to |G.V|-1
3.          for each edge (u,v) ∈ G.E
4.              RELAX(u, v, w)
5.      for each edge (u,v) ∈ G.E
6.          if  v.d > u.d + w(u,v)
7.              return FALSE
8.      return TRUE
```

i = 4

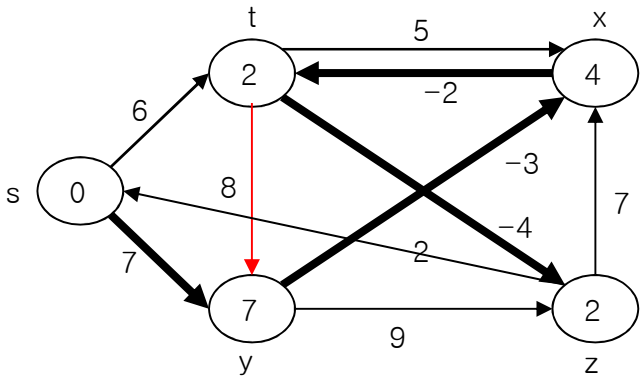



| G.E | (t,x) | (t,y) | (t,z) | (x,t) | (y,x) | (y,z) | (z,x) | (z,s) | (s,t) | (s,y) |
|---|---|---|---|---|---|---|---|---|---|---|

# Bellman-Ford Algorithm

```
BELLMAN-FORD(G, w, s)
1.      INITIALIZE-SINGLE-SOURCE(G,s)
2.      for i=1 to |G.V|-1
3.          for each edge (u,v) ∈ G.E
4.              RELAX(u, v, w)
5.      for each edge (u,v) ∈ G.E
6.          if  v.d > u.d + w(u,v)
7.              return FALSE
8.      return TRUE
```

i = 4

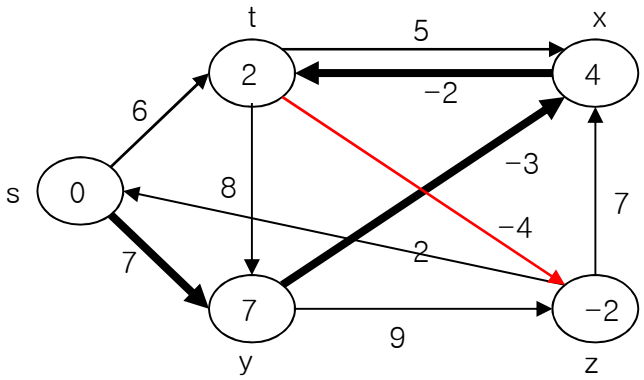



| G.E | (t,x) | (t,y) | (t,z) | (x,t) | (y,x) | (y,z) | (z,x) | (z,s) | (s,t) | (s,y) |
|---|---|---|---|---|---|---|---|---|---|---|

# Bellman-Ford Algorithm

```
BELLMAN-FORD(G, w, s)
1.      INITIALIZE-SINGLE-SOURCE(G,s)
2.      for i=1 to |G.V|-1
3.          for each edge (u,v) ∈ G.E
4.              RELAX(u, v, w)
5.      for each edge (u,v) ∈ G.E
6.          if  v.d > u.d + w(u,v)
7.              return FALSE
8.      return TRUE
```

i = 4

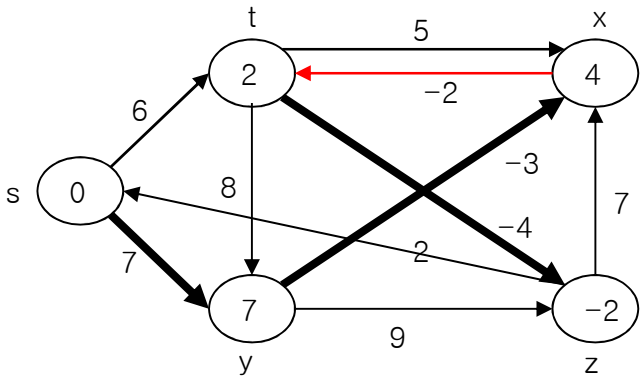



| G.E | (t,x) | (t,y) | (t,z) | (x,t) | (y,x) | (y,z) | (z,x) | (z,s) | (s,t) | (s,y) |
|---|---|---|---|---|---|---|---|---|---|---|

# Bellman-Ford Algorithm

```
BELLMAN-FORD(G, w, s)
1.      INITIALIZE-SINGLE-SOURCE(G,s)
2.      for i=1 to |G.V|-1
3.          for each edge (u,v) ∈ G.E
4.              RELAX(u, v, w)
5.      for each edge (u,v) ∈ G.E
6.          if  v.d > u.d + w(u,v)
7.              return FALSE
8.      return TRUE
```

i = 4

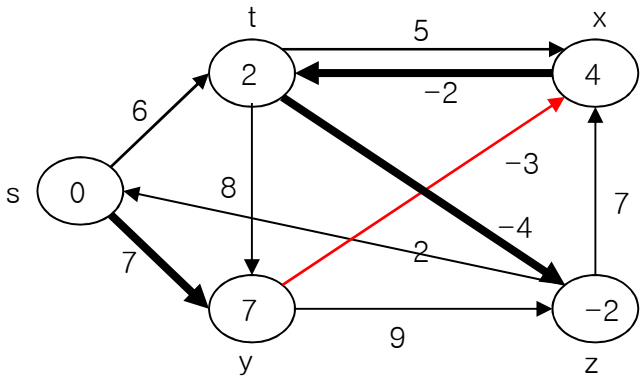



| G.E | (t,x) | (t,y) | (t,z) | (x,t) | (y,x) | (y,z) | (z,x) | (z,s) | (s,t) | (s,y) |
|---|---|---|---|---|---|---|---|---|---|---|

# Bellman-Ford Algorithm

```
BELLMAN-FORD(G, w, s)
1.      INITIALIZE-SINGLE-SOURCE(G,s)
2.      for i=1 to |G.V|-1
3.          for each edge (u,v) ∈ G.E
4.              RELAX(u, v, w)
5.      for each edge (u,v) ∈ G.E
6.          if  v.d > u.d + w(u,v)
7.              return FALSE
8.      return TRUE
```

i = 4

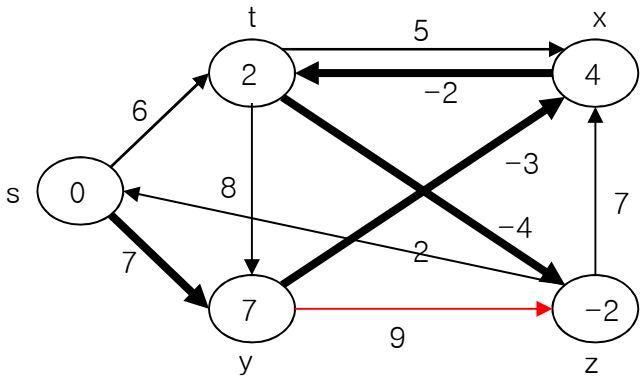



| G.E | (t,x) | (t,y) | (t,z) | (x,t) | (y,x) | (y,z) | (z,x) | (z,s) | (s,t) | (s,y) |
|---|---|---|---|---|---|---|---|---|---|---|

# Bellman-Ford Algorithm

```
BELLMAN-FORD(G, w, s)
1.      INITIALIZE-SINGLE-SOURCE(G,s)
2.      for i=1 to |G.V|-1
3.          for each edge (u,v) ∈ G.E
4.              RELAX(u, v, w)
5.      for each edge (u,v) ∈ G.E
6.          if  v.d > u.d + w(u,v)
7.              return FALSE
8.      return TRUE
```

i = 4

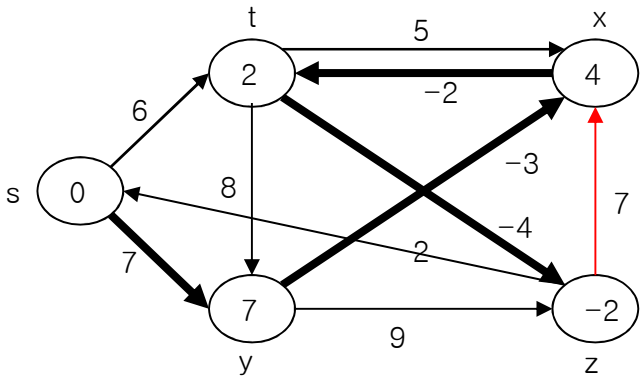



| G.E | (t,x) | (t,y) | (t,z) | (x,t) | (y,x) | (y,z) | (z,x) | (z,s) | (s,t) | (s,y) |
|---|---|---|---|---|---|---|---|---|---|---|

# Bellman-Ford Algorithm

```
BELLMAN-FORD(G, w, s)
1.      INITIALIZE-SINGLE-SOURCE(G,s)
2.      for i=1 to |G.V|-1
3.          for each edge (u,v) ∈ G.E
4.              RELAX(u, v, w)
5.      for each edge (u,v) ∈ G.E
6.          if  v.d > u.d + w(u,v)
7.              return FALSE
8.      return TRUE
```

i = 4

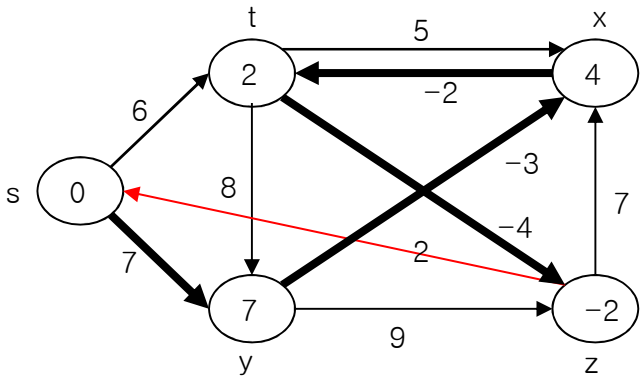



| G.E | (t,x) | (t,y) | (t,z) | (x,t) | (y,x) | (y,z) | (z,x) | (z,s) | (s,t) | (s,y) |
|---|---|---|---|---|---|---|---|---|---|---|

# Bellman-Ford Algorithm

```
BELLMAN-FORD(G, w, s)
1.      INITIALIZE-SINGLE-SOURCE(G,s)
2.      for i=1 to |G.V|-1
3.          for each edge (u,v) ∈ G.E
4.              RELAX(u, v, w)
5.      for each edge (u,v) ∈ G.E
6.          if  v.d > u.d + w(u,v)
7.              return FALSE
8.      return TRUE
```

i = 4

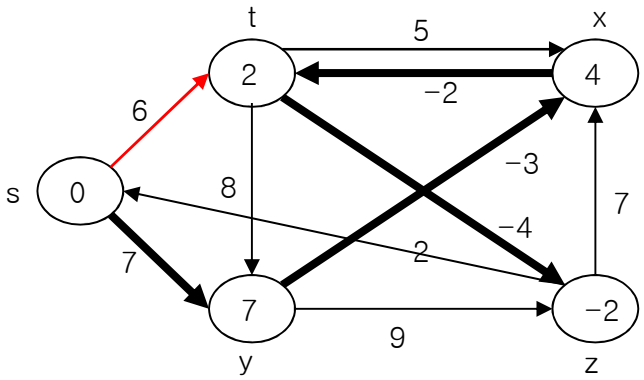



| G.E | (t,x) | (t,y) | (t,z) | (x,t) | (y,x) | (y,z) | (z,x) | (z,s) | (s,t) | (s,y) |
|---|---|---|---|---|---|---|---|---|---|---|

# Bellman-Ford Algorithm

```
BELLMAN-FORD(G, w, s)
1.      INITIALIZE-SINGLE-SOURCE(G,s)
2.      for i=1 to |G.V|-1
3.          for each edge (u,v) ∈ G.E
4.              RELAX(u, v, w)
5.      for each edge (u,v) ∈ G.E
6.          if  v.d > u.d + w(u,v)
7.              return FALSE
8.      return TRUE
```

i = 4

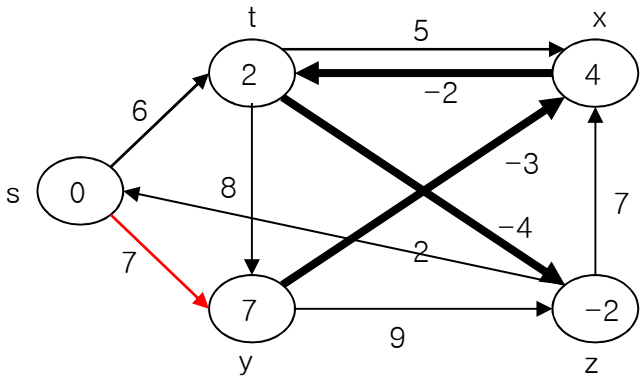



| G.E | (t,x) | (t,y) | (t,z) | (x,t) | (y,x) | (y,z) | (z,x) | (z,s) | (s,t) | (s,y) |
|---|---|---|---|---|---|---|---|---|---|---|

# Bellman-Ford Algorithm

```
BELLMAN-FORD(G, w, s)
1.      INITIALIZE-SINGLE-SOURCE(G,s)
2.      for i=1 to |G.V|-1
3.          for each edge (u,v) ∈ G.E
4.              RELAX(u, v, w)
5.      for each edge (u,v) ∈ G.E
6.          if  v.d > u.d + w(u,v)
7.              return FALSE
8.      return TRUE
```

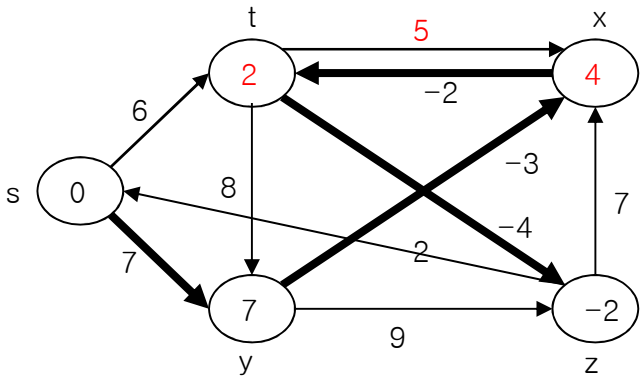
i = 4




| G.E | (t,x) | (t,y) | (t,z) | (x,t) | (y,x) | (y,z) | (z,x) | (z,s) | (s,t) | (s,y) |
|---|---|---|---|---|---|---|---|---|---|---|

# Bellman-Ford Algorithm

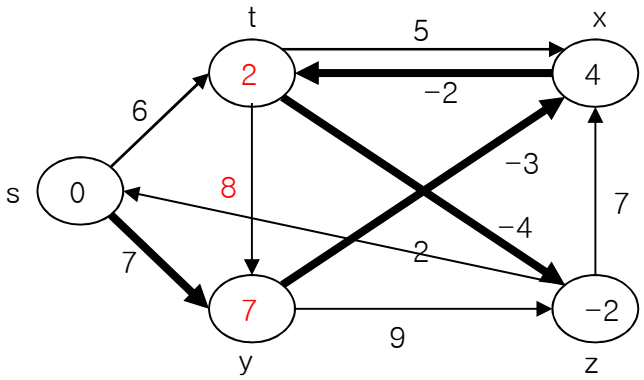
```
BELLMAN-FORD(G, w, s)
1.      INITIALIZE-SINGLE-SOURCE(G,s)
2.      for i=1 to |G.V|-1
3.          for each edge (u,v) ∈ G.E
4.              RELAX(u, v, w)
5.      for each edge (u,v) ∈ G.E
6.          if  v.d > u.d + w(u,v)
7.              return FALSE
8.      return TRUE
```




| G.E | (t,x) | (t,y) | (t,z) | (x,t) | (y,x) | (y,z) | (z,x) | (z,s) | (s,t) | (s,y) |
|---|---|---|---|---|---|---|---|---|---|---|

# Bellman-Ford Algorithm

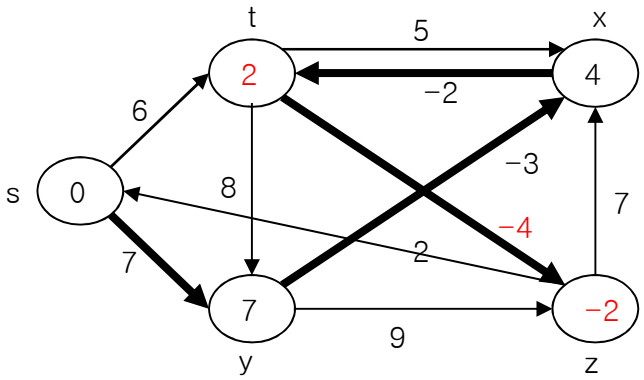
```
BELLMAN-FORD(G, w, s)
1.      INITIALIZE-SINGLE-SOURCE(G,s)
2.      for i=1 to |G.V|-1
3.          for each edge (u,v) ∈ G.E
4.              RELAX(u, v, w)
5.      for each edge (u,v) ∈ G.E
6.          if  v.d > u.d + w(u,v)
7.              return FALSE
8.      return TRUE
```




| G.E | (t,x) | (t,y) | (t,z) | (x,t) | (y,x) | (y,z) | (z,x) | (z,s) | (s,t) | (s,y) |
|---|---|---|---|---|---|---|---|---|---|---|

# Bellman-Ford Algorithm

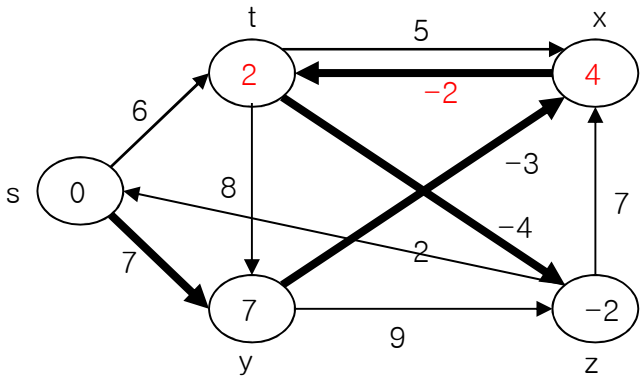
```
BELLMAN-FORD(G, w, s)
1.      INITIALIZE-SINGLE-SOURCE(G,s)
2.      for i=1 to |G.V|-1
3.          for each edge (u,v) ∈ G.E
4.              RELAX(u, v, w)
5.      for each edge (u,v) ∈ G.E
6.          if  v.d > u.d + w(u,v)
7.              return FALSE
8.      return TRUE
```




| G.E | (t,x) | (t,y) | (t,z) | (x,t) | (y,x) | (y,z) | (z,x) | (z,s) | (s,t) | (s,y) |
|---|---|---|---|---|---|---|---|---|---|---|

# Bellman-Ford Algorithm

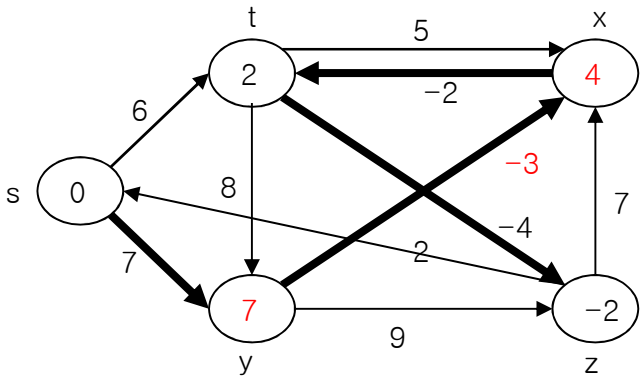
```
BELLMAN-FORD(G, w, s)
1.      INITIALIZE-SINGLE-SOURCE(G,s)
2.      for i=1 to |G.V|-1
3.          for each edge (u,v) ∈ G.E
4.              RELAX(u, v, w)
5.      for each edge (u,v) ∈ G.E
6.          if  v.d > u.d + w(u,v)
7.              return FALSE
8.      return TRUE
```




| G.E | (t,x) | (t,y) | (t,z) | (x,t) | (y,x) | (y,z) | (z,x) | (z,s) | (s,t) | (s,y) |
|---|---|---|---|---|---|---|---|---|---|---|

# Bellman-Ford Algorithm

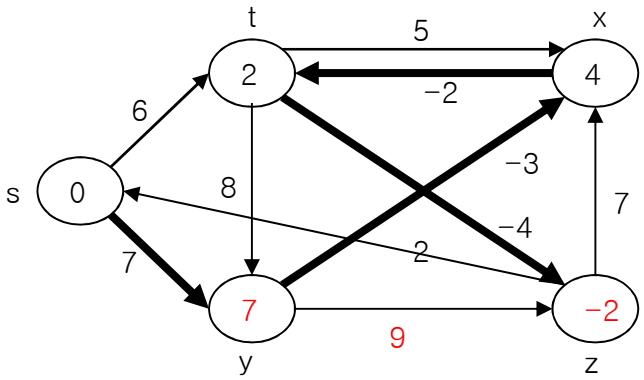
```
BELLMAN-FORD(G, w, s)
1.      INITIALIZE-SINGLE-SOURCE(G,s)
2.      for i=1 to |G.V|-1
3.          for each edge (u,v) ∈ G.E
4.              RELAX(u, v, w)
5.      for each edge (u,v) ∈ G.E
6.          if  v.d > u.d + w(u,v)
7.              return FALSE
8.      return TRUE
```




| G.E | (t,x) | (t,y) | (t,z) | (x,t) | (y,x) | (y,z) | (z,x) | (z,s) | (s,t) | (s,y) |
|---|---|---|---|---|---|---|---|---|---|---|

# Bellman-Ford Algorithm

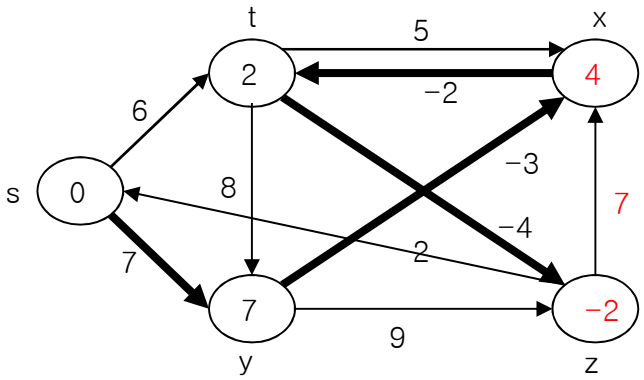
```
BELLMAN-FORD(G, w, s)
1.      INITIALIZE-SINGLE-SOURCE(G,s)
2.      for i=1 to |G.V|-1
3.          for each edge (u,v) ∈ G.E
4.              RELAX(u, v, w)
5.      for each edge (u,v) ∈ G.E
6.          if  v.d > u.d + w(u,v)
7.              return FALSE
8.      return TRUE
```




| G.E | (t,x) | (t,y) | (t,z) | (x,t) | (y,x) | (y,z) | (z,x) | (z,s) | (s,t) | (s,y) |
|---|---|---|---|---|---|---|---|---|---|---|

# Bellman-Ford Algorithm

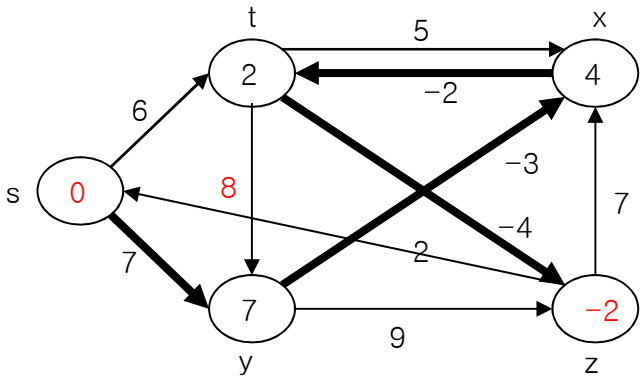
```
BELLMAN-FORD(G, w, s)
1.      INITIALIZE-SINGLE-SOURCE(G,s)
2.      for i=1 to |G.V|-1
3.          for each edge (u,v) ∈ G.E
4.              RELAX(u, v, w)
5.      for each edge (u,v) ∈ G.E
6.          if  v.d > u.d + w(u,v)
7.              return FALSE
8.      return TRUE
```




| G.E | (t,x) | (t,y) | (t,z) | (x,t) | (y,x) | (y,z) | (z,x) | (z,s) | (s,t) | (s,y) |
|---|---|---|---|---|---|---|---|---|---|---|

# Bellman-Ford Algorithm

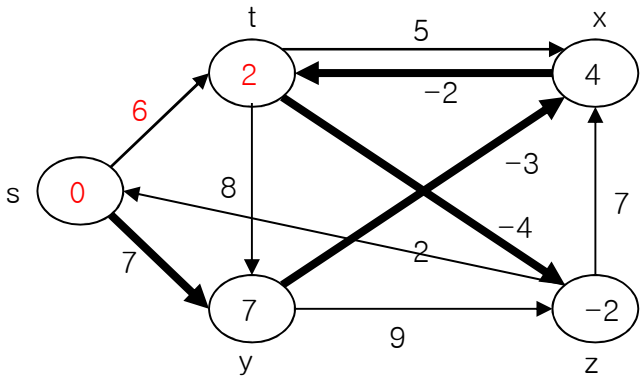
```
BELLMAN-FORD(G, w, s)
1.      INITIALIZE-SINGLE-SOURCE(G,s)
2.      for i=1 to |G.V|-1
3.          for each edge (u,v) ∈ G.E
4.              RELAX(u, v, w)
5.      for each edge (u,v) ∈ G.E
6.          if  v.d > u.d + w(u,v)
7.              return FALSE
8.      return TRUE
```




| G.E | (t,x) | (t,y) | (t,z) | (x,t) | (y,x) | (y,z) | (z,x) | (z,s) | (s,t) | (s,y) |
|---|---|---|---|---|---|---|---|---|---|---|

# Bellman-Ford Algorithm

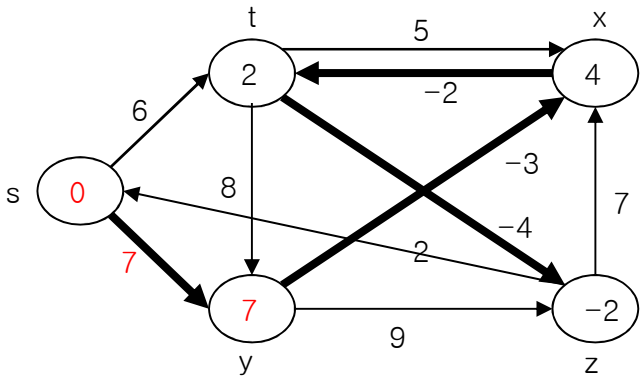
```
BELLMAN-FORD(G, w, s)
1.      INITIALIZE-SINGLE-SOURCE(G,s)
2.      for i=1 to |G.V|-1
3.          for each edge (u,v) ∈ G.E
4.              RELAX(u, v, w)
5.      for each edge (u,v) ∈ G.E
6.          if  v.d > u.d + w(u,v)
7.              return FALSE
8.      return TRUE
```




| G.E | (t,x) | (t,y) | (t,z) | (x,t) | (y,x) | (y,z) | (z,x) | (z,s) | (s,t) | (s,y) |
|---|---|---|---|---|---|---|---|---|---|---|

# Bellman-Ford Algorithm

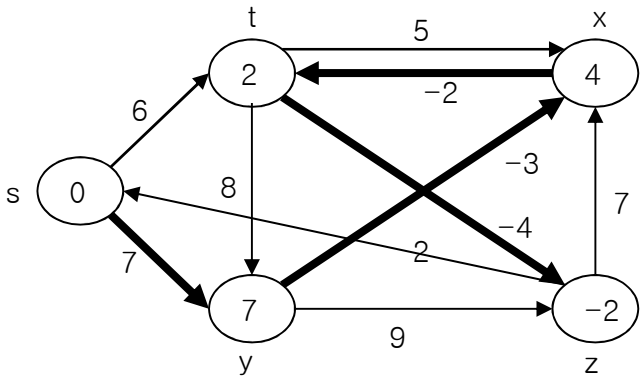
```
BELLMAN-FORD(G, w, s)
1.      INITIALIZE-SINGLE-SOURCE(G,s)
2.      for i=1 to |G.V|-1
3.          for each edge (u,v) ∈ G.E
4.              RELAX(u, v, w)
5.      for each edge (u,v) ∈ G.E
6.          if  v.d > u.d + w(u,v)
7.              return FALSE
8.      return TRUE
```



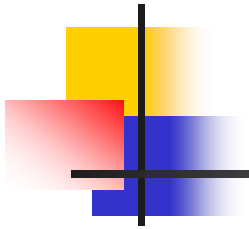

| G.E | (t,x) | (t,y) | (t,z) | (x,t) | (y,x) | (y,z) | (z,x) | (z,s) | (s,t) | (s,y) |
|---|---|---|---|---|---|---|---|---|---|---|

# Bellman-Ford Algorithm

```
BELLMAN-FORD(G, w, s)
1.      INITIALIZE-SINGLE-SOURCE(G,s)
2.      for i=1 to |G.V|-1
3.          for each edge (u,v) ∈ G.E
4.              RELAX(u, v, w)
5.      for each edge (u,v) ∈ G.E
6.          if  v.d > u.d + w(u,v)
7.              return FALSE
8.      return TRUE
```




| G.E | (t,x) | (t,y) | (t,z) | (x,t) | (y,x) | (y,z) | (z,x) | (z,s) | (s,t) | (s,y) |
|---|---|---|---|---|---|---|---|---|---|---|



# Single Source/All Destinations: Bellman-Ford Algorithm

- $dist^k[u]$
  - The length of a shortest path from the source vertex v to vertex u under the constraint that the shortest path contains at most k edges

- When there are no cycles of negative length, we can find exact shortest path which has at most n-1 edges

# Single Source/All Destinations: Bellman-Ford Algorithm

- Goal
  - Compute $dist^{n-1}[u]$ for all u
  - This can be done using dynamic programming methodology
- Observation
  - If the shortest path from v to u with at most k, k>1, edges has no more than k-1 edges
    - $dist^{k}[u] = dist^{k-1}[u]$
  - If the shortest path from v to u with at most k, k>1, edges has exactly k edges
    - It is comprised of a shortest path from v to some vertex j followed by the edge <j,u>. The path from v to j has k-1 edges, and its length is $dist^{k-1}[j]$
    - All vertices i such that the edge <i,u> is in the graph are candidates for j
    - $dist^{k}[u] = \min\{dist^{k-1}[u], \min\{dist^{k-1}[i]+length[i][u]\}\}$

# Single Source/All Destinations: Bellman-Ford Algorithm

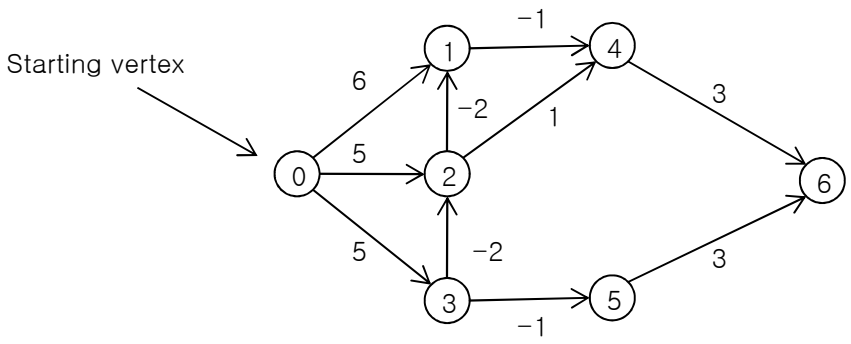
```
1.  void MatrixWDigraph::BellmanFord(const int n, const int v)
2.  { // Single source all destination shortest paths with negative edge lengths
3.        for(int i=0; i<n; i++) dist[i] = length[v][i]; // initialize dist

4.        for(int k=2; k<=n-1; k++)
5.              for(each u such that u != v and u has at least one incoming edge)
6.                    for(each <i,u> in the graph)
7.                          if(dist[u] > dist[i] + length[i][u]) dist[u] = dist[i] + length[i][u];
8.  }
```

Program 6.9: Bellman and Ford algorithm to compute shortest paths

- Time Complexity
  - lines 5 to 7
    - $O(n^2)$ when adjacency matrices are used
    - O(e) when adjacency lists are used
  - Overall
    - $O(n^3)$ when adjacency matrices are used
    - O(ne) when adjacency lists are used

# Single Source/All Destinations: Bellman-Ford Algorithm

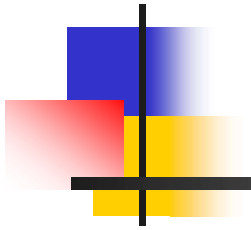



(a) A directed graph

| k | $dist^k[7]$ | | | | | | |
|---|---|---|---|---|---|---|---|
| | 0 | 1 | 2 | 3 | 4 | 5 | 6 |
| 1 | 0 | 6 | 5 | 5 | ∞ | ∞ | ∞ |
| 2 | 0 | 3 | 3 | 5 | 5 | 4 | ∞ |
| 3 | 0 | 1 | 3 | 5 | 2 | 4 | 7 |
| 4 | 0 | 1 | 3 | 5 | 0 | 4 | 5 |
| 5 | 0 | 1 | 3 | 5 | 0 | 4 | 3 |
| 6 | 0 | 1 | 3 | 5 | 0 | 4 | 3 |

(b) $dist^k$

Figure 6.31: Shortest paths with negative edge lengths

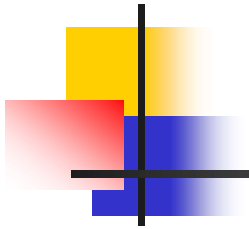


# Single-source Shortest Paths

# Single-source Shortest-Paths Problem

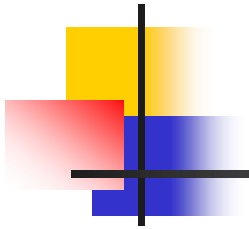
- Given a weighted directed graph G=(V,E) with weight function w:E→R mapping edges to real-valued weights, find the minimum-weight path from a given source vertex s to another vertex v.
  - The weight w(p) of a path p=<$v_0$,$v_1$,…,$v_k$> is the sum of the weights of its constituent edges: $w(p) = \sum_{i=1}^{k} w(v_{i-1}, v_i)$
  - The shortest-path weight δ(s,v) from s to v by

$$\delta(s,v) = \begin{cases} \min\{w(p): s \rightsquigarrow v\} & \text{if there is a path } p \text{ from u to v} \\ \infty & \text{otherwise} \end{cases}$$



# Optimal Substructure Property

- A shortest path between two vertices contains other shortest paths within it.
- Lemma 24.1 (Subpaths of shortest paths are shortest paths)
  - Consider a weighted graph G, with weight function $w:E \rightarrow R$,
  - Let $p=\langle v_0, v_1, v_2, v_4, \ldots, v_{k-1}, v_k \rangle$ be a shortest path from vertex $v_0$ to vertex $v_k$.
  - For any i and j such that $1 \leq i \leq j \leq k$ , let $p_{ij}=\langle v_i, v_{i+1}, \ldots, v_j \rangle$ be the subpath of p from vertex $v_i$ to vertex $v_j$.
  - Then, $p_{ij}$ is a shortest path from $v_i$ to $v_j$.
- Proof is straightforward.

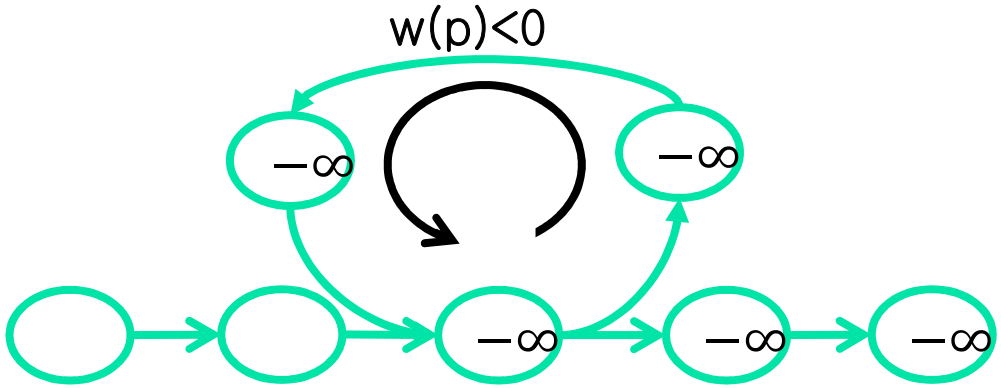


# Shortest Path Properties

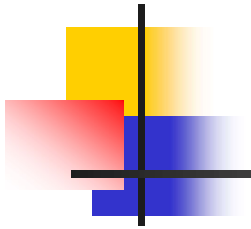
- Some instances of the single-source shortest-paths problem may include edges whose weights are negative.
- If the graph G=(V,E) contains no negative weight cycles reachable from the source s, then for all $v \in V$, the shortest-path weight $\delta(s,v)$ remains well defined, even if it has a negative value.
- If the graph contains a negative-weight cycle reachable from s, however, shortest-path weights are not well defined.

# Shortest Path Properties

- No path from s to a vertex on the cycle can be a shortest path—we can always find a path with lower weight by following the proposed shortest path and then traversing the negative-weight cycle.
- If there is a negative-weight cycle on some path from s to v, we define $\delta(s,v) = -\infty$.



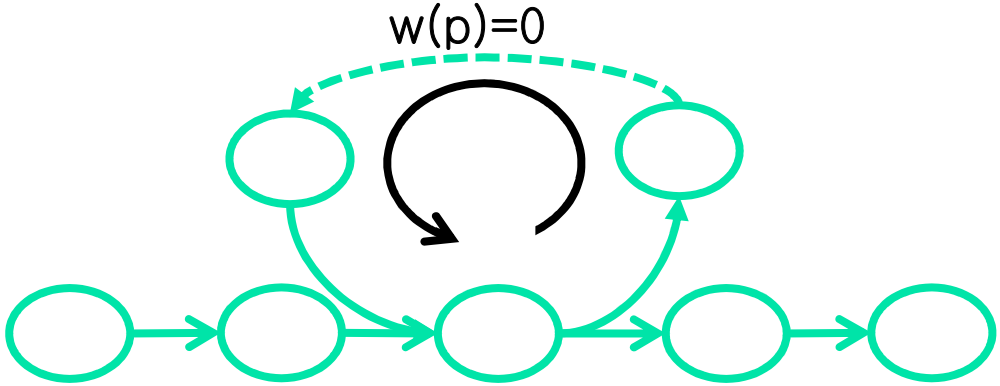
# Can a Shortest Path Contain a Cycle?

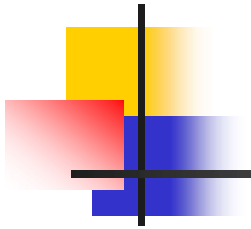


- As we have just seen, it cannot contain a negative-weight cycle.
- Nor can it contain a positive-weight cycle, since removing the cycle from the path produces a path with the same source and destination vertices and a lower path weight.
- We can remove a 0-weight cycle from any path to produce another path whose weight is the same.
- Thus, if there is a shortest path from a source vertex s to a destination vertex  that contains a 0-weight cycle, then there is another shortest path from s to  without this cycle.

# Shortest Path Properties

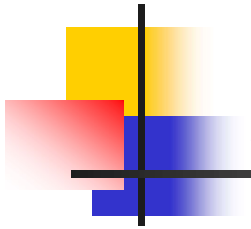
- Without loss of generality, we can assume that when we are finding shortest paths, they have no cycles.
- Since any acyclic path in a graph G=(V,E), contains at most |V| distinct vertices, it also contains at most |V|-1 edges.
- If have only 0-weight cycles, we can restrict our attention to shortest paths of at most |V|-1 edges.





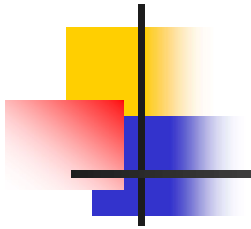
# Representing Shortest Paths

- We represent shortest paths similarly to how we represented breadth-first trees.
- Shortest-paths tree:
  - Predecessor subgraph $G_\pi = (V_\pi, E_\pi)$
    - $V_\pi = \{v \in V : v.\pi \neq \text{NIL}\} \cup \{s\}$
    - $E_\pi = \{(v.\pi, v) \in E : v \in V_\pi - \{s\}$
  - $V_\pi$ is the set of vertices of G with non-NIL predecessors, plus the source s.
  - $E_\pi$ is the the set of edges induced by the $\pi$ values for vertices in $V_\pi$.
  - $G_\pi$ forms a rooted tree with root s containing a shortest path from the source s to every vertex that is reachable from s.



# A Shortest-path Tree

- Let G=(V, E) be a weighted, directed graph with weight function w:E→R.
- Assume that G contains no negative-weight cycles reachable from the source vertex s ∈ V, so that shortest paths are well defined.
- A shortest-path tree rooted at s is a directed subgraph G′=(V′, E′), where V′⊂V and E′ ⊂ E, such that
  - V′ is the set of vertices reachable from s in G.
  - G′ forms a rooted tree with root s.
  - For all v ∈ V′, the unique simple path from s to v in G′ is a shortest path from s to v in G.

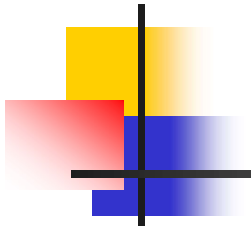


# Relaxation

- The algorithms in this chapter use the technique of relaxation.
- For all $v \in V$, we maintain an attribute v.d which is an upper bound on the weight of a shortest path from source s to v.
- We call v.d a shortest-path estimate.

INITIALIZE-SINGLE-SOURCE(G,s)

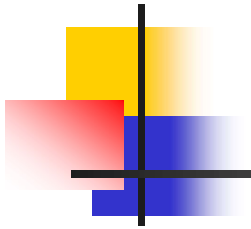
1. **for** each vertex $v \in G.V$
2. $v.d = \infty$
3. $v.\pi$ = NIL
4. s.d = 0



# Relaxation

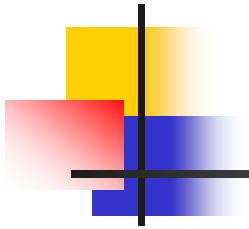
- The process of relaxing an edge (u,v) consists of testing whether we can improve the shortest path to v found so far by going through u and, if so, updating v.d and $v.\pi$.
- The following code performs a relaxation step on edge (u,v) in O(1) time.

```
RELAX(u, v, w)
1.    if  v.d>u.d + w(u,v)
2.          v.d = u.d + w(u,v)
3.          v.π = u
```



# Relaxation

- Each algorithm in this chapter calls INITIALIZE-SINGLE-SOURCE and then repeatedly relaxes edges.
- Moreover, relaxation is the only means by which shortest path estimates and predecessors change.
- The algorithms in this chapter differ in how many times they relax each edge and the order in which they relax edges.
  - The Bellman-Ford algorithm relaxes each edge |V|-1 times.
  - Dijkstra's algorithm for directed acyclic graphs relaxes each edge exactly once.



# Initialize

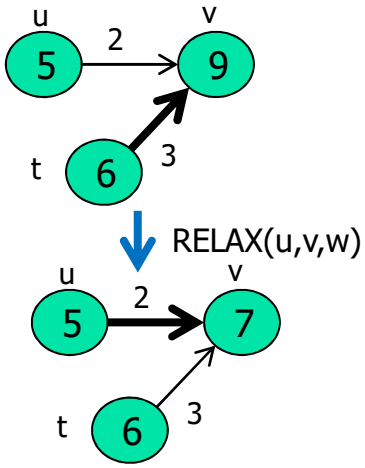
INITIALIZE-SINGLE-SOURCE(G,s)

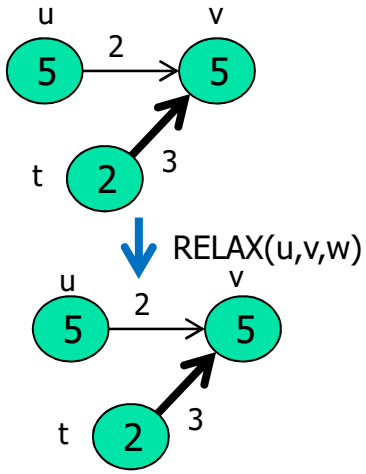
1. **for** each vertex $v \in G.V$
2. $v.d = \infty$
3. $v.\pi = NIL$
4. $s.d = 0$

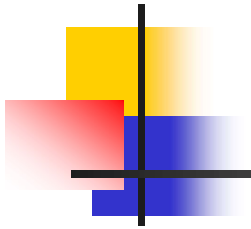
# Relaxation

RELAX(u, v, w)

1. **if** v.d>u.d + w(u,v)
2. v.d = u.d + w(u,v)
3. v.$\pi$ = u



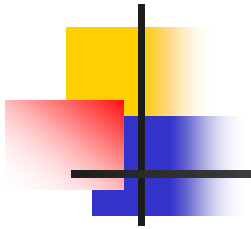





# Properties of Shortest Paths and Relaxation

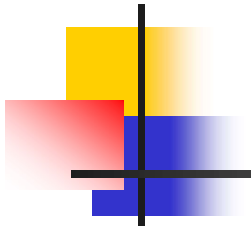
- Triangle inequality (Lemma 24.10)
- Upper-bound property (Lemma 24.11)
- No-path property (Corollary 24.12)
- Convergence property (Lemma 24.14)
- Path-relaxation property (Lemma 24.15)
- Predecessor-subgraph property (Lemma 24.17)

# Triangle Inequality (Lemma 24.10)



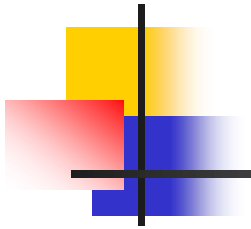
- Let G=(V,E) be a weighted, directed graph with weight function $w : E \rightarrow R$ and source vertex s. Then, for all edges $(u,v) \in E$, we have $\delta(s,v) \leq \delta(s,u)+w(u,v)$
- Proof:
  - Suppose that there is a shortest path p from source s to v. Then p has no more weight than any other path from s to v. Specifically, path p has no more weight than the particular path that takes a shortest path from source s to vertex u and then takes edge (u,v).
  - Otherwise, (there is no shortest path from s to v). Exercise 24.5-3.

# Upper-bound Property (Lemma 24.11)



- Let
  - G=(V,E) be a weighted, directed graph with weight function w : $E \rightarrow R$
  - $s \in V$ the source vertex
- The graph G is initialized by INITILIZE-SINGLE-SOURCE(G,s).
- Then, we have $v.d \geq \delta(s,v)$ for all $v \in V$. This invariant is maintained over any sequence of relaxation steps on the edges of G. Moreover, once v.d achieves its lower bound $\delta(s,v)$, it never changes.

# Upper-bound Property (Lemma 24.11)

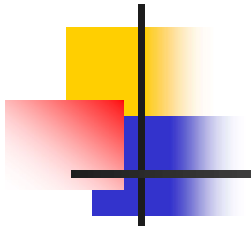


- Proof by induction over the number of relaxation steps
  - Basis case (0 relaxation)
    - $v.d = \infty \geq \delta(s,v)$ for all vertices $v \in V-\{s\}$
    - $s.d = 0 = \delta(s,s)$

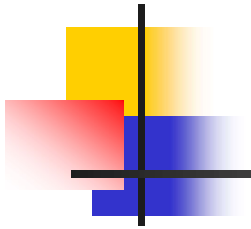
# Upper-bound Property (Lemma 24.11)

- Proof by induction over the number of relaxation steps
  - Induction step
    - Induction hypothesis: $v.d \geq \delta(s,v)$ for all $v \in V$ prior to the relaxation of an edge $(u,v)$
    - The only d value that may change is v.d. If it changes, we have
      $v.d = u.d + w(u,v) \geq \delta(s,u) + w(u,v) \geq \delta(s,v)$
  - To see that the value of v.d never changes once $v.d = \delta(s,v)$
    - v.d cannot decrease because $v.d \geq \delta(s,v)$
    - v.d cannot increase because relaxation steps do not increase d values

# No-path Property (Corollary 24.12)



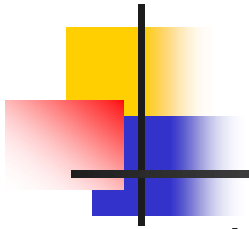
- Suppose that in a weighted, directed graph $G=(V,E)$ with weight function $w:E \rightarrow R$, no path connects a source $s$ to a given vertex $v$.
- Then, after the graph is initialized by INITIALIZE-SINGLE-SOURCE(G,s), we have $v.d = \delta(s,v) = \infty$ and this equality is maintained as an invariant over any sequence of relaxation steps on the edges of G.
- Proof: By the upper-bound property, we always have $\infty = \delta(s,v) \leq v.d$ and thus we have $v.d = \infty = \delta(s,v)$.



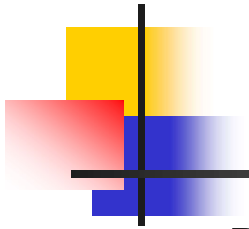
# Lemma 24.13

- Let G=(V,E) be a weighted, directed graph with weight function $w:E \rightarrow R$, and let $(u,v) \in E$.
- Then, immediately after relaxing edge (u,v) by executing RELAX(u,v,w), we have $v.d \leq u.d+w(u,v)$.
- Proof: if, just prior to relaxing edge (u,v), we have $v.d>u.d+w(u,v)$, then before $v.d=u.d+w(u,v)$ afterward. If, instead, $v.d \leq u.d+w(u,v)$ just before the relaxation, then neither u.d nor v.d changes, and so $v.d \leq u.d+w(u,v)$ afterward.

# Convergence Property (Lemma 24.14)

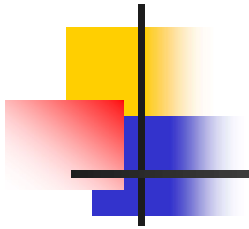


- Let G=(V,E) be a weighted, directed graph with weight function $w : E \rightarrow R$.
- Let $s \in V$ the source vertex.
- Let $s \rightsquigarrow u \rightarrow v$ be a shortest path in G for some vertices $u,v \in V$.
- Let the graph is initialized by INITILIZE-SINGLE-SOURCE(G,s) and then a sequence of relaxation steps that includes the call RELAX(u,v,w) is executed on the edge of G.
- If $u.d = \delta(s,u)$ at any time prior to the call, then we have $v.d = \delta(s,v)$ at all times after the call.



# Convergence Property (Proof)

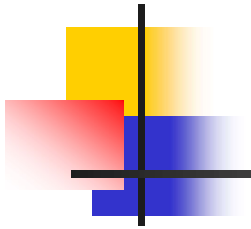
- By the upper-bound property, if $u.d = \delta(s,u)$ at some point prior to relaxing edge (u,v), this equality holds thereafter.
- In particular, after relaxing edge (u,v), we have $v.d \leq u.d+w(u,v) = \delta(s,u)+w(u,v) = \delta(s,v)$.
- By the upper-bound property, $v.d \geq \delta(s,v)$ from which we conclude that $v.d = \delta(s,v)$.
- Thus, this equality is maintained thereafter.



# Shortest-paths Properties

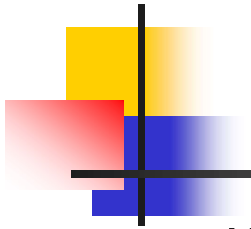
- Triangle Inequality
  - For any edge $(u,v) \in E$ , we have $\delta(s,v) \leq \delta(s,u)+w(u,v)$.
- Upper-bound property
  - We always have $v.d \geq \delta(s,v)$ for all $v \in V$ , and once v.d achieves the value $\delta(s,v)$, it never changes.
- No-path property
  - If there is no path from s to v, then we always have $v.d = \infty = \delta(s,v)$.
- Convergence property
  - If $s \rightsquigarrow u \rightarrow v$ be a shortest path in G for some vertices $u,v \in V$, and if $u.d = \delta(s,u)$ at any time prior to relaxing edge (u,v), then $v.d = \delta(s,v)$ at all times afterward.

# Path-relaxation Property (Lemma 24.15)



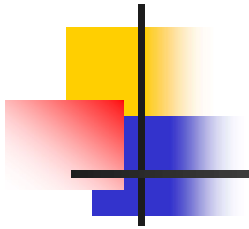
- If $p=\langle v_0,v_1,\ldots,v_k\rangle$ is a shortest path from $s=v_0$ to $v_k$, and the edges of p are relaxed in the order $(v_0,v_1),(v_1,v_2),\ldots,(v_{k-1},v_k)$, then $v_k.d=\delta(s,v_k)$.
- This property holds regardless of any other relaxation steps that occur, even if they are intermixed with relaxations of the edges of p.

# Path-relaxation Property (Proof)

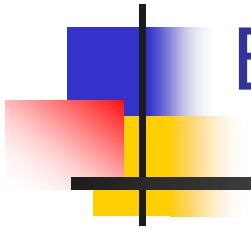


- We show by induction that after i-th edge of path $p=\langle v_0,v_1,\ldots,v_k\rangle$ is relaxed, we have $v_i.d=\delta(s,v_i)$.
- For the basis, i=0, and before any edges of p have been relaxed, we have from the initialization that $v_0.d=s.d=0=\delta(s,s)$. By the upper-bound property, the value of s.d never changes after initialization.
- For the induction step, we assume that $v_{i-1}.d = \delta(s,v_{i-1})$, and we examine the relaxation of edge $(v_{i-1},v_i)$. By the convergence property, after relaxation, we have $v_i.d=\delta(s,v_i)$, and this equality is maintained at all times thereafter.

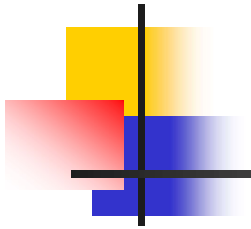
# Predecessor-subgraph Property (Lemma 24.17)



- Let G=(V,E) be a weighted, directed graph with weight function $w:E \rightarrow R$, let $s \in V$ be a source vertex, and assume that G contains no negative-weight cycles that are reachable from s.
- Let us call INITIALIZE-SINGLE-SOURCE(G, s) and then execute any sequence of relaxation steps on edges of G that produces $v.d=\delta(s,v)$ for all $v \in V$.
- Then, the predecessor subgraph G is a shortest-path tree rooted at s.
- Proof is omitted.



# Bellman-Ford Algorithm



# Bellman-Ford Algorithm

```
BELLMAN-FORD(G, w, s)
1.      INITIALIZE-SINGLE-SOURCE(G,s)
2.      for i=1 to |G.V|-1
3.          for each edge (u,v) ∈ G.E
4.              RELAX(u, v, w)
5.      for each edge (u,v) ∈ G.E
6.          if  v.d > u.d + w(u,v)
7.              return FALSE
8.      return TRUE
```