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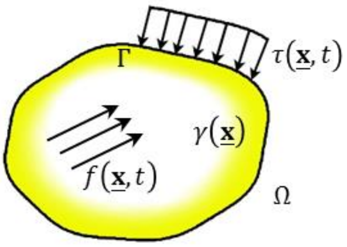
457.646 Topics in Structural Reliability

In-Class Material: Class 27

IX. Finite Element Reliability Analysis (Haukaas, 2006)

→ summary and good findings

◎ Equations of Motion and Randomness



"Weak" form of equilibrium:

$$\int_\Omega \delta u_i \gamma \ddot{u}_i d\Omega + \int_\Omega \delta u_{i,j} \sigma_{ij} d\Omega - \int_\Omega \delta u_i f_i d\Omega - \int_\Gamma \delta u_i \tau_i d\Gamma = 0$$

γ : density, $\ddot{u}_i$: acc, $u_{i,j}$: strain, σ_{ij} : stress, f_i : body force, τ_i : traction

① Basic random fields

$C_{ijkl}(\mathbf{x})$, $\gamma(\mathbf{x})$: material properties (constants)

Tensor of material elastic constants, $\sigma_{ij} = C_{ijkl}\varepsilon_{kl}$

$f_i(\mathbf{x}, t)$, $\tau_i(\mathbf{x}; t)$: loads

Ω, Γ : geometry

$\Rightarrow$ Discretized to a random vector $\mathbf{v}$

② Derived response is a function of $\mathbf{v}$

$u_i(\mathbf{x}, t, \mathbf{v})$: displacement

$\varepsilon_{ij}(\mathbf{x}, t, \mathbf{v})$: strain

$\varepsilon_{ij}^P(\mathbf{x}, t, \mathbf{v})$: plastic strain

$\sigma_{ij}(\mathbf{x}, t, \mathbf{v})$: stress

$$g(S(\mathbf{v}), \mathbf{v}) \le 0$$

$\vdots$

$\mathbf{S}(\mathbf{x}, t, \mathbf{v})$: generic response vector

③ FE models and r.v's

i. Nonlinear & Dynamic problem

$$\mathbf{M}(\mathbf{v})\ddot{\mathbf{u}}(t,\mathbf{v})+\mathbf{C}(\mathbf{v})\dot{\mathbf{u}}(t,\mathbf{v})+\mathbf{R}(\mathbf{u}(t,\mathbf{v}),\mathbf{v})=\mathbf{P}(t,\mathbf{v})$$

ii. Static problem

$$\mathbf{R}(\mathbf{u}(t,\mathbf{v}),\mathbf{v})=\mathbf{P}(t,\mathbf{v})$$

iii. Linear Static problem

$$\mathbf{K}(\mathbf{v})\cdot\mathbf{u}(\mathbf{v})=\mathbf{P}(\mathbf{v})$$

④ FE reliability analysis

i. MCS $\mathbf{v}_i,\ \ i=1,\cdots,N$

ii. Importance Sampling

iii. Response Surface $g\approx\eta(\mathbf{x})$

iv. Form (HLRF)

Initialize $\mathbf{u}_1=\mathbf{u}(\mathbf{v}_1)$

$\downarrow$

$\mathbf{v}_i=\mathbf{v}(\mathbf{u}_i)$ skip if $i=1$

$G(\mathbf{u}_i)=g(\mathbf{S}(\mathbf{v}_i),\mathbf{v}_i)$

$$\nabla_{\mathbf{u}}G(\mathbf{u}_i)=\nabla_{\mathbf{v}}g(\mathbf{v})J_{\mathbf{v},\mathbf{u}}$$
$$=(\nabla_{\mathbf{s}}g\cdot J_{\mathbf{s},\mathbf{v}}+\nabla_{\mathbf{v}}g)\cdot J_{\mathbf{v},\mathbf{u}}$$

$\leftarrow$ $\mathbf{S}(\mathbf{v}_i), J_{\mathbf{s},\mathbf{v}}$ FE code

$\downarrow$ The same procedure

$\vdots$

e.g. FERUM-ABAQUS

(Young Joo Lee, 2012)

◎ Gradient $J_{\mathbf{s},\mathbf{v}}$?

e.g. $\dfrac{\partial u_i}{\partial E},\dfrac{\partial \sigma_i}{\partial P},\cdots$

Methods to get sensitivity $J_{\mathbf{s},\mathbf{v}}$

e.g. Linear Static Problem (suppose there is only one r.v. $\mathbf{v}=v$)

$$\underbrace{\mathbf{K}(v)}_{\text{Stiffness}}\cdot\underbrace{\mathbf{u}(v)}_{\text{displacement}}=\underbrace{\mathbf{P}(v)}_{\text{loads}}\ \rightarrow\ \mathbf{u}=\mathbf{K}^{-1}(v)\cdot\mathbf{P}(v)$$

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① Finite Difference Method ("FFD" option of FERUM)

$\mathbf{u}(v) = \mathbf{K}^{-1}(v) \cdot \mathbf{P}(v)$ original FE

$\mathbf{u}(v+\Delta v) = \mathbf{K}^{-1}(v+\Delta v) \cdot \mathbf{P}(v+\Delta v)$ (i.e. additional FE for each v_i in $\mathbf{v}$)

$$\frac{\partial \mathbf{u}}{\partial v} \cong \frac{\mathbf{u}(v+\Delta v) - \mathbf{u}(v)}{\Delta v}$$

⇒ Need to solve FE again (for each r.v)

⇒ Can cause numerical errors

② Perturbation Method

$$\mathbf{Ku} = \mathbf{P}$$

$$\Delta\mathbf{K} = \mathbf{K}(v+\Delta v) - \mathbf{K}(v)$$
$$\Delta\mathbf{P} = \mathbf{P}(v+\Delta v) - \mathbf{P}(v)$$

$$(\mathbf{K}+\Delta\mathbf{K})(\mathbf{u}+\Delta\mathbf{u}) = \mathbf{P}+\Delta\mathbf{P}$$

$$\mathbf{Ku}+\mathbf{K}\Delta\mathbf{u}+\Delta\mathbf{Ku}+\Delta\mathbf{K}\Delta\mathbf{u} = \mathbf{P}+\Delta\mathbf{P}$$

$$\therefore \Delta\mathbf{u} \cong \mathbf{K}^{-1}(\Delta\mathbf{P}-\Delta\mathbf{Ku})$$

⇒ Do not have to re-solve FE

⇒ Error ($\Delta\mathbf{K}\Delta\mathbf{u} \approx 0$)

③ Direct Differentiation Method ('DDM' option for FERUM)

$$\mathbf{Ku} = \mathbf{P}$$

$$\frac{\partial \mathbf{K}}{\partial v}\mathbf{u} + \mathbf{K}\frac{\partial \mathbf{u}}{\partial v} = \frac{\partial \mathbf{P}}{\partial v}$$

$$\frac{\partial \mathbf{u}}{\partial v} = \mathbf{K}^{-1}\left(\frac{\partial \mathbf{P}}{\partial v} - \frac{\partial \mathbf{K}}{\partial v}\mathbf{u}\right)$$

→ Do not need to solve FEM again

→ No error

→ $\frac{\partial \mathbf{K}}{\partial v} = \sum_e \frac{\partial \mathbf{K}^e}{\partial v}$ ($\frac{\partial \mathbf{K}^e}{\partial v}$ ← direct stiffness method)

→ Nonlinear static, nonlinear dynamic

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④ Adjoint method

➔ Tutorial by Prof. Andrew M. Bradley at Stanford University: http://cs.stanford.edu/~ambrad/adjoint_tutorial.pdf)

$x \in \Re^{n_x},\ p \in \Re^{n_p},\ f(x(p)):\ \Re^{n_x} \to \Re$

Subject to $h(x(p), p) = 0$ for $h:\ \Re^{n_x} \times \Re^{n_p} \to \Re$

e.g. h=0 → PDE equilibrium (mass ∈ p, displacement ∈ x, member force = f)

$d_p f$? (total derivative of f w.r.t p)

Consider the Lagrangian

$$L(x, p, \lambda) = f(x(p)) + \lambda^{\mathrm{T}} h(x(p), p)$$

$$\begin{aligned} d_p f &= d_p L \qquad (\because \text{ only on } h = \qquad\qquad) \\ &= \partial_x f d_p x + d_p \lambda^{\mathrm{T}} h + \lambda^{\mathrm{T}} (\partial_x h d_p x + \partial_p h) \\ &= f_x x_p + \lambda^{\mathrm{T}} (h_x x_p + h_p) \qquad (\because \qquad\qquad) \\ &= (f_x + \lambda^T h_x) x_p + \lambda^{\mathrm{T}} h_p \end{aligned}$$

Choose λ such that $h_x^{\mathrm{T}} \lambda = -f_x^{\mathrm{T}}$ ("adjoint equation") → λ^*

Then we can avoid calculating ( )

Then compute $d_p f$ as ___________

⇒ Used for RBTO of structures under stochastic excitations (Chun, Song and Paulino, 2016)

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In-Class Material: Class 28

X-1. Probability-Based Structural Design Code

→ Cornell. C.A (1969) A probability-based structural code (J. ACI)

→ Ravindara & Galambos (1978) Load & resistance factor design for steel structures

(J. Str. Eng, Div. ASCE)

◎ Load & Resistance Factor Design (LRFD)

Replaced allowable stress design (ASD) (→safety factor)

⇒ Probability-based code

$$\phi R_n \geq \sum \gamma_k Q_{km} = \gamma_D Q_{Dm} + \gamma_L Q_{Lm} \quad \cdots\cdots\cdots\cdots\cdots (1)$$

Dead load Live load

i. R_n : " " resistance

→ code formula (e.g. $V_c = \frac{1}{6}\sqrt{f_c^{'}} b_w d$)

→ nominal values used (material & dimension)

: given in " " force, e.g. bending moment, axial force, shear force

ii. ϕ : " " Factor ~ ϕ 1

(Dimensionless) conservatism due to the uncertainties in R

iii. Q_m : mean load effect

→ in generalized force (structural analysis)

iv. γ : "Load" factor~ γ 1

Conservatism due to

① Potential overload

② Uncertainty in load effect calculation

v. Limit-State

"U " limit-states

e.g. frame instability, plastic mechanism formed incremental collapse

"S " limit-states

e.g. excessive deflection, excessive vibration, premature yielding or slip

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LRFD codes suggest formulas for ( ), methods to compute ( ) from loads

provide ( ) & ( )

for each structural element (Q_m) from loads

to satisfy the ( ) reliability level

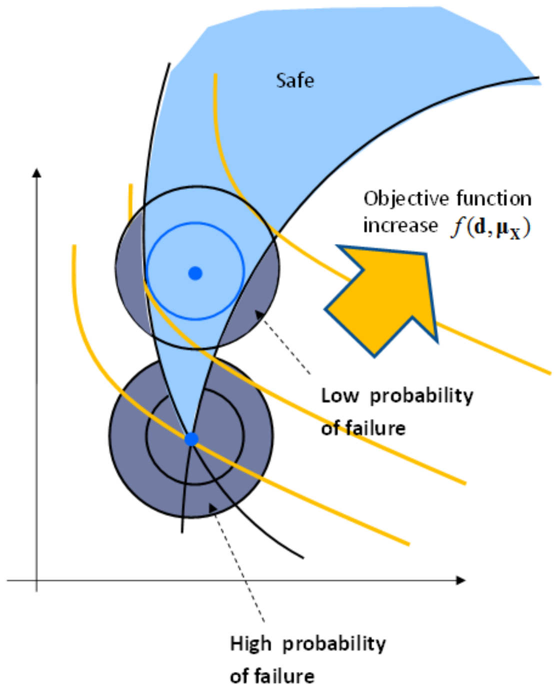
More details on Reliability-Based Design Codes and Calibration:

Covered by the course "Topics in Structural Reliability (Code Calibration)"

Review in Nguyen, Song & Paulino (2010)

X-2. Reliability-Based Design Optimization (RBDO)

◎ RBDO formulation



$$\min_{\mathbf{d},\boldsymbol{\mu}_\mathbf{x}} f(\mathbf{d},\boldsymbol{\mu}_\mathbf{x})$$

s.t. $P[g(\mathbf{d},\boldsymbol{\mu}_\mathbf{x}) \le 0] \le P_f^t$

$$\mathbf{d}^L \le \mathbf{d} \le \mathbf{d}^u$$

$$\boldsymbol{\mu}_\mathbf{x}^L \le \boldsymbol{\mu}_\mathbf{x} \le \boldsymbol{\mu}_\mathbf{x}^u$$

Where

$f(\mathbf{d},\boldsymbol{\mu}_\mathbf{x})$

$\mathbf{d}$

$\mathbf{x}$

$\boldsymbol{\mu}_\mathbf{x}$

P_f^t

$\mathbf{d}^L, \mathbf{d}^u$

$\boldsymbol{\mu}_\mathbf{x}^L, \boldsymbol{\mu}_\mathbf{x}^u$

◎ Reliability Index Approach (RIA; Enevaldsen & Sorensen 1994)

$$\min_{\mathbf{d},\boldsymbol{\mu}_\mathbf{x}} f(\mathbf{d},\boldsymbol{\mu}_\mathbf{x})$$

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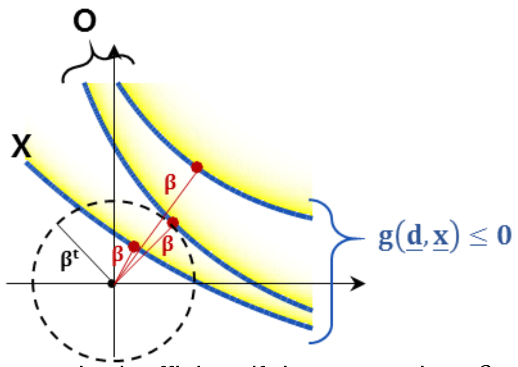
s.t. β $\qquad\qquad$ β^t

$\beta^t \leftarrow$ target reliability index $-\Phi^{-1}[P_f^t]$

$\beta \leftarrow$ generalized reliability index

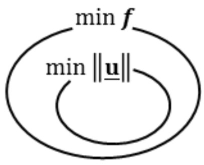
$\beta = -\Phi^{-1}[\qquad\qquad\qquad\qquad]$

↖ By FORM analysis (or others)

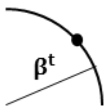


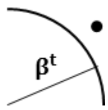
⇒ compute P_f for each interation of $\mathbf{d}$ to check if the constraint is satisfied

⇒ double loop approach



⇒ can be inefficient if the constraint $\beta \ge \beta^t$ is inactive





⇒ may not be able to provide an optimal solution if the failure does not occur in the feasible domain

◎ Performance Measure Approach (PMA; Tu et al., 1999) ※ double-loop

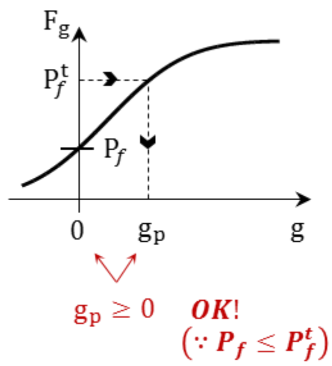
$$\min_{\mathbf{d}, \boldsymbol{\mu}_\mathbf{x}} f(\mathbf{d}, \boldsymbol{\mu}_\mathbf{x})$$

s.t. $\underline{g_p} = F_g^{-1}\left[\Phi\left(-\beta^t\right)\right] \ge 0$ $\quad(\Phi^{-1}[-\beta^t] = P^t)$

"Performance function" = quantile of g at P^t

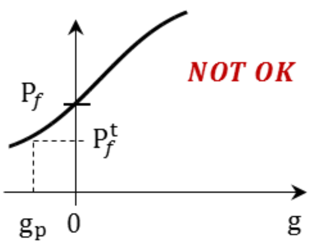
$g_p \ge 0 \iff P_f \le P_f^t$

$\qquad \iff \beta \ge \beta^t$



Equivalent RBDO

How to find g_p?



They proposed (instead of solving FORM target β)

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$$g_p = \min_{\mathbf{u}} G(\mathbf{d}, \mathbf{u}) \qquad \cdots\cdots\cdots\cdots\cdots (1)$$

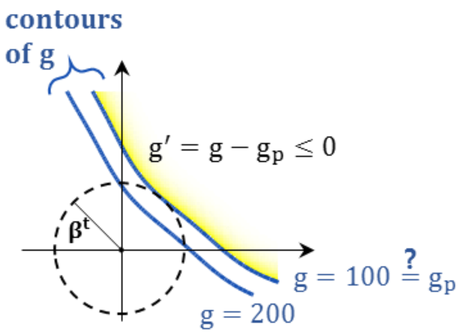
s.t. $\|\mathbf{u}\| = \beta^t$ $\Rightarrow$ Minimizes g instead of $\|\mathbf{u}\|$

~ facilitates gradient-based optimization (using $\dfrac{\partial g}{\partial \mathbf{d}}$)

$\Rightarrow$ Overcomes the problems in RIA

(↑)

Is this g_p really $F_g^{-1}\left[P_f^t\right]$?



Set a new limit-state function

$g'(x) = g(\mathbf{x}) - g_p$

$P(g' \le 0) \cong \Phi(-\beta^t) = P_f^t$

||

$P(g \le g_p)$

||

$F_g(g_p)$ $\quad g_p = F_g^{-1}[P_f^t]$

◎ Single-Loop PMA (Liang et al., 2004)

Replace the optimization in (1) with an approximation (but non-iterative)

system equation, i.e, Karush-Kuhn-Tucker (KKT)

condition

$\nabla_{\mathbf{u}} G(\mathbf{d}, \mathbf{u}) + \lambda \nabla_{\mathbf{u}}(\|\mathbf{u}\| - \beta^t) = 0$ (λ →Lagrange Multiplier)

$\|\mathbf{u}\| - \beta^t = 0$

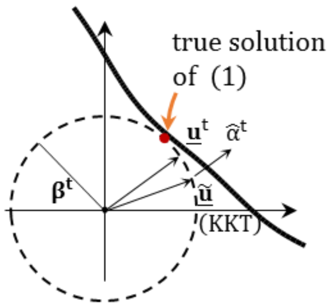
i. Solve KKT to get $\mathbf{u} = \tilde{\mathbf{u}}$

ii. Evaluate $\hat{\alpha}$ at $\mathbf{u} = \tilde{\mathbf{u}}$

iii. Approximate design point by

$\mathbf{u}^t = \beta^t \cdot \hat{\alpha}^t$

iv. Check $g(\mathbf{u}^t) - g_p \ge 0$



Single loop RBDO

$$\min_{\mathbf{d},\boldsymbol{\mu}_\mathbf{x}} \quad f(\mathbf{d},\boldsymbol{\mu}_\mathbf{x})$$

s.t. $g_p \equiv g(\mathbf{d},\mathbf{x}(\mathbf{u}^t)) \geq 0$

Nguyen, T.H., J. Song, and G.H. Paulino (2010). Single-loop system reliability-based design optimization using matrix-based system reliability method: theory and applications. *J. of Mechanical Design, ASME*, Vol. 132, 011005-1~11.

Nguyen, T.H., J. Song, and G.H. Paulino (2011). Single-loop system reliability-based topology optimization considering statistical dependence between limit states. *Structural and Multidisciplinary Optimization*, Vol. 44(5), 593-611.

Many thanks for your hard work in this semester to learn theories of structural reliability and their applications. I wish you the very best on your course work, research and future career.

Best,
Junho