Time-Varying Fields

Introduction to Electromagnetism with Practice

Theory & Applications

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Ampère-Maxwell Law

Derivation of Ampère-Maxwell Law

Magnetostatic Case

$$\nabla \times \mathbf{H} = \mathbf{J}$$

Consider the continuity Eq.

$$\nabla \cdot \mathbf{J} = -\frac{d\rho}{dt}$$

$$\nabla \cdot \left(\nabla \times \mathbf{H} \right) = 0 = \nabla \cdot \mathbf{J} = -\frac{d\rho}{dt}$$

Magnetostatics already assumes that
there is no source/sink of charges…

Derivation of Ampère-Maxwell Law

$$\nabla \times \mathbf{H} = \mathbf{J} + \mathbf{K}$$

$$\nabla \cdot \left(\nabla \times \mathbf{H} \right) = 0 = \nabla \cdot \mathbf{J} + \nabla \cdot \mathbf{K}$$

$$= -\frac{d\rho}{dt} + \nabla \cdot \mathbf{K}$$

$$\nabla \cdot \mathbf{D} = \rho$$

$$-\frac{d\nabla \cdot \mathbf{D}}{dt} + \nabla \cdot \mathbf{K} = \mathbf{O}$$

$$\mathbf{K} = \frac{d\mathbf{D}}{dt}$$

Ampère-Maxwell Law

Ampère-Maxwell Law

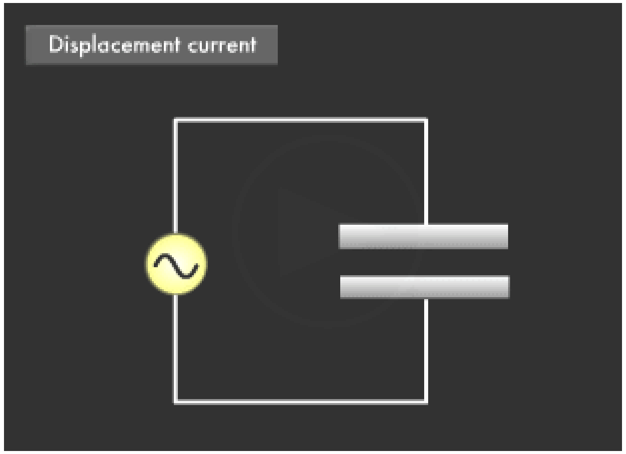
$$\boxed{\nabla \times \mathbf{H} = \mathbf{J} + \frac{d\mathbf{D}}{dt}}$$

Displacement Current Density

Ampère-Maxwell Law

$$\boxed{\nabla \times \mathbf{H} = \mathbf{J} + \frac{d\mathbf{D}}{dt}}$$

Displacement Current Density



https://global.canon/en/technology/s_labo/light/001/11.html

Faraday’s Law

Remind: Electromotive Force (emf)

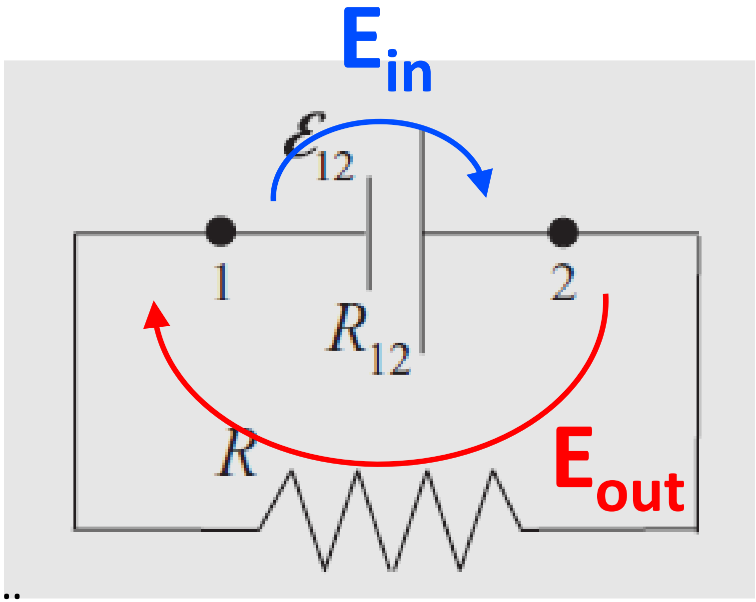
$$\oint_C \mathbf{E}\cdot d\mathbf{l} = \int_1^2 \mathbf{E}_{\text{in}}\cdot d\mathbf{l} + \int_2^1 \mathbf{E}_{\text{out}}\cdot d\mathbf{l} = 0$$

$$-\int_1^2 \mathbf{E}_{\text{in}}\cdot d\mathbf{l} = +\int_2^1 \mathbf{E}_{\text{out}}\cdot d\mathbf{l}$$

Electromotive Force (emf) ➔ Voltage Source

$$\boxed{\gamma = -\int_1^2 \mathbf{E}_{\text{in}}\cdot d\mathbf{l} = +\int_2^1 \mathbf{E}_{\text{out}}\cdot d\mathbf{l}}$$

Allowing potential drop from (+) to (–) electrode



Electric Battery, Electric Generator, Solar Cells, …

Remind: Lorentz's Force Equation

Lorentz's Force Equation

$$\boxed{\mathbf{F} = \mathbf{F}_e + \mathbf{F}_m = q\left(\mathbf{E} + \mathbf{v} \times \mathbf{B}\right)}$$

Faraday's Law of Induction

Differential Form

$$\boxed{\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}}$$

historical development

easy to derive…

Integral Form

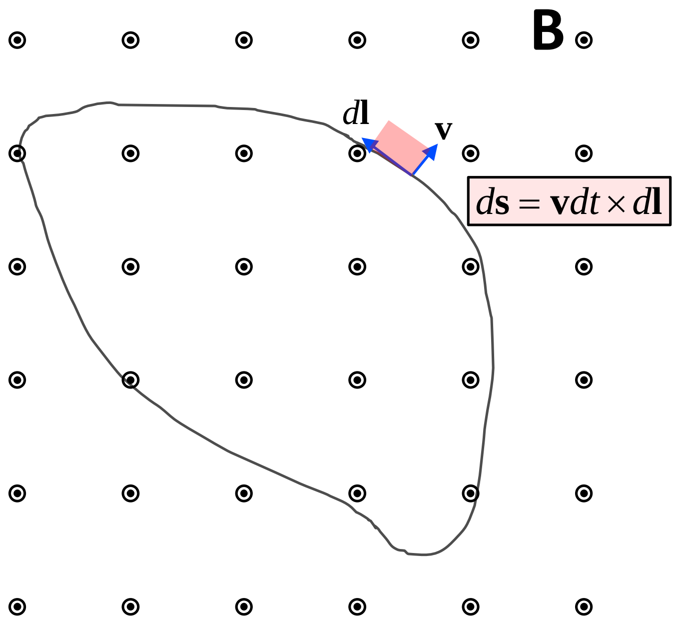
$$\boxed{\gamma_{\mathrm{F}} = -\frac{d\Phi}{dt}}$$

From the differential form,

proving the integral form & introducing its experimental evidences

The key of derivation is on the definition of γ_{F} = Faraday's EMF

Toward Faraday's EMF



$$\Phi = \int_S \mathbf{B} \cdot d\mathbf{s}$$

$$\frac{d\Phi}{dt} = \frac{d}{dt}\int_{S(t)} \mathbf{B}(t) \cdot d\mathbf{s}$$

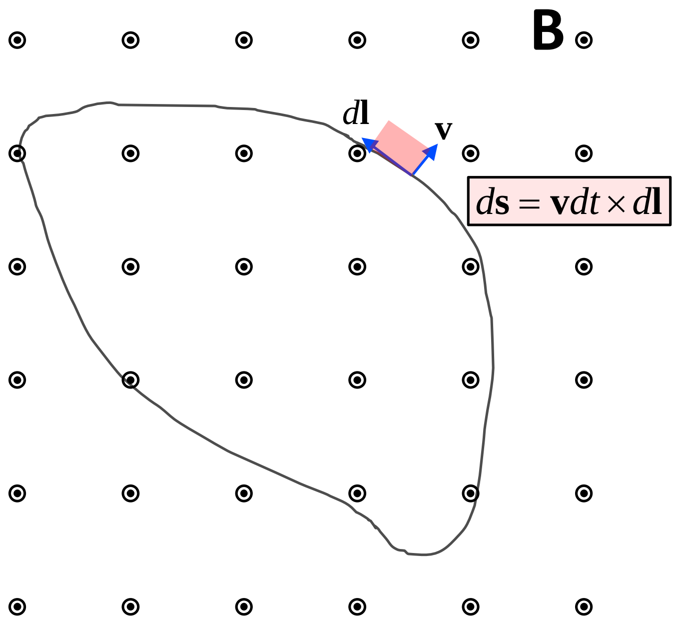
$$\left.\frac{d\Phi}{dt}\right|_{t=t_0} = \int_{S(t=t_0)} \left.\frac{\partial \mathbf{B}(t)}{\partial t}\right|_{t=t_0} \cdot d\mathbf{s} + \frac{d}{dt}\int_{S(t)} \mathbf{B}(t_0) \cdot d\mathbf{s}$$

Transformer EMF

Motional EMF

Toward Faraday's EMF

$$\boxed{\nabla\times\mathbf{E}=-\frac{\partial\mathbf{B}}{\partial t}}$$



$$\int_{S(t=t_0)}\left.\frac{\partial\mathbf{B}(t)}{\partial t}\right|_{t=t_0}\cdot d\mathbf{s}=-\oint_{C(t=t_0)}\mathbf{E}(t_0)\cdot d\mathbf{l}$$

$$\begin{aligned}\frac{d}{dt}\int_{S(t)}\mathbf{B}(t_0)\cdot d\mathbf{s}&=\frac{d}{dt}\oint_{C(t=t_0)}\mathbf{B}(t_0)\cdot\left(\mathbf{v}dt\times d\mathbf{l}\right)\\&=\frac{d}{dt}\oint_{C(t=t_0)}d\mathbf{l}\cdot\left(\mathbf{B}(t_0)\times\mathbf{v}dt\right)\\&=-\oint_{C(t=t_0)}\left(\mathbf{v}\times\mathbf{B}(t_0)\right)\cdot d\mathbf{l}\end{aligned}$$

$$\left.\frac{d\Phi}{dt}\right|_{t=t_0}=\underbrace{\int_{S(t=t_0)}\left.\frac{\partial\mathbf{B}(t)}{\partial t}\right|_{t=t_0}\cdot d\mathbf{s}}_{\text{Transformer EMF}}+\underbrace{\frac{d}{dt}\int_{S(t)}\mathbf{B}(t_0)\cdot d\mathbf{s}}_{\text{Motional EMF}}$$

Toward Faraday's EMF

Transformer EMF Motional EMF

$$\left.\frac{d\Phi}{dt}\right|_{t=t_0} = -\oint_{C(t=t_0)} \mathbf{E}(t_0)\cdot d\mathbf{l} - \oint_{C(t=t_0)} \left(\mathbf{v}\times\mathbf{B}(t_0)\right)\cdot d\mathbf{l}$$

$$= -\oint_{C(t=t_0)} \left[\mathbf{E}(t_0) + \mathbf{v}\times\mathbf{B}(t_0)\right]\cdot d\mathbf{l}$$

$$\boxed{\frac{d\Phi}{dt} = -\oint_C \left[\mathbf{E} + \mathbf{v}\times\mathbf{B}\right]\cdot d\mathbf{l} = -\gamma_{\mathrm{F}}}$$

$$\boxed{\gamma_{\mathrm{F}} = \oint_C \left[\mathbf{E} + \mathbf{v}\times\mathbf{B}\right]\cdot d\mathbf{l} = -\frac{d\Phi}{dt}}$$

γ_F: Faraday's EMF

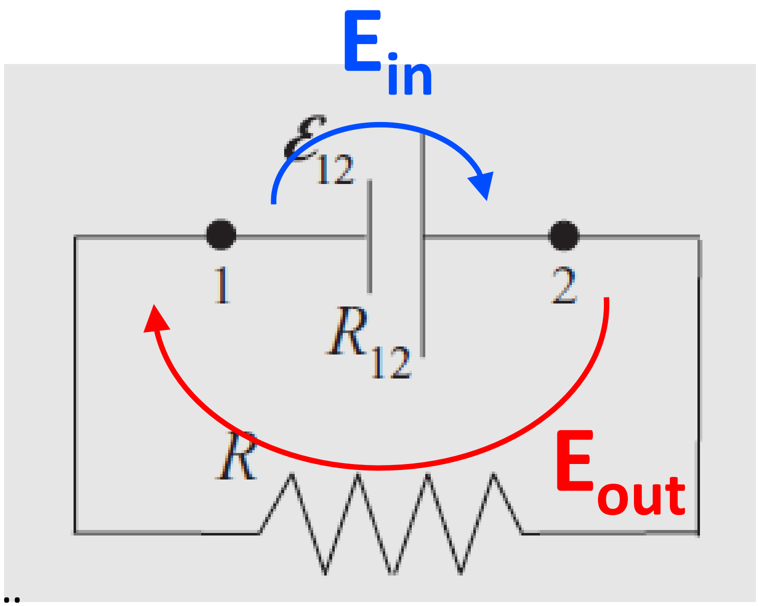
Understanding Faraday's Law from EMF Viewpoint

Electromotive Force (emf) ➔ Voltage Source

$$\boxed{\gamma = -\int_1^2 \mathbf{E}_{\text{in}} \cdot d\mathbf{l} = +\int_2^1 \mathbf{E}_{\text{out}} \cdot d\mathbf{l}}$$

Allowing potential drop from (+) to (–) electrode

Electric Battery, Electric Generator, Solar Cells, …



Source **Work**

$$\boxed{-\frac{d\Phi}{dt} = \gamma_{\text{F}} = \oint_C [\mathbf{E} + \mathbf{v} \times \mathbf{B}] \cdot d\mathbf{l} = \oint_C \frac{q[\mathbf{E} + \mathbf{v} \times \mathbf{B}]}{q} \cdot d\mathbf{l} = \oint_C \frac{\mathbf{F}}{q} \cdot d\mathbf{l}}$$

Understanding Faraday's Law from Experiments

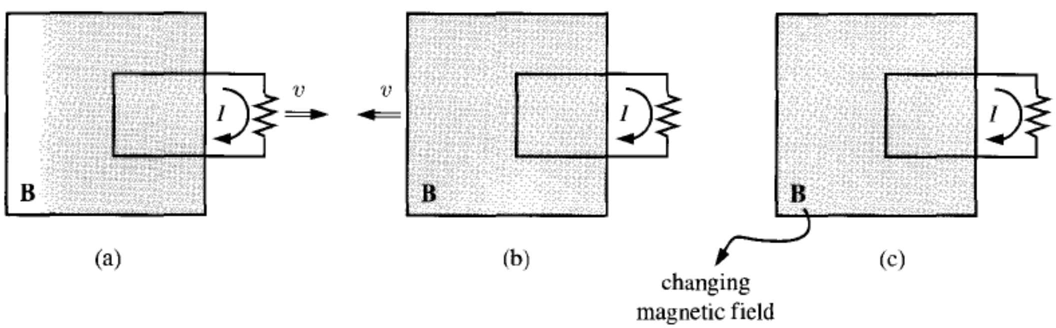
Experimental Evidences

In 1831 Michael Faraday reported on a series of experiments, including three that (with some violence to history) can be characterized as follows:

Experiment 1. He pulled a loop of wire to the right through a magnetic field (Fig. 7.20a). A current flowed in the loop.

Experiment 2. He moved the *magnet* to the *left*, holding the loop still (Fig. 7.20b). Again, a current flowed in the loop.

Experiment 3. With both the loop and the magnet at rest (Fig. 7.20c), he changed the *strength* of the field (he used an electromagnet, and varied the current in the coil). Once again, current flowed in the loop.



Proof!

$$\gamma_F = -\frac{d\Phi}{dt}$$

We prove it!
(or showing mathematical identity)

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$

Faraday's Law of Electromagnetic Induction

Faraday's Law of Electromagnetic Induction

$$\boxed{\gamma_{\mathrm{F}} = -\frac{d\Phi}{dt}}$$

Lenz's law

The induced current in a circuit from a changing magnetic field is directed to oppose the change in flux and to exert a mechanical force which opposes the motion.

$$\gamma_{\mathrm{F}} = \oint_C \left[\mathbf{E} + \mathbf{v} \times \mathbf{B}\right] \cdot d\mathbf{l}$$

$$\Phi = \int_S \mathbf{B} \cdot d\mathbf{s}$$

Solving Faraday's Law Problems

$$\gamma_{\mathrm{F}} = \oint_C \left[\mathbf{E} + \mathbf{v} \times \mathbf{B}\right] \cdot d\mathbf{l} = -\frac{d\Phi}{dt}$$

γ_{F}: Faraday's EMF

Time-Varying Magnetic Fields	Time-Varying Area
Transformer EMF	Motional EMF

$$\gamma_{\mathrm{F}} = \oint_C \mathbf{E} \cdot d\mathbf{l} = -\int_S \frac{d\mathbf{B}}{dt} \cdot d\mathbf{s} = -\frac{d\Phi}{dt}$$

$$\gamma_{\mathrm{F}} = \oint_C \left(\mathbf{v} \times \mathbf{B}\right) \cdot d\mathbf{l} = -\frac{d\Phi}{dt}$$

Example 030: Transformer EMF

EXAMPLE 7–1 A circular loop of N turns of conducting wire lies in the xy-plane with its center at the origin of a magnetic field specified by $\mathbf{B} = \mathbf{a}_z B_0 \cos(\pi r/2b) \sin \omega t$, where b is the radius of the loop and ω is the angular frequency. Find the emf induced in the loop.

Solution The problem specifies a stationary loop in a time-varying magnetic field; hence Eq. (7–6) can be used directly to find the induced emf, $\mathscr{V}$. The magnetic flux linking each turn of the circular loop is

$$\begin{aligned}\Phi &= \int_S \mathbf{B} \cdot d\mathbf{s} \\ &= \int_0^b \left[\mathbf{a}_z B_0 \cos\frac{\pi r}{2b} \sin \omega t\right] \cdot (\mathbf{a}_z 2\pi r\, dr) \\ &= \frac{8b^2}{\pi}\left(\frac{\pi}{2} - 1\right) B_0 \sin \omega t.\end{aligned}$$

Since there are N turns, the total flux linkage is $N\Phi$, and we obtain

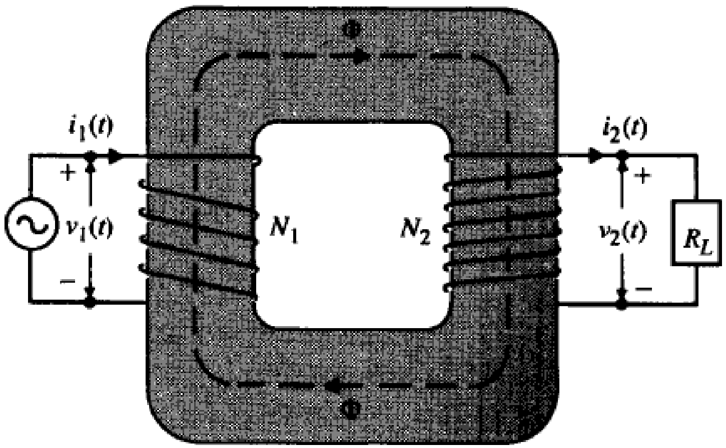
$$\begin{aligned}\mathscr{V} &= -N\frac{d\Phi}{dt} \\ &= -\frac{8N}{\pi} b^2 \left(\frac{\pi}{2} - 1\right) B_0 \omega \cos \omega t \qquad \text{(V)}.\end{aligned}$$

The induced emf is seen to be 90° out of time phase with the magnetic flux. ∎

Example 031: Transformer EMF

Ratio between *v* & *i*?

A transformer is an alternating-current (a-c) device that transforms voltages, currents, and impedances. It usually consists of two or more coils coupled magnetically through a common ferromagnetic core, such as that sketched in Fig. 7–1. Faraday's law of electromagnetic induction is the principle of operation of transformers.



(a) Schematic diagram of a transformer.

For the closed path in the magnetic circuit in Fig. 7–1(a) traced by magnetic flux Φ, we have, from Eq. (6–101),

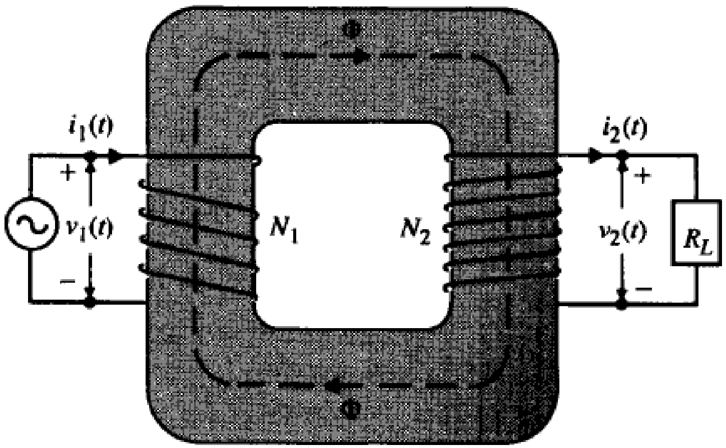
$$N_1 i_1 - N_2 i_2 = \mathscr{R}\Phi, \tag{7–7}$$

where N_1, N_2 and i_1, i_2 are the numbers of turns and the currents in the primary and secondary circuits, respectively, and $\mathscr{R}$ denotes the reluctance of the magnetic circuit. In Eq. (7–7) we have noted, in accordance with Lenz's law, that the induced mmf in the secondary circuit, $N_2 i_2$, opposes the flow of the magnetic flux Φ created by the mmf in the primary circuit, $N_1 i_1$. From Section 6–8 we know that the reluctance of the ferromagnetic core of length ℓ, cross-sectional area S, and permeability μ is

$$\mathscr{R} = \frac{\ell}{\mu S}. \tag{7–8}$$

Example 031: Transformer EMF

Ratio between *v* & *i*?



(a) Schematic diagram of a transformer.

Substituting Eq. (7–8) in Eq. (7–7), we obtain

$$N_1 i_1 - N_2 i_2 = \frac{\ell}{\mu S}\Phi. \tag{7–9}$$

a) *Ideal transformer*. For an ideal transformer we assume that $\mu \to \infty$, and Eq. (7–9) becomes

$$\boxed{\frac{i_1}{i_2} = \frac{N_2}{N_1}.} \tag{7–10}$$

Equation (7–10) states that ***the ratio of the currents in the primary and secondary windings of an ideal transformer is equal to the inverse ratio of the numbers of turns***. Faraday's law tells us that

$$v_1 = N_1 \frac{d\Phi}{dt} \tag{7–11}$$

and

$$v_2 = N_2 \frac{d\Phi}{dt}, \tag{7–12}$$

the proper signs for v_1 and v_2 having been taken care of by the designated polarities in Fig. 7–1(a). From Eqs. (7–11) and (7–12) we have

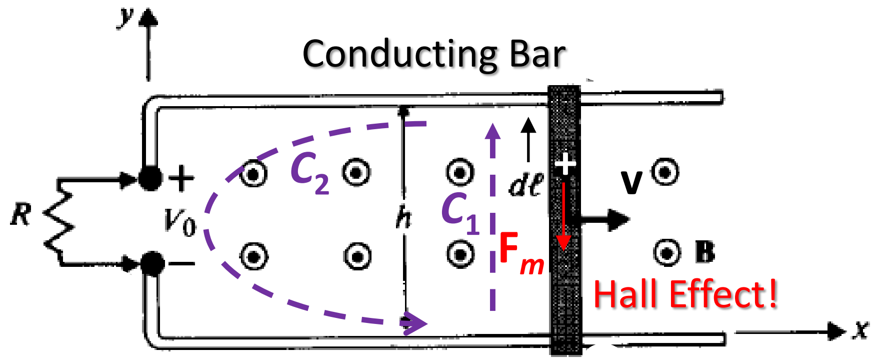
$$\boxed{\frac{v_1}{v_2} = \frac{N_1}{N_2}.} \tag{7–13}$$

Thus, ***the ratio of the voltages across the primary and secondary windings of an an ideal transformer is equal to the turns ratio.***

Example 032: Motional EMF

Find the emf in the following system

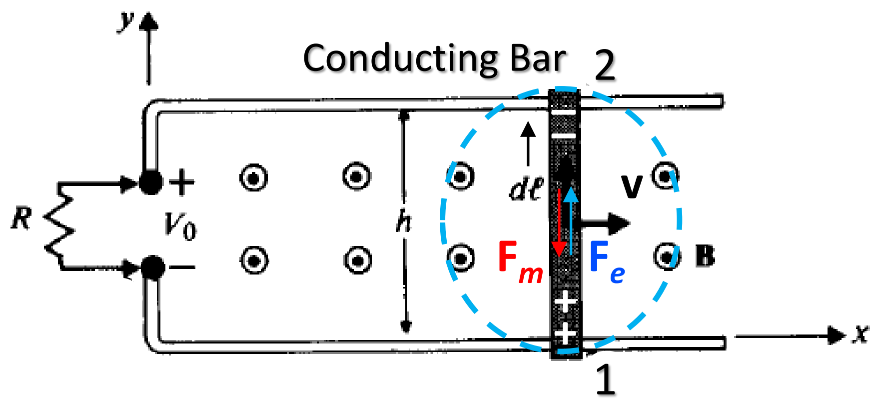
$$\gamma_F = \oint_C \left[\mathbf{E} + \mathbf{v}\times\mathbf{B}\right]\cdot d\mathbf{l} = -\frac{d\Phi}{dt}$$



$$\frac{d\Phi}{dt} = Blv$$

$$\mathbf{F} = \mathbf{F}_e + \mathbf{F}_m = q\left(\mathbf{E} + \mathbf{v}\times\mathbf{B}\right)$$

$$\gamma_F = -Blv$$



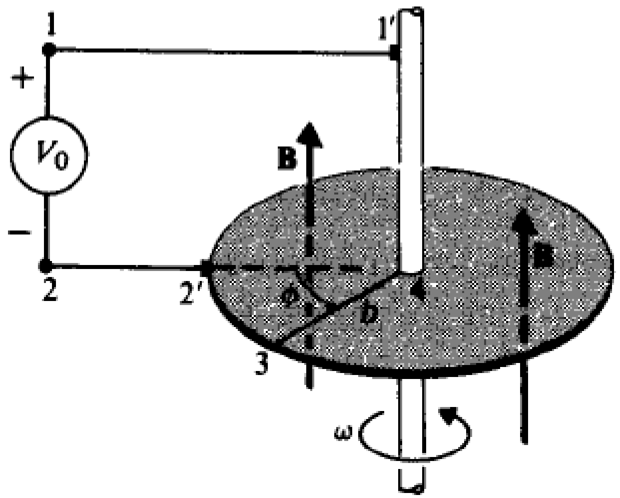
$$E = vB$$

$$\gamma_F = -El$$

Accumulating until $F_e = F_m$

Example 033: Motional EMF

EXAMPLE 7–3 The ***Faraday disk generator*** consists of a circular metal disk rotating with a constant angular velocity ω in a uniform and constant magnetic field of flux density $\mathbf{B} = \mathbf{a}_z B_0$ that is parallel to the axis of rotation. Brush contacts are provided at the axis and on the rim of the disk, as depicted in Fig. 7–4. Determine the open-circuit voltage of the generator if the radius of the disk is b.



Solution Let us consider the circuit 122′341′1. Of the part 2′34 that moves with the disk, only the straight portion 34 "cuts" the magnetic flux. We have, from Eq. (7–24),

$$\begin{aligned} V_0 &= \oint (\mathbf{u} \times \mathbf{B}) \cdot d\boldsymbol{\ell} \\ &= \int_3^4 [(\mathbf{a}_\phi r\omega) \times \mathbf{a}_z B_0] \cdot (\mathbf{a}_r\, dr) \qquad (7\text{–}30) \\ &= \omega B_0 \int_b^0 r\, dr = -\frac{\omega B_0 b^2}{2} \qquad \text{(V)}, \end{aligned}$$

which is the emf of the Faraday disk generator. To measure V_0, we must use a voltmeter of a very high resistance so that no appreciable current flows in the circuit to modify the externally applied magnetic field. ■

Maxwell’s Equations

General Form: Light in Media

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$

$$\nabla \times \mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t}$$

$$\nabla \cdot \mathbf{D} = \rho$$

$$\nabla \cdot \mathbf{B} = 0$$

Maxwell's Equations
in ***General Media***

$$\mathbf{D} = \mathbf{D}(\mathbf{E}, \mathbf{H})$$

$$\mathbf{B} = \mathbf{B}(\mathbf{E}, \mathbf{H})$$

Also, functions of $\mathbf{x}, t, \mathbf{k}, \omega$

$$\mathbf{D} = \varepsilon_0 \varepsilon_{\mathrm{r}} \mathbf{E}$$

$$\mathbf{B} = \mu_0 \mu_{\mathrm{r}} \mathbf{H}$$

Lorentz's Force Equation

$$\mathbf{F} = \mathbf{F}_e + \mathbf{F}_m = q\left(\mathbf{E} + \mathbf{v} \times \mathbf{B}\right)$$

General Form: Light in the Vacuum

$$\nabla \times \mathbf{E} = -\mu_0 \frac{\partial \mathbf{H}}{\partial t}$$

$$\nabla \times \mathbf{H} = \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t}$$

$$\nabla \cdot \mathbf{E} = 0$$

$$\nabla \cdot \mathbf{H} = 0$$

Maxwell's Equations
in the ***Vacuum***

$\mu_0 = 4\pi \times 10^{-7}$ (H/m)
$\varepsilon_0 = 8.854 \times 10^{-12}$ (F/m)

Integral Forms of Maxwell's Equations

TABLE 7–2
Maxwell's Equations

Differential Form	Integral Form	Significance
$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$	$\oint_C \mathbf{E} \cdot d\boldsymbol{\ell} = -\frac{d\Phi}{dt}$	Faraday's law
$\nabla \times \mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t}$	$\oint_C \mathbf{H} \cdot d\boldsymbol{\ell} = I + \int_S \frac{\partial \mathbf{D}}{\partial t} \cdot d\mathbf{s}$	Ampère's circuital law
$\nabla \cdot \mathbf{D} = \rho$	$\oint_S \mathbf{D} \cdot d\mathbf{s} = Q$	Gauss's law
$\nabla \cdot \mathbf{B} = 0$	$\oint_S \mathbf{B} \cdot d\mathbf{s} = 0$	No isolated magnetic charge

Electromagnetic Boundary Conditions

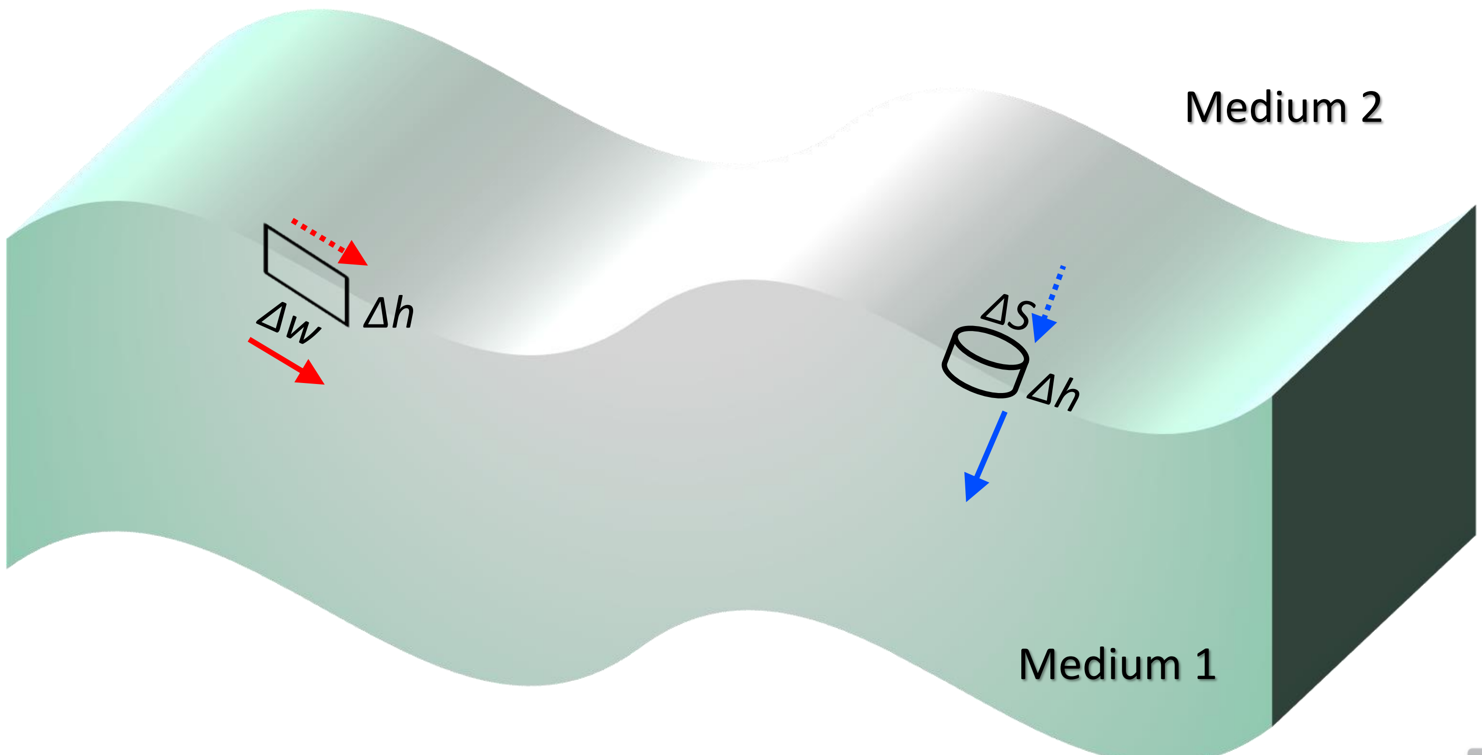
Strategy for Boundary Conditions

I. Boundary includes "different" materials ➔ Integral forms are proper

II. Stokes ➔ "Closed Loop" across materials
Gauss ➔ "Closed Surface" across materials

$$\oint_C \mathbf{E} \cdot d\mathbf{l} = -\int_S \frac{d\mathbf{B}}{dt} \cdot d\mathbf{s}, \qquad \oint_S \mathbf{D} \cdot d\mathbf{S} = Q$$

III. Loop measures tangential fields & Surface measures normal fields



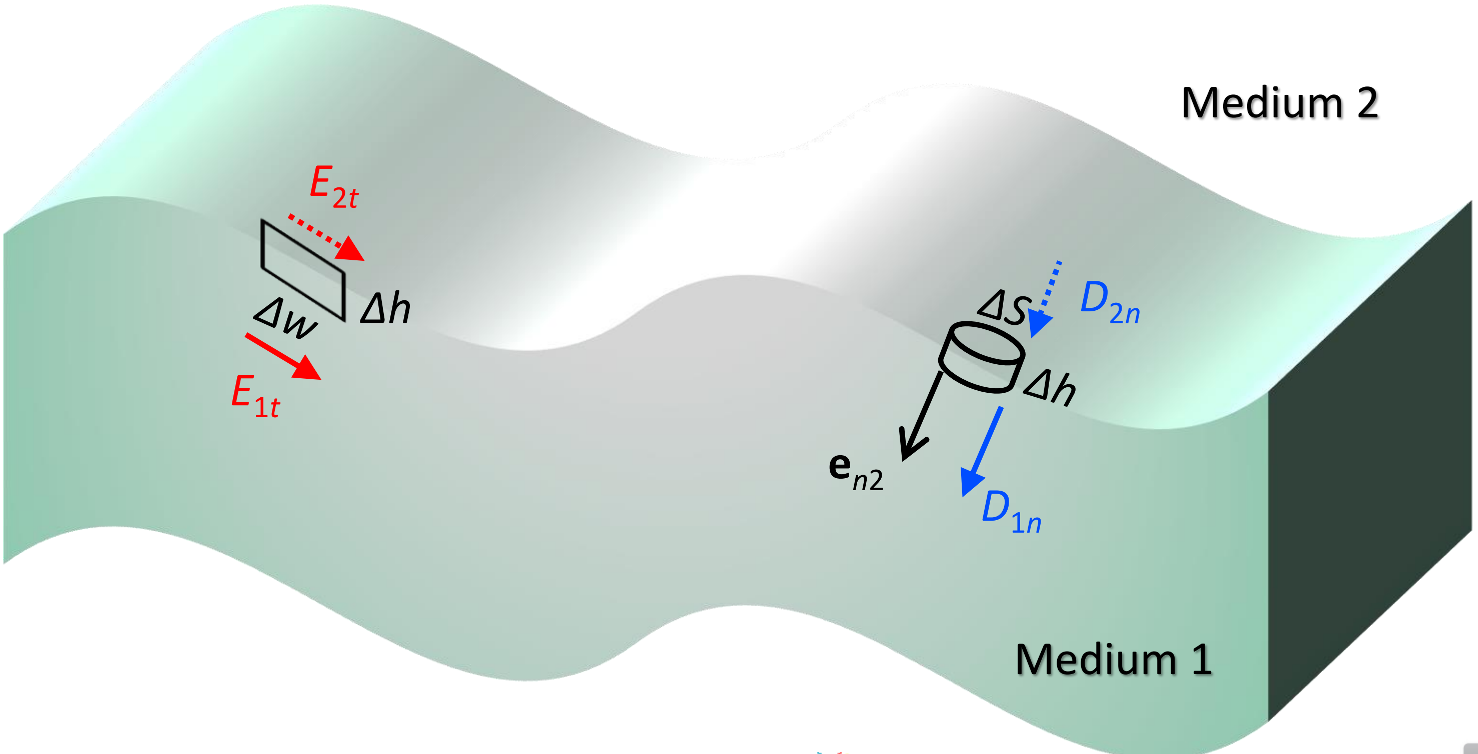
Analyzing Boundary Conditions

$\Delta h \to 0$ to characterize the "boundary"

$$\oint_C \mathbf{E}\cdot d\mathbf{l}\Big|_{\Delta h=0} = E_{1t}\Delta w - E_{2t}\Delta w$$

$$= -\int_S \frac{d\mathbf{B}}{dt}\cdot d\mathbf{s}\Big|_{\Delta h=0} = 0$$

$$\oint_S \mathbf{D}\cdot d\mathbf{S} = \mathbf{e}_{n2}\cdot\left(\mathbf{D}_1 - \mathbf{D}_2\right)\Delta S = \rho_s \Delta S$$



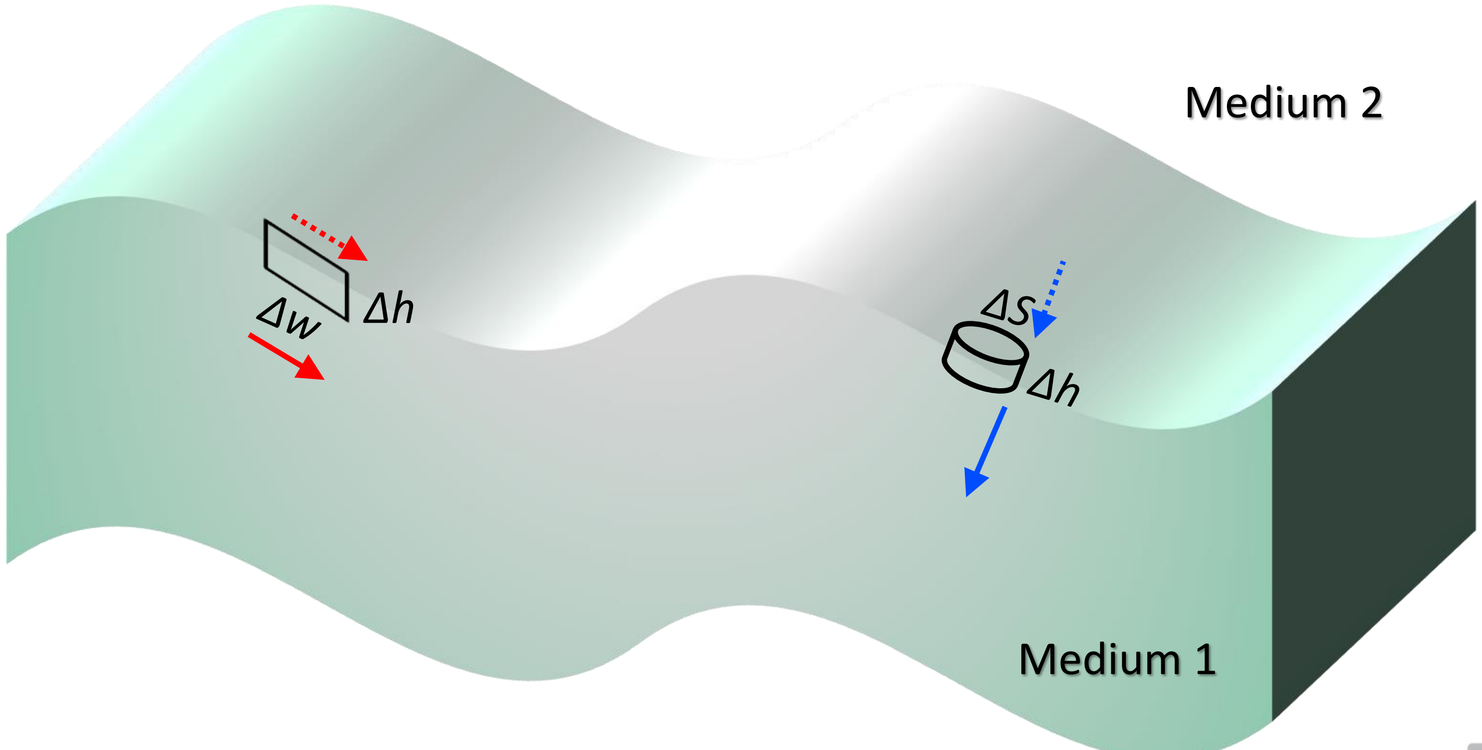
Strategy for Boundary Conditions

I. Boundary includes "different" materials ➔ Integral forms are proper

II. Stokes ➔ "Closed Loop" across materials
Gauss ➔ "Closed Surface" across materials

$$\oint_C \mathbf{H} \cdot d\mathbf{l} = \int_S \mathbf{J} \cdot d\mathbf{s} + \int_S \frac{d\mathbf{D}}{dt} \cdot d\mathbf{s}, \qquad \oint_S \mathbf{B} \cdot d\mathbf{S} = 0$$

III. Loop measures tangential fields & Surface measures normal fields



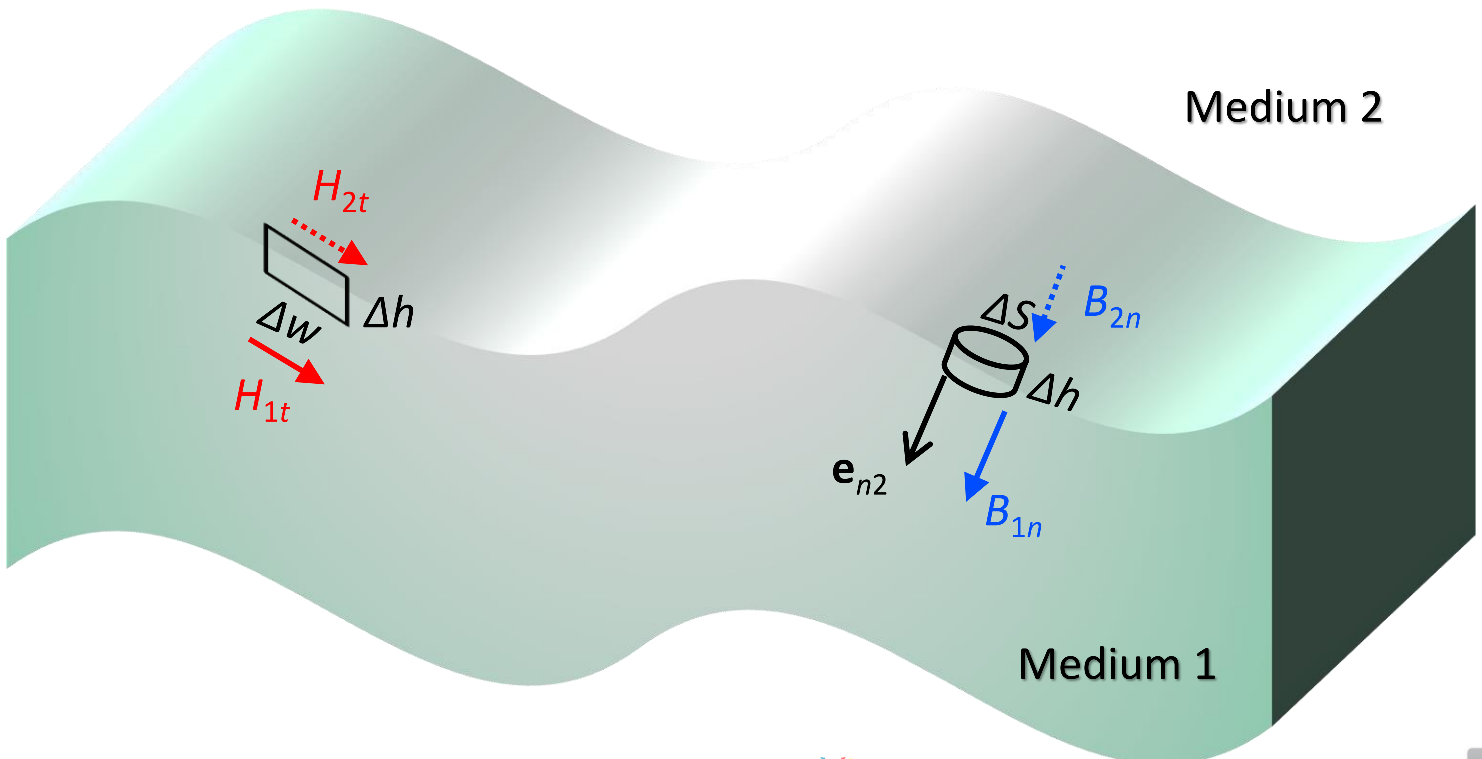
Analyzing Boundary Conditions

$\Delta h \to 0$ to characterize the "boundary"

$$\oint_C \mathbf{H}\cdot d\mathbf{l}\Big|_{\Delta h=0} = H_{1t}\Delta w - H_{2t}\Delta w$$

$$= J_{\mathrm{sn}}\Delta w + \int_S \frac{d\mathbf{D}}{dt}\cdot d\mathbf{s}\Big|_{\Delta h=0} = J_{\mathrm{sn}}\Delta w$$

$$\oint_S \mathbf{B}\cdot d\mathbf{S} = \mathbf{e}_{n2}\cdot(\mathbf{B}_1 - \mathbf{B}_2)\Delta S = 0$$



(General) EM Boundary Conditions

Tangential Fields

$$E_{1t} = E_{2t}$$

Normal Fields

$$\mathbf{e}_{n2} \cdot (\mathbf{D}_1 - \mathbf{D}_2) = \rho_s$$

Tangential Fields

$$\mathbf{e}_{n2} \times (\mathbf{H}_1 - \mathbf{H}_2) = \mathbf{J}_s$$

Normal Fields

$$B_{1n} = B_{2n}$$

Tangential Fields

$$\frac{J_{1t}}{\sigma_1} = \frac{J_{2t}}{\sigma_2}$$

Normal Fields

$$\mathbf{e}_{n2} \cdot (\mathbf{J}_1 - \mathbf{J}_2) = -\frac{d\rho}{dt}$$

Time-Varying Fields and Maxwell's Equations

Introduction to Electromagnetism with Practice

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Potential Functions

Remind: General Form: Light in Media

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$

$$\nabla \times \mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t}$$

$$\nabla \cdot \mathbf{D} = \rho$$

$$\nabla \cdot \mathbf{B} = 0$$

Maxwell's Equations
in ***General Media***

$$\mathbf{D} = \mathbf{D}(\mathbf{E}, \mathbf{H})$$

$$\mathbf{B} = \mathbf{B}(\mathbf{E}, \mathbf{H})$$

Also, functions of $\mathbf{x}$,t,$\mathbf{k}$,ω

$$\mathbf{D} = \varepsilon_0 \varepsilon_{\mathrm{r}} \mathbf{E}$$

$$\mathbf{B} = \mu_0 \mu_{\mathrm{r}} \mathbf{H}$$

Lorentz's Force Equation

$$\mathbf{F} = \mathbf{F}_e + \mathbf{F}_m = q\left(\mathbf{E} + \mathbf{v} \times \mathbf{B}\right)$$

Remind: General Form: Light in the Vacuum

$$\nabla \times \mathbf{E} = -\mu_0 \frac{\partial \mathbf{H}}{\partial t}$$

$$\nabla \times \mathbf{H} = \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t}$$

$$\nabla \cdot \mathbf{E} = 0$$

$$\nabla \cdot \mathbf{H} = 0$$

Maxwell's Equations in the ***Vacuum***

$\mu_0 = 4\pi \times 10^{-7}$ (H/m)
$\varepsilon_0 = 8.854 \times 10^{-12}$ (F/m)

Generalization of Electromagnetic Potentials

$$\nabla \times \mathbf{E} = \mathbf{O} \Rightarrow \mathbf{E} = -\nabla V$$

$$\nabla \cdot \mathbf{B} = 0 \Rightarrow \boxed{\mathbf{B} = \nabla \times \mathbf{A}}$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \Rightarrow \boxed{\mathbf{E} = -\nabla V - \frac{\partial \mathbf{A}}{\partial t}}$$

$$\Rightarrow \nabla \times \mathbf{E} = -\nabla \times \nabla V - \frac{\partial \nabla \times \mathbf{A}}{\partial t} = -\frac{\partial \mathbf{B}}{\partial t}$$

Electromagnetic Potentials

$$\boxed{\mathbf{E} = -\nabla V - \frac{\partial \mathbf{A}}{\partial t}}$$

$$\boxed{\mathbf{B} = \nabla \times \mathbf{A}}$$

Gauge Transformation & Gauge Freedom

Gauge Invariance ~ Observables

$$\boxed{\mathbf{E} = -\nabla V - \frac{\partial \mathbf{A}}{\partial t}}$$

$$\boxed{\mathbf{B} = \nabla \times \mathbf{A}}$$

Gauge Dependent

$$\boxed{V \rightarrow V - \frac{\partial \chi}{\partial t}}$$

$$\boxed{\mathbf{A} \rightarrow \mathbf{A} + \nabla \chi}$$

In Quantum Mechanics …

$$\boxed{\psi \rightarrow \psi \exp(iq\chi / \hbar)}$$

χ: Gauge Function

Gauge Transformation & Gauge Freedom

$$\boxed{\mathbf{E} = -\nabla V - \frac{\partial \mathbf{A}}{\partial t}}$$

$$\mathbf{E}' = -\nabla\left(V - \frac{\partial \chi}{\partial t}\right) - \frac{\partial\left(\mathbf{A} + \nabla\chi\right)}{\partial t} = -\nabla V - \frac{\partial \mathbf{A}}{\partial t} = \mathbf{E}$$

$$\boxed{\mathbf{B} = \nabla \times \mathbf{A}}$$

$$\mathbf{B}' = \nabla \times \left(\mathbf{A} + \nabla\chi\right) = \nabla \times \mathbf{A} = \mathbf{B}$$

$$V \rightarrow V - \frac{\partial \chi}{\partial t}$$

$$\mathbf{A} \rightarrow \mathbf{A} + \nabla\chi$$

Types of Gauges: Determining Potentials

Discussed later…

$$c^2 = \frac{1}{\mu_0 \varepsilon_0}$$

Coulomb Gauge

$$\boxed{\nabla \cdot \mathbf{A} = 0}$$

Lorentz Gauge

$$\boxed{\nabla \cdot \mathbf{A} + \frac{\mu_r \varepsilon_r}{c^2} \frac{\partial V}{\partial t} = 0}$$

$$V \to V - \frac{\partial \chi}{\partial t} \qquad \mathbf{A} \to \mathbf{A} + \nabla \chi$$

$$\nabla^2 \chi - \frac{\mu_r \varepsilon_r}{c^2} \frac{\partial^2 \chi}{\partial t^2} = 0$$

Why?

Interpreting Maxwell's Eqs with EM Potentials

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$

$$\Rightarrow \quad \nabla \times \left(\nabla V + \frac{\partial \mathbf{A}}{\partial t} \right) = \frac{\partial}{\partial t}(\nabla \times \mathbf{A})$$

$$\nabla \times \mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t}$$

$$\Rightarrow \quad \nabla \times \nabla \times \mathbf{A} = \mu_0 \mu_r \mathbf{J} - \mu_0 \varepsilon_0 \mu_r \varepsilon_r \frac{\partial}{\partial t} \left(\nabla V + \frac{\partial \mathbf{A}}{\partial t} \right)$$

$$\nabla \cdot \mathbf{D} = \rho$$

$$\Rightarrow \quad \varepsilon_0 \varepsilon_r \nabla \cdot \left(\nabla V + \frac{\partial \mathbf{A}}{\partial t} \right) = -\rho$$

$$\nabla \cdot \mathbf{B} = 0$$

$$\Rightarrow \quad \nabla \cdot (\nabla \times \mathbf{A}) = 0$$

$$\boxed{\mathbf{E} = -\nabla V - \frac{\partial \mathbf{A}}{\partial t}} \qquad \boxed{\mathbf{B} = \nabla \times \mathbf{A}}$$

$$\nabla^2 \mathbf{A} - \mu_0 \varepsilon_0 \mu_r \varepsilon_r \frac{\partial^2 \mathbf{A}}{\partial t^2} = \nabla \left(\nabla \cdot \mathbf{A} + \frac{\mu_r \varepsilon_r}{c^2} \frac{\partial V}{\partial t} \right) - \mu_0 \mu_r \mathbf{J}$$

$$\nabla^2 V + \frac{\partial (\nabla \cdot \mathbf{A})}{\partial t} = -\frac{\rho}{\varepsilon_0 \varepsilon_r}$$

Coulomb Gauge

Coulomb Gauge

$$\boxed{\nabla \cdot \mathbf{A} = 0}$$

$$\nabla^2 \mathbf{A} - \mu_0 \varepsilon_0 \mu_r \varepsilon_r \frac{\partial^2 \mathbf{A}}{\partial t^2} = \nabla\left(\nabla \cdot \mathbf{A} + \frac{\mu_r \varepsilon_r}{c^2}\frac{\partial V}{\partial t}\right) - \mu_0 \mu_r \mathbf{J}$$

$$\nabla^2 V + \frac{\partial(\nabla \cdot \mathbf{A})}{\partial t} = -\frac{\rho}{\varepsilon_0 \varepsilon_r}$$

$$\nabla^2 \mathbf{A} - \mu_0 \varepsilon_0 \mu_r \varepsilon_r \frac{\partial^2 \mathbf{A}}{\partial t^2} = \frac{\mu_r \varepsilon_r}{c^2}\frac{\partial \nabla V}{\partial t} - \mu_0 \mu_r \mathbf{J}$$

$$\nabla^2 V = -\frac{\rho}{\varepsilon_0 \varepsilon_r}$$

Poisson Eq. (~Static)

$$\boxed{V = \frac{1}{4\pi\varepsilon_0}\int_V \frac{1}{|\mathbf{x} - \mathbf{x}'|}\rho(\mathbf{x}', t)\, d^3x'}$$

Lorentz Gauge

Lorentz Gauge $$c^2 = \frac{1}{\mu_0 \varepsilon_0}$$

$$\boxed{\nabla \cdot \mathbf{A} + \frac{\mu_r \varepsilon_r}{c^2}\frac{\partial V}{\partial t} = 0}$$

$$\nabla^2 \mathbf{A} - \mu_0 \varepsilon_0 \mu_r \varepsilon_r \frac{\partial^2 \mathbf{A}}{\partial t^2} = \nabla\left(\nabla \cdot \mathbf{A} + \frac{\mu_r \varepsilon_r}{c^2}\frac{\partial V}{\partial t}\right) - \mu_0 \mu_r \mathbf{J}$$

$$\nabla^2 V + \frac{\partial(\nabla \cdot \mathbf{A})}{\partial t} = -\frac{\rho}{\varepsilon_0 \varepsilon_r}$$

Decoupled **A** & *V*!

$$\nabla^2 \mathbf{A} - \frac{\mu_r \varepsilon_r}{c^2}\frac{\partial^2 \mathbf{A}}{\partial t^2} = -\mu_0 \mu_r \mathbf{J}$$

$$\nabla^2 V - \frac{\mu_r \varepsilon_r}{c^2}\frac{\partial^2 V}{\partial t^2} = -\frac{\rho}{\varepsilon_0 \varepsilon_r}$$

d'Alembertian

$$\square^2 = \nabla^2 - \frac{\mu_r \varepsilon_r}{c^2}\frac{\partial^2}{\partial t^2}$$

$$\boxed{\square^2 \mathbf{A} = -\mu_0 \mu_r \mathbf{J}, \quad \square^2 V = -\frac{\rho}{\varepsilon_0 \varepsilon_r}}$$

Time-Harmonic Fields & Phasors

Time-Varying Fields: Linearity ➔ "Superposition"

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$

$$\nabla \times \mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t}$$

$$\nabla \cdot \mathbf{D} = \rho$$

$$\nabla \cdot \mathbf{B} = 0$$

Also hold for

$$\mathbf{E} \to a\mathbf{E}$$

$$\mathbf{B} \to a\mathbf{B}$$

$$\rho \to a\rho$$

$$\mathbf{J} \to a\mathbf{J}$$

Maxwell's Equations in ***General Media***

$$\mathbf{D} = \mathbf{D}(\mathbf{E}, \mathbf{H})$$

$$\mathbf{B} = \mathbf{B}(\mathbf{E}, \mathbf{H})$$

Also, functions of $\mathbf{x}, t, \mathbf{k}, \omega$

$$\mathbf{D} = \varepsilon_0 \varepsilon_\mathrm{r} \mathbf{E}$$

$$\mathbf{B} = \mu_0 \mu_\mathrm{r} \mathbf{H}$$

Lorentz's Force Equation

$$\mathbf{F} = \mathbf{F}_e + \mathbf{F}_m = q\left(\mathbf{E} + \mathbf{v} \times \mathbf{B}\right)$$

Concept of Phasors: Proper to Time-Invariant System

$$\boxed{\mathbf{E}(\mathbf{x},t)=\mathbf{E}_0(\mathbf{x})\cos\left(\omega t\right)=\mathrm{Re}[\mathbf{E}_0(\mathbf{x})e^{i\omega t}]}$$

$$\frac{\partial\mathbf{E}(\mathbf{x},t)}{\partial t}=\mathrm{Re}[i\omega\mathbf{E}_0(\mathbf{x})e^{i\omega t}]$$

$$\nabla\cdot\mathbf{E}(\mathbf{x},t)=\left[\nabla\cdot\mathbf{E}_0(\mathbf{x})\right]\cos\left(\omega t\right)=\mathrm{Re}[\left(\nabla\cdot\mathbf{E}_0(\mathbf{x})\right)e^{i\omega t}]$$

Wave Equations

Potential Fields for Source Terms: Homogeneous Cases

$$\nabla^2 V - \frac{\mu_r \varepsilon_r}{c^2}\frac{\partial^2 V}{\partial t^2} = -\frac{\rho}{\varepsilon_0 \varepsilon_r}$$

$$\nabla^2 \mathbf{A} - \frac{\mu_r \varepsilon_r}{c^2}\frac{\partial^2 \mathbf{A}}{\partial t^2} = -\mu_0 \mu_r \mathbf{J}$$

Static: $\nabla^2 V = -\dfrac{\rho}{\varepsilon_0 \varepsilon_r}$

$$\boxed{V = \frac{1}{4\pi\varepsilon_0}\int_V \frac{1}{|\mathbf{x}-\mathbf{x}'|}\rho(\mathbf{x}',t)d^3x'}$$

Static: $\nabla^2 \mathbf{A} = -\mu_0 \mu_r \mathbf{J}$

$$\boxed{\mathbf{A} = \frac{\mu_0 \mu_r}{4\pi}\int_V \frac{1}{|\mathbf{x}-\mathbf{x}'|}\mathbf{J}(\mathbf{x}',t)d^3x'}$$

$$V = \frac{1}{4\pi\varepsilon_0}\int_V \frac{1}{|\mathbf{x}-\mathbf{x}'|}\rho\left(\mathbf{x}',t \pm \frac{\sqrt{\mu_r\varepsilon_r}}{c}|\mathbf{x}-\mathbf{x}'|\right)d^3x'$$

$$\mathbf{A} = \frac{\mu_0 \mu_r}{4\pi}\int_V \frac{1}{|\mathbf{x}-\mathbf{x}'|}\mathbf{J}\left(\mathbf{x}',t \pm \frac{\sqrt{\mu_r\varepsilon_r}}{c}|\mathbf{x}-\mathbf{x}'|\right)d^3x'$$

$$V = \frac{1}{4\pi\varepsilon_0}\int_V \frac{1}{|\mathbf{x}-\mathbf{x}'|}\rho\left(\mathbf{x}',t - \frac{\sqrt{\mu_r\varepsilon_r}}{c}|\mathbf{x}-\mathbf{x}'|\right)d^3x'$$

$$\mathbf{A} = \frac{\mu_0 \mu_r}{4\pi}\int_V \frac{1}{|\mathbf{x}-\mathbf{x}'|}\mathbf{J}\left(\mathbf{x}',t - \frac{\sqrt{\mu_r\varepsilon_r}}{c}|\mathbf{x}-\mathbf{x}'|\right)d^3x'$$

Source-Free Wave Equations: Homogeneous Cases

$$\nabla \times (\nabla \times \mathbf{E}) = -\frac{\partial}{\partial t}\nabla \times \mathbf{B} = -\mu_0 \mu_r \frac{\partial}{\partial t}\nabla \times \mathbf{H} = -\mu_0 \varepsilon_0 \mu_r \varepsilon_r \frac{\partial^2}{\partial t^2}\mathbf{E}$$

$$\nabla(\nabla \cdot \mathbf{E}) - \nabla^2 \mathbf{E} = -\mu_0 \varepsilon_0 \mu_r \varepsilon_r \frac{\partial^2}{\partial t^2}\mathbf{E}$$

$$\nabla^2 \mathbf{E} - \mu_0 \varepsilon_0 \mu_r \varepsilon_r \frac{\partial^2}{\partial t^2}\mathbf{E} = \mathbf{O}$$

$$\boxed{\nabla^2 \mathbf{E} - \frac{\mu_r \varepsilon_r}{c^2}\frac{\partial^2 \mathbf{E}}{\partial t^2} = \mathbf{O}}$$

Source-Free Wave Equations: Homogeneous Cases

$$\boxed{\nabla^2 \mathbf{E} - \frac{\mu_r \varepsilon_r}{c^2} \frac{\partial^2 \mathbf{E}}{\partial t^2} = \mathbf{O}}$$

$$\frac{\partial^2 E_z}{\partial x^2} - \frac{\mu_r \varepsilon_r}{c^2} \frac{\partial^2 E_z}{\partial t^2} = 0$$

$$E_z = u\left(x - \frac{c}{\sqrt{\mu_r \varepsilon_r}} t\right)$$

$$x - \frac{c}{\sqrt{\mu_r \varepsilon_r}} t = 0 \rightarrow \frac{x}{t} = \boxed{\frac{c}{\sqrt{\mu_r \varepsilon_r}}}$$

Speed of Light inside a material
c: Light speed inside a vacuum!

Retarded Potentials

$$V = \frac{1}{4\pi\varepsilon_0}\int_V \frac{1}{|\mathbf{x}-\mathbf{x}'|}\rho(\mathbf{x}', t \pm \frac{\sqrt{\mu_r\varepsilon_r}}{c}|\mathbf{x}-\mathbf{x}'|)d^3x'$$

$$\mathbf{A} = \frac{\mu_0\mu_r}{4\pi}\int_V \frac{1}{|\mathbf{x}-\mathbf{x}'|}\mathbf{J}(\mathbf{x}', t \pm \frac{\sqrt{\mu_r\varepsilon_r}}{c}|\mathbf{x}-\mathbf{x}'|)d^3x'$$

1. Light Speed: The effect of sources to a distant point requires a “finite” time!
2. Causality: EM fields cannot be determined by the source at a “future” time!

$$V = \frac{1}{4\pi\varepsilon_0}\int_V \frac{1}{|\mathbf{x}-\mathbf{x}'|}\rho(\mathbf{x}', t - \frac{\sqrt{\mu_r\varepsilon_r}}{c}|\mathbf{x}-\mathbf{x}'|)d^3x'$$

$$\mathbf{A} = \frac{\mu_0\mu_r}{4\pi}\int_V \frac{1}{|\mathbf{x}-\mathbf{x}'|}\mathbf{J}(\mathbf{x}', t - \frac{\sqrt{\mu_r\varepsilon_r}}{c}|\mathbf{x}-\mathbf{x}'|)d^3x'$$

Wave Equations: Harmonic & Inhomogeneous Materials

Maxwell's Equations in ***Simple + Source-Free Media + Harmonic Condition***

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$

$$\nabla \times \mathbf{H} = \frac{\partial \mathbf{D}}{\partial t}$$

$$\nabla \cdot \mathbf{D} = 0$$

$$\nabla \cdot \mathbf{B} = 0$$

$$\mathbf{J} = \mathbf{O}, \quad \rho = 0$$

$$\mathbf{D} = \varepsilon_0 \varepsilon_{\mathrm{r}}(\omega)\mathbf{E}$$

$$\mathbf{B} = \mu_0 \mu_{\mathrm{r}}(\omega)\mathbf{H}$$

$$\begin{aligned} \nabla \times (\nabla \times \mathbf{E}) = -i\omega \nabla \times \mathbf{B} &= -i\omega\mu_0 \nabla \times (\mu_r \mathbf{H}) \\ &= -i\omega\mu_0 (\mu_r \nabla \times \mathbf{H} - \mathbf{H} \times \nabla \mu_r) \\ &= -i\omega\mu_0 (\mu_r i\omega\varepsilon_0\varepsilon_r \mathbf{E} - \mathbf{H} \times \nabla \mu_r) \end{aligned}$$

$$\nabla(\nabla \cdot \mathbf{E}) - \nabla^2 \mathbf{E} = k_0{}^2 \mu_r \varepsilon_r \mathbf{E} + i\omega\mu_0 (\mathbf{H} \times \nabla \mu_r) \qquad \boxed{k_0{}^2 = \frac{\omega^2}{c^2}}$$

$$\nabla \cdot (\varepsilon_0 \varepsilon_r \mathbf{E}) = \varepsilon_0 (\nabla \varepsilon_r) \cdot \mathbf{E} + \varepsilon_0 \varepsilon_r \nabla \cdot \mathbf{E} = 0$$

$$\boxed{\nabla^2 \mathbf{E} + k_0{}^2 \mu_r \varepsilon_r \mathbf{E} = -\nabla \frac{(\nabla \varepsilon_r) \cdot \mathbf{E}}{\varepsilon_r} - i\omega\mu_0 (\mathbf{H} \times \nabla \mu_r)}$$

Wave Equations: Harmonic & Inhomogeneous Materials

Maxwell's Equations in ***Simple + Source-Free Media + Harmonic Condition***

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$

$$\nabla \times \mathbf{H} = \frac{\partial \mathbf{D}}{\partial t}$$

$$\nabla \cdot \mathbf{D} = 0$$

$$\nabla \cdot \mathbf{B} = 0$$

$$\mathbf{J} = \mathbf{O}, \quad \rho = 0$$

$$\mathbf{D} = \varepsilon_0 \varepsilon_{\mathrm{r}}(\omega)\mathbf{E}$$

$$\mathbf{B} = \mu_0 \mu_{\mathrm{r}}(\omega)\mathbf{H}$$

$$\begin{aligned}
\nabla^2\mathbf{E} + k_0{}^2\mu_r\varepsilon_r\mathbf{E} &= -\nabla\frac{(\nabla\varepsilon_r)\cdot\mathbf{E}}{\varepsilon_r} - i\omega\frac{1}{\mu_r}(\mathbf{B}\times\nabla\mu_r) \\
&= -\nabla\frac{(\nabla\varepsilon_r)\cdot\mathbf{E}}{\varepsilon_r} - i\omega\frac{1}{\mu_r}\left(\frac{1}{-i\omega}(\nabla\times\mathbf{E})\times\nabla\mu_r\right) \\
&= -\nabla\frac{(\nabla\varepsilon_r)\cdot\mathbf{E}}{\varepsilon_r} + \frac{(\nabla\times\mathbf{E})\times(\nabla\mu_r)}{\mu_r} \\
&= -\nabla\frac{\nabla\varepsilon_r}{\varepsilon_r}\cdot\mathbf{E} + (\nabla\times\mathbf{E})\times\frac{\nabla\mu_r}{\mu_r} \\
&= -\nabla\frac{\nabla\varepsilon_r}{\varepsilon_r}\cdot\mathbf{E} - \frac{\nabla\mu_r}{\mu_r}\times(\nabla\times\mathbf{E})
\end{aligned}$$

$$\boxed{\nabla^2\mathbf{E} + k_0{}^2\mu_r\varepsilon_r\mathbf{E} = -\nabla\frac{\nabla\varepsilon_r}{\varepsilon_r}\cdot\mathbf{E} - \frac{\nabla\mu_r}{\mu_r}\times(\nabla\times\mathbf{E})}$$

Wave Equations: Harmonic & Homogeneous Materials

$$\boxed{\nabla^2 \mathbf{E} + k_0{}^2 \mu_r \varepsilon_r \mathbf{E} = -\nabla \frac{\nabla \varepsilon_r}{\varepsilon_r} \cdot \mathbf{E} - \frac{\nabla \mu_r}{\mu_r} \times \left(\nabla \times \mathbf{E} \right)}$$

$$\Big\Downarrow \quad \nabla \mu_r = 0$$

$$\boxed{\nabla^2 \mathbf{E} + k_0{}^2 \mu_r \varepsilon_r \mathbf{E} = -\nabla \frac{\left(\nabla \varepsilon_r \right) \cdot \mathbf{E}}{\varepsilon_r}}$$

$$\Big\Downarrow \quad \nabla \varepsilon_r = 0$$

$$\boxed{\nabla^2 \mathbf{E} + k_0{}^2 \mu_r \varepsilon_r \mathbf{E} = \mathbf{O}}$$

Maxwell's Equations in ***Simple + Source-Free + Homogeneous Media***

Wave Equations: Scalar Waves

$$\boxed{\nabla^2 \mathbf{E} + k_0{}^2 \mu_r \varepsilon_r \mathbf{E} = \mathbf{O}}$$

Without polarization mixing due to the symmetry in structures

$$\boxed{\nabla^2 \psi + k_0{}^2 \mu_r \varepsilon_r \psi = 0}$$

Wave Equations: Scalar Waves – Solution

$$\boxed{\nabla^2\psi + k_0{}^2\mu_r\varepsilon_r\psi = 0}$$

$$k_0{}^2 = \frac{\omega^2}{c^2}$$

Assume *x*-axis dependency only...

$$\boxed{E = E_0 \exp(i\omega t \pm ikx)}$$

$$k^2 = k_0{}^2\mu_r\varepsilon_r$$

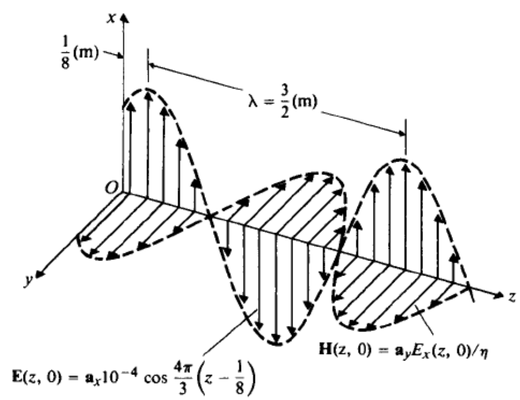

FIGURE 8–2
E and **H** fields of a uniform plane wave at $t = 0$ (Example 8–1).