1.4.1 Root shear load

The vertical, in-plane, and radial root shear are as follows

$$
\begin{aligned}
s_z &= \int_e^R \left[\frac{dF_z}{dr} - m(r-e)\ddot{\beta} \right] dr \\
s_x &= \int_e^R \left[\frac{dF_x}{dr} \right] dr \\
s_r &= \int_e^R \left[m\Omega^2 r - \beta \frac{dF_r}{dr} \right] dr
\end{aligned}
$$

1.4.2 Root bending load

Like in the case of root shears, the root bending loads are obtained by integrating the moments generated by the sectional forces about the root. The flap bending moment n_f is as follows. Recall, that the same expression was derived in equation (1.19).

$$
\begin{aligned}
n_f &= \int_e^R (r-e)dF_z - \int_e^R (mdr)\Omega^2 r(r-e)\beta - \int_e^R m(r-e)^2\ddot{\beta}dr \\
&= k_\beta(\beta - \beta_p)
\end{aligned}
\tag{1.36}
$$

Now use the non-dimensional form of the flap frequency as given in equation (1.23) to replace k_β in terms of the flap frequency ν_β.

$$
\begin{aligned}
n_f &= k_\beta(\beta - \beta_p) \\
&= \left(\nu_\beta^2 - 1 - \frac{3}{2}\frac{e}{R} \right) I_\beta \Omega^2 (\beta - \beta_p) \\
&= \left(\nu_\beta^2 - 1 \right) I_\beta \Omega^2 (\beta - \beta_p) \qquad \text{for hinge offset } e/R = 0 \\
&= \left(\nu_\beta^2 - 1 \right) I_\beta \Omega^2 \beta \qquad \text{for } e/R = 0, \text{ and precone } \beta_p = 0
\end{aligned}
\tag{1.37}
$$

Thus, the flap bending moment at the root is related to the flap frequency, and flap dynamics. Similarly, later we shall see that the lag and torsion moments depend on lag and torsion frequencies, and lag and torsion dynamics. Here, we have considered only the flap motion. The lag and torsion moments then simply become

$$
n_l = \int_e^R \left[(r-e)\frac{dF_x}{dr} \right] dr \tag{1.38}
$$

$$
n_t = \int_e^R \left[\frac{dM_x}{dr} \right] dr \tag{1.39}
$$

where dM_x is the nose-up aerodynamic pitching moment acting on the airfoils over each section of length dr. dM_x is about the elastic axis, which is generally close to quarter-chord.

1.4.3 Rotating frame hub loads

The rotating frame hub loads are obtained by simply transfering the root loads to the hub. By hub we mean the center of rotation, i.e. the rotor shaft axis. Note that in the case of zero hinge offset, $e/R = 0$, then the root loads are directly the rotating frame hub loads.

$$
\begin{aligned}
f_x &= s_x & m_x &= n_f \\
f_y &= s_r & m_y &= n_t \\
f_z &= s_z & m_z &= -n_l
\end{aligned}
\tag{1.40}
$$

For a non-zero hinge offset

$$\begin{aligned} f_x &= s_x & m_x &= n_f + e s_z \\ f_y &= s_r & m_y &= n_t \\ f_z &= s_z & m_z &= -n_l - e s_x \end{aligned} \tag{1.41}$$

In the case of non-zero hinge offset, m_x and m_z can be obtained directly by integrating the moments generated by the blade forces about the hub, instead of about the hinge.

It is important to note that the rotating frame hub loads are associated with each blade. At any instant of time, each blade produces six rotating frame hub loads. For each blade, they act in three directions along an axis system stuck to its root. This local axis system rotates with the blade. Thus, before the contribution from all blades at the hub can be added up, the rotating frame loads from each blade must be resolved into three fixed directions which do not rotate with any of the blades. This is called a fixed frame.

1.4.4 Fixed frame hub loads

The fixed frame hub loads are often simply called the hub loads. They are obtained from the rotating frame loads by the following two steps.

1. Resolve the rotating frame loads of each blade in a fixed frame.

2. Sum the fixed frame loads from all N_b blades.

Let $m = 1, 2, ...N_b$ be the blade number. ψ_m be the azimuthal location of each blade m. Then we have

$$\begin{aligned} F_x &= \sum_{m=1}^{N_b} (f_y \cos\psi_m + f_x \sin\psi_m) \\ F_y &= \sum_{m=1}^{N_b} (f_y \sin\psi_m - f_x \cos\psi_m) \\ F_z &= \sum_{m=1}^{N_b} f_z \\ M_x &= \sum_{m=1}^{N_b} (m_x \sin\psi_m + m_y \cos\psi_m) \\ M_y &= \sum_{m=1}^{N_b} (-m_x \cos\psi_m + m_y \sin\psi_m) \\ M_z &= \sum_{m=1}^{N_b} m_z \end{aligned} \tag{1.42}$$

In general f_x, f_y, f_z and m_x, m_y, m_z contain all harmonics $1, 2, 3...\infty$/rev.

Step 1 redistributes the magnitudes of individual harmonics, but retains all harmonics. For example in the calculation of F_x, the $f_y \sin\psi$ term would re-distribute a steady f_y component into a 1/rev harmonic, a 1/rev f_y component into 0/rev (steady) and 1/rev components. In general, a p/rev component in the rotating frame loads can, when resolved in a fixed frame, give rise to $p \pm 1$/rev components. F_z, and M_z are exceptions. Here f_z, and m_z are not multiplied with sine or cosine components. Thus p/rev loads remain p/rev loads when resolved in a fixed frame.

Step2, i.e. the summation over all blades, filters out all non-pN_b/rev harmonics. For example in the case of a four bladed rotor, $N_b = 4$, the fixed frame hub loads contain only $0, 4, 8, 12, ...$/rev harmonics. The N_b/rev harmonic is called the blade passage frequency. Thus the fixed frame hub loads contain only integral multiples of the blade passage frequency. Consider for example

$$\begin{aligned}
f_z(\psi) &= a_0 + a_1 \sin\psi + a_2 \sin 2\psi + a_3 \sin 3\psi + a_4 \sin 4\psi \\
F_z(\psi) &= f_z(\psi_1) + f_z(\psi_2) + f_z(\psi_3) + f_z(\psi_4) \\
&= f_z(\psi) + f_z(\psi + 90^\circ) + f_z(\psi + 180^\circ) + f_z(\psi_4 + 270^\circ) \\
&= 4a_0 + 4a_4 \sin 4\psi
\end{aligned}$$

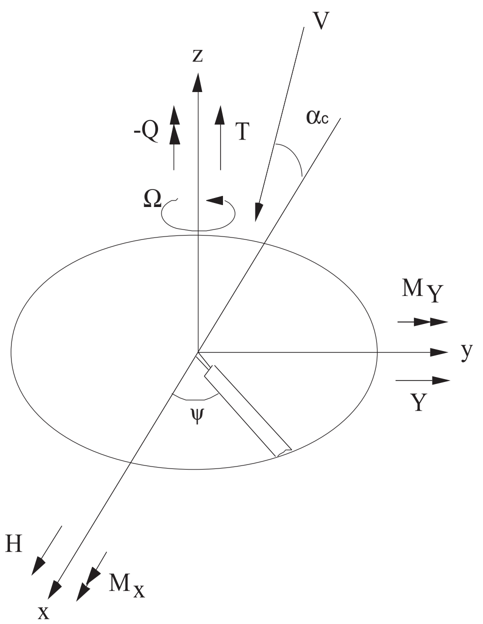
Note that the assumption here is that all blades have identical root loads, only shifted in phase. In case the blades are dissimilar this assumption does not hold. The hub loads in that case transmit all harmonics. Such is the case for damaged or dissimilar rotors. The goal is to make all the blades identical.

The pN_b/rev harmonics of the hub loads, e.g. the $4a_4 \sin 4\psi$ component, create enormous vibration in the fuselage. The steady component, e.g. the $4a_0$ component is used to trim the helicopter. The steady component is the average force generated by the rotor. In this case $4a_0$ was the rotor thrust. The steady components of F_x, F_y, F_z are often denoted as H, Y, T, the rotor drag, side force, and thrust. The steady components of M_x and M_y are denoted as M_X and M_Y, the roll-left, and pitch-up moments. The steady component of $-M_z$ is denoted by Q, the rotor torque.

The steady components can be more easily obtained by averaging the rotating frame loads over the rotor disk, and then multiplying by N_b to account for all blades. Using the same example as above, the thrust can be calculated as

$$\begin{aligned}
T &= \frac{N_b}{2\pi}\int_0^{2\pi} f_z(\psi)d\psi \\
&= \frac{4}{2\pi}\int_0^{2\pi} a_0 + a_1 \sin\psi + a_2 \sin 2\psi + a_3 \sin 3\psi + a_4 \sin 4\psi \\
&= \frac{4}{2\pi} 2\pi a_0 \\
&= 4a_0
\end{aligned}$$

Thus in general we have the steady rotor forces H, Y, T, and moments M_X, M_Y, Q as follows.



Rotor Thrust T:

$$
\begin{aligned}
T &= \frac{N_b}{2\pi}\int_0^{2\pi} f_z d\psi \\
&= \frac{N_b}{2\pi}\int_0^{2\pi} s_z d\psi \\
&= \frac{N_b}{2\pi}\int_0^{2\pi}\int_e^R \left[\frac{dF_z}{dr} - m(r-e)\ddot{\beta}\right] dr d\psi \\
&= \frac{N_b}{2\pi}\int_0^{2\pi}\int_e^R dF_z d\psi
\end{aligned} \tag{1.43}
$$

This is because $\ddot{\beta}$ cannot have a steady component, and all harmonics integrate to zero over the azimuth.

Rotor Drag H:

$$
\begin{aligned}
H &= \frac{N_b}{2\pi}\int_0^{2\pi} (f_y \cos\psi + f_x \sin\psi) d\psi \\
&= \frac{N_b}{2\pi}\int_0^{2\pi} (s_r \cos\psi + s_x \sin\psi) d\psi \\
&= \frac{N_b}{2\pi}\int_0^{2\pi}\int_e^R (dF_r \cos\psi + dF_x \sin\psi) d\psi
\end{aligned} \tag{1.44}
$$

where the centrifugal component of s_r integrates to zero.

Rotor Side Force Y:

$$
\begin{aligned}
Y &= \frac{N_b}{2\pi}\int_0^{2\pi} (f_y \sin\psi - f_x \cos\psi) d\psi \\
&= \frac{N_b}{2\pi}\int_0^{2\pi} (s_r \sin\psi - s_x \cos\psi) d\psi \\
&= \frac{N_b}{2\pi}\int_0^{2\pi}\int_e^R (dF_r \sin\psi - dF_x \cos\psi) d\psi
\end{aligned} \tag{1.45}
$$

Rotor Torque Q:

$$
\begin{aligned}
Q &= -\frac{N_b}{2\pi}\int_0^{2\pi} m_z d\psi \\
&= \frac{N_b}{2\pi}\int_0^{2\pi} (n_l + e s_x) d\psi \\
&= \frac{N_b}{2\pi}\int_0^{2\pi}\int_e^R \left[(r-e) dF_x + e dF_x\right] d\psi \\
&= \frac{N_b}{2\pi}\int_0^{2\pi}\int_e^R r dF_x d\psi
\end{aligned} \tag{1.46}
$$

Rotor Roll Moment Mx: Assume that the torsion moment is zero, i.e. $m_y = n_t \cong 0$.

$$
\begin{aligned}
M_X &= \frac{N_b}{2\pi}\int_0^{2\pi} m_x \sin\psi d\psi \\
&= \frac{N_b}{2\pi}\int_0^{2\pi} (n_f + e s_z) \sin\psi d\psi \\
&= \frac{N_b}{2\pi}\int_0^{2\pi} \left(\nu_\beta^2 - 1 - \frac{3}{2}\frac{e}{R}\right) I_\beta \Omega^2 (\beta - \beta_p) \sin\psi d\psi + \frac{N_b}{2\pi}\int_0^{2\pi}\int_e^R e\, dF_z \sin\psi d\psi
\end{aligned} \tag{1.47}
$$

For $e = 0$ and $\beta_p = 0$ an useful expression is obtained

$$\begin{aligned} M_X &= \frac{N_b}{2\pi}\int_0^{2\pi}\left(\nu_\beta^2 - 1\right) I_\beta \Omega^2 \beta \sin\psi d\psi \\ &= N_b\left(\nu_\beta^2 - 1\right) I_\beta \Omega^2 \frac{1}{N_b}\int_0^{2\pi}\beta \sin\psi d\psi \\ &= N_b\left(\nu_\beta^2 - 1\right) I_\beta \Omega^2 \beta_{1s} \end{aligned} \tag{1.48}$$

In non-dimensional form we have

$$C_{MX} = \frac{M_X}{\rho A(\Omega R)^2 R} = \frac{\sigma a}{2\gamma}\left(\nu_\beta^2 - 1\right)\beta_{1s} \tag{1.49}$$

Rotor Pitch Moment My: Assume that the torsion moment is zero, i.e. $m_y = n_t \cong 0$.

$$\begin{aligned} M_Y &= \frac{N_b}{2\pi}\int_0^{2\pi} -m_x \cos\psi d\psi \\ &= \frac{N_b}{2\pi}\int_0^{2\pi} -(n_f + e s_z)\cos\psi d\psi \\ &= -\frac{N_b}{2\pi}\int_0^{2\pi}\left(\nu_\beta^2 - 1 - \frac{3}{2}\frac{e}{R}\right) I_\beta \Omega^2 (\beta - \beta_p)\cos\psi d\psi - \frac{N_b}{2\pi}\int_0^{2\pi}\int_e^R e\, dF_z \cos\psi d\psi \end{aligned} \tag{1.50}$$

For $e = 0$ and $\beta_p = 0$ an useful expression is obtained

$$\begin{aligned} M_Y &= -\frac{N_b}{2\pi}\int_0^{2\pi}\left(\nu_\beta^2 - 1\right) I_\beta \Omega^2 \beta \cos\psi d\psi \\ &= -N_b\left(\nu_\beta^2 - 1\right) I_\beta \Omega^2 \frac{1}{N_b}\int_0^{2\pi}\beta \cos\psi d\psi \\ &= -N_b\left(\nu_\beta^2 - 1\right) I_\beta \Omega^2 \beta_{1c} \end{aligned} \tag{1.51}$$

In non-dimensional form we have

$$C_{MY} = \frac{M_Y}{\rho A(\Omega R)^2 R} = -\frac{\sigma a}{2\gamma}\left(\nu_\beta^2 - 1\right)\beta_{1c} \tag{1.52}$$

1.5 Rotor planes of reference

There are various physical planes which can be used to describe the rotor motion. Researchers and engineers use different planes for different purposes. For example, the expressions for inflow derived earlier were perpendicular to the plane of the disk tilt. This plane is also called the tip path plane (TPP). The tip of the blades lie in this plane, hence the name. For the purposes of rotor dynamic analysis, the hub plane (HP) is the most convenient plane. The hub plane is perpendicular to the rotor shaft. The rotor RPM, Ω, is along the shaft. Recall fig. 6.2. The vertical axis z was perpendicular to the hub plane. The inflow λ used in the expression for U_P was along z, i.e., it was perpendicular to the hub plane. This inflow must be calculated from the inflow expression derived using momentum theory earlier by transformation between TPP and HP. In general, it is often necessary to transform variables from one type of axes system to another.

For hover and vertical flight, the control is the thrust level which is obtained by the collective pitch setting. There is no variation of pitch or flap angle along the azimuth.

$\theta(\psi) = \theta_0$ collective

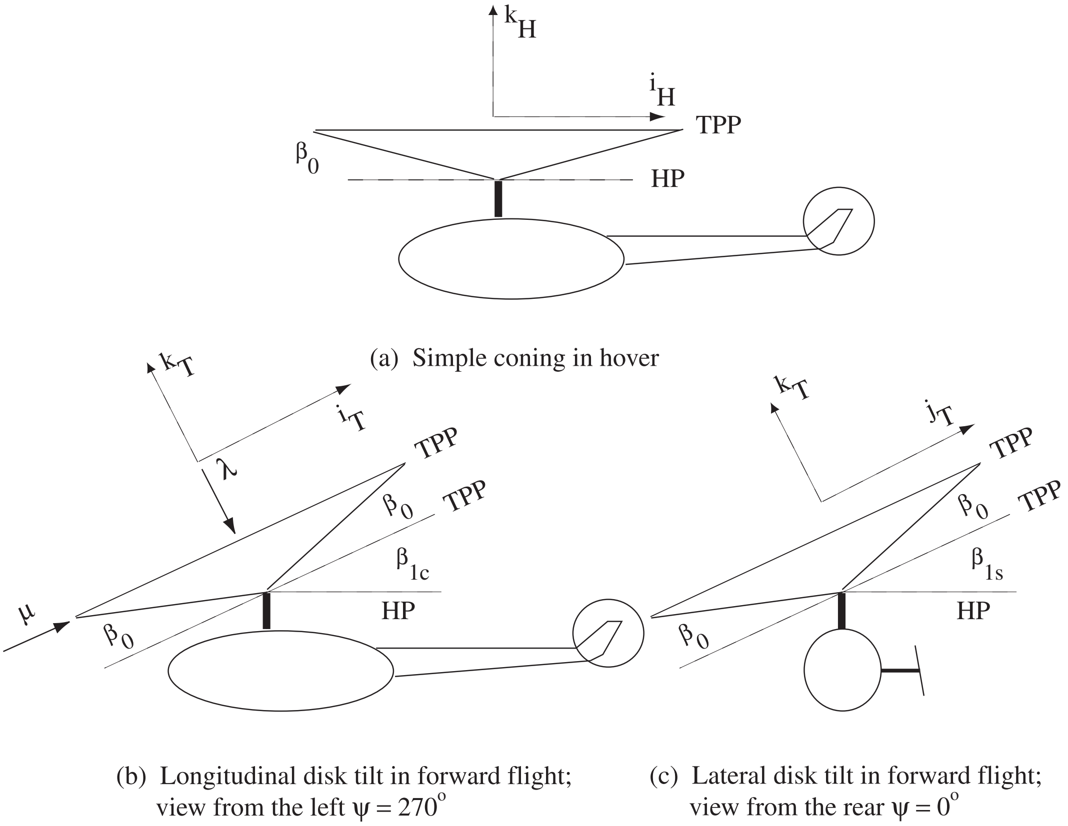


Figure 1.13: **Definition of tip path plane (TPP) and hub plane (HP)**

$\beta(\psi) = \beta_0$ coning

TPP is parallel to HP, see Fig. 1.13(a). Both are perpendicular to the shaft axis. The thrust vector acts along the shaft and is normal to both planes. In forward flight, the TPP is tilted longitudinally and laterally. Consider the following flapping motion.

$$\beta(\psi) = \underset{\text{coning}}{\beta_0} + \underset{\substack{\text{longitudinal}\\ \text{TPP tilt}}}{\beta_{1c}\cos\psi} + \underset{\substack{\text{lateral}\\ \text{TPP tilt}}}{\beta_{1s}\sin\psi}$$

Figures 1.13(b) and (c) show the longitudinal and lateral tilts for positive β_{1c} and β_{1s}. The tilt of the tip path plane tilts the thrust vector. The longitudinal tilt is forward. The vertical component of the thrust balances the weight and the horizontal component of the thrust provides a propulsive force. The lateral tilt is to the left or right depending on the roll moment requirement to trim the rotor. The transformation between the TPP and HP is obtained by subsequent rotations of the hub plane by β_{1c} and β_{1s}. If i_H, j_H, k_H and i_T, j_T, k_T are the unit vectors in HP and TPP, we have

$$\begin{Bmatrix} i_T \\ j_T \\ k_T \end{Bmatrix} = \begin{bmatrix} c\beta_{1c} & 0 & s\beta_{1c} \\ -s\beta_{1c}s\beta_{1s} & c\beta_{1s} & s\beta_{1s}c\beta_{1c} \\ -s\beta_{1c}c\beta_{1s} & -s\beta_{1s} & c\beta_{1s}c\beta_{1s} \end{bmatrix} \begin{Bmatrix} i_H \\ j_H \\ k_H \end{Bmatrix} \cong \begin{bmatrix} 1 & 0 & \beta_{1c} \\ 0 & 1 & \beta_{1s} \\ -\beta_{1c} & -\beta_{1s} & 1 \end{bmatrix} \begin{Bmatrix} i_H \\ j_H \\ k_H \end{Bmatrix} \tag{1.53}$$

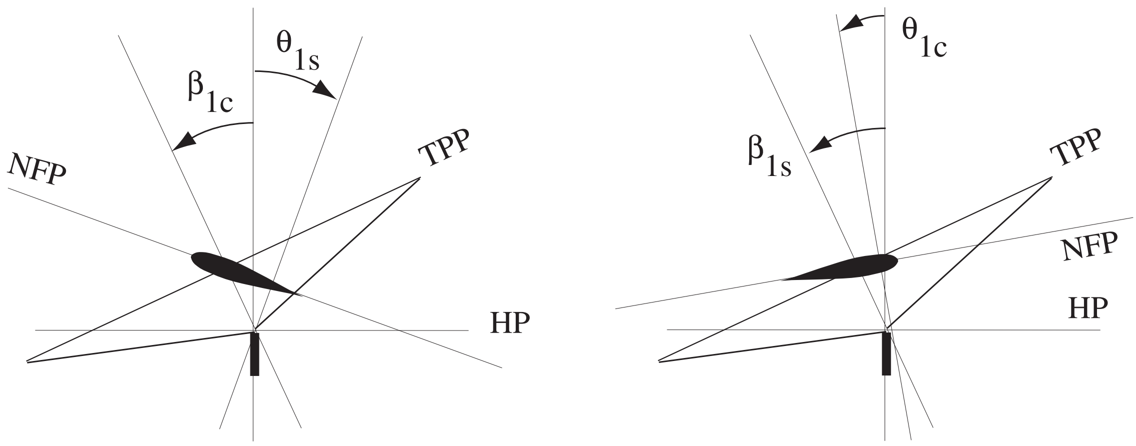
It follows for example,

$$\begin{aligned} \lambda_H &= \lambda_{TPP} - \mu\beta_{1c} \\ H_H &= H_{TPP} - \beta_{1c}T_T \\ Y_H &= Y_{TPP} - \beta_{1s}T_T \end{aligned} \tag{1.54}$$

The flapping motion is controlled by introducing collective and cyclic pitch angles through the swashplate.

$$\theta(\psi) = \underset{\text{collective}}{\theta_0} + \underset{\substack{\text{lateral}\\ \text{cyclic}}}{\theta_{1c}\cos\psi} + \underset{\substack{\text{longitudinal}\\ \text{cyclic}}}{\theta_{1s}\sin\psi}$$

The cyclic pitch angles lie in a plane. This is a plane from which one observes no variation of cyclic pitch. The longitudinal and lateral tilts of this plane are shown in Figs. 1.14(a) and (b). The



(b) Longitudinal tilt of no feathering plane view from the left $\psi = 270^{\circ}$

(b) Lateral tilt of no feathering plane view from the left $\psi = 0^{\circ}$

Figure 1.14: **Definition of no feathering plane (NFP), tip path plane (TPP) and hub plane (HP)**

transformation between the NFP and HP is obtained by subsequent rotations of the hub plane by θ_{1s} and θ_{1c}. If i_H, j_H, k_H and i_N, j_N, k_N are the unit vectors in HP and TPP, we have

$$\begin{Bmatrix} i_N \\ j_N \\ k_N \end{Bmatrix} = \begin{bmatrix} c\theta_{1s} & 0 & -s\theta_{1s} \\ s\theta_{1s}s\theta_{1c} & c\theta_{1c} & s\theta_{1c}c\theta_{1s} \\ c\theta_{1c}s\theta_{1s} & -s\theta_{1c} & c\theta_{1c}c\theta_{1s} \end{bmatrix} \begin{Bmatrix} i_H \\ j_H \\ k_H \end{Bmatrix} \cong \begin{bmatrix} 1 & 0 & \beta_{1c} \\ 0 & 1 & \theta_{1c} \\ \theta_{1s} & -\theta_{1c} & 1 \end{bmatrix} \begin{Bmatrix} i_H \\ j_H \\ k_H \end{Bmatrix} \tag{1.55}$$

It follows for example,

$$\begin{aligned} \lambda_H &= \lambda_{NFP} + \mu\theta_{1c} \\ H_H &= H_{NFP} + \theta_{1s}T_T \\ Y_H &= Y_{NFP} - \theta_{1c}T_T \end{aligned} \tag{1.56}$$

It is important to keep in mind the reference frame from which the flap and cyclic pitch angles are measured. From the hub plane, the flap and pitch angles are β_{1c}, β_{1s} and θ_{1c}, θ_{1s}. From the tip path plane, the flap angles are zero. Similarly, from the no feathering plane, the cyclic pitch angles are zero. Note that the angle between any two planes remain the same, irrespective of the plane from which they are measured. For example, the longitudinal tilt angle between NFP and TPP when measured from the hub plane is $(\beta_{1c} + \theta_{1s})$, see fig. 1.14(a). The same angle is only β_{1c} when measured from NFP. However this β_{1c} is different from the β_{1c} measured from the HP, but is equal to $(\beta_{1c} + \theta_{1s})$ as measured from the HP. Thus,

$$(\beta_{1c} + \theta_{1s})_H = (\beta_{1c})_N = (\theta_{1s})_T$$

Similarly for the lateral tilt, we have from fig. 1.14(b),

$$(\beta_{1s} - \theta_{1c})_H = (\beta_{1s})_N = -(\theta_{1c})_T$$

In addition to TPP, HP, and NFP, another plane can be defined. This is the plane of the swashplate, called the control plane (CP). See Fig. 1.15. As shown in the figure, if the pitch links are connected

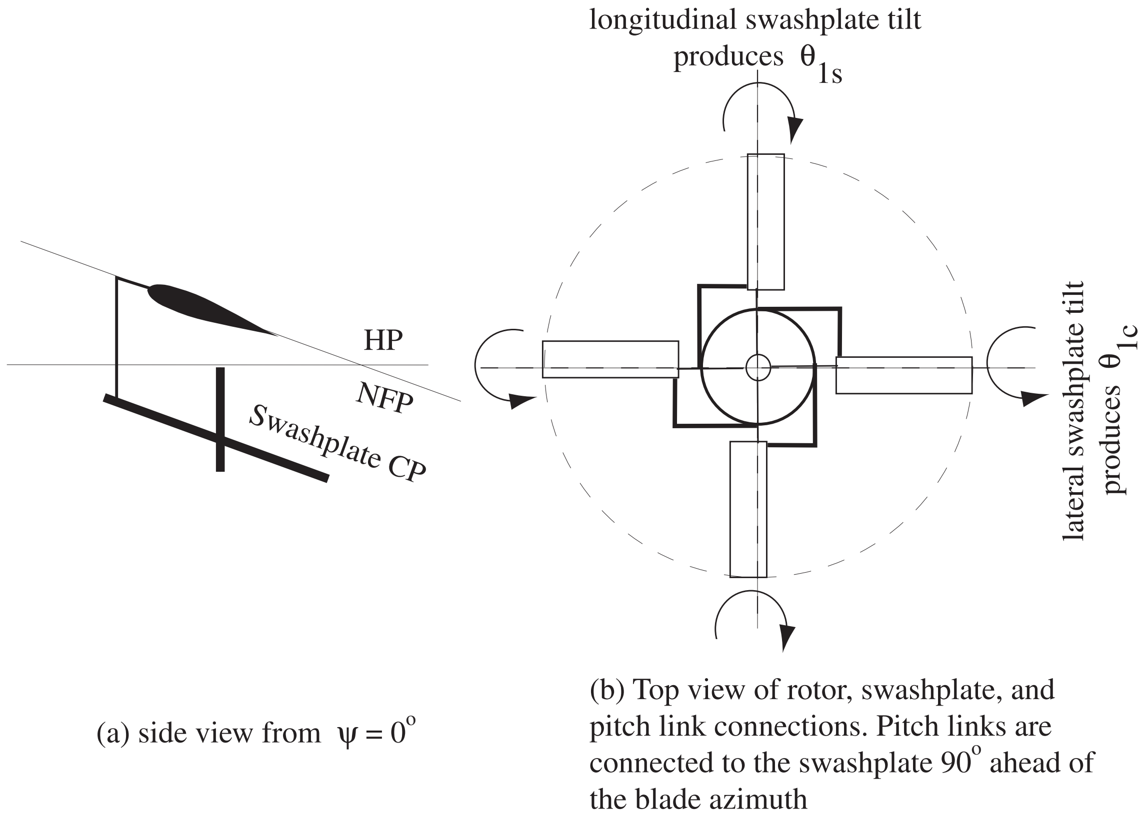


Figure 1.15: **Definition of control plane (CP)**

90^o ahead of the blade azimuth, the CP is parallel to the NFP. In addition, the pitch flap coupling must be zero for this condition to hold. The different rotor reference planes, and their use are briefly summarized below.

(a) Tip Path Plane (TPP): This is a plane described by blade tips, so that there is no cyclic variation of flap angles when measured from this plane. This plane is frequently used for wake studies and acoustic studies. The expressions for inflow derived earlier using moment theory were with respect to this plane. The TPP is same as the disk tilt plane.

(b) No Feather Plane (NFP): This is a plane from which there is no cyclic variation of control pitch. This is often used for performance and stability analysis, especially for autogyros. In Gessow and Myers' book, this plane is used for performance studies.

(c) Control Plate (CP): It represents the swashplate plane. This plane is important for servo-actuators.

(d) Hub Plane (HP): This plane is normal to the rotor shaft. Both cyclic flap and cyclic pitch control angles are non-zero when measured from this plane. This plane is routinely adopted for the blade dynamic analysis.

Finally, note that the concept of TPP and NFP is applicable only with the assumption of 1/rev variations of flap and cyclic pitch. In reality the flapping motion contains all harmonics, the 2/rev and higher harmonics create ripples over the tip path plane. Similarly in the case of higher harmonic control, when the swashplate is used to input higher harmonics of pitch angle, the NFP is no longer defined. Also note that, in Gessow and Myers book, the pitch and flap angles (including higher harmonics) are defined as

$$\theta(\psi) = A_0 - A_1 \cos\psi - B_1 \sin\psi - A_2 \cos 2\psi - B_2 \sin 2\psi \ldots$$

$$\beta(\psi) = a_0 - a_1 \cos\psi - b_1 \sin\psi - a_2 \cos 2\psi - b_2 \sin 2\psi \ldots$$

where

$$\theta_0 = A_0$$

$$\theta_{nc} = -A_n$$

$$\theta_{ns} = -B_n$$

$$\beta_0 = a_0$$

$$\beta_{nc} = -a_n$$

$$\beta_{ns} = -b_n$$

1.6 Helicopter Trim

Trimming an helicopter means maintaining equilibrium in space. The steady forces and moments generated by the rotor should be equal and opposite to those generated by the other parts of the helicopter, e.g. the tail rotor, the fuselage, the horizontal stabilizer etc. The steady forces and moments generated by the rotor should remain the same from one rotor revolution to another. In order to satisfy this condition it is necessary that the blades exhibit periodic motion. Therefore, helicopter trim involves two steps:

1. Achieving periodic blade response. Also called *uncoupled trim*.

2. Achieving periodic blade response such that specific targets are met. Also called *coupled trim*.

A trimmed flight can be achieved under any steady condition – axial flight, ascent and descent along a coordinated banked turn, and straight and level flight. In this section we consider a straight and level flight. Coupled trim is broadly classified into two types:

1. Isolated rotor trim.

2. Full aircraft trim.

For an isolated rotor trim, the three rotor control angles are determined based on three specified targets, e.g. the thrust, and rotor pitch and roll moments. When the targetted moments are zero, it is called moment trim. Alternatively, the thrust, and the first harmonic flapping motions, β_{1c} and β_{1s}, are specified. One popular approach is to specify zero first harmonic flapping. This procedure is widely used in wind tunnel trim. Isolated rotor trim is used in wind-tunnels to achieve specific flight conditions in a controlled environment.

Full aircraft trim is also called propulsive trim. The only assumption is that sufficient rotor power is available from the powerplants. The target rotor forces and moments are equal and opposite to those produced by the rest of the aircraft. The three rotor controls, the tail rotor collective, and the two aircraft attitude angles, longitudinal and lateral, are determined using the six vehicle equilibrium equations.

The trim procedures require the calculation of rotor forces and moments.

1.6.1 Rotor Forces and Moments

The steady rotor forces and moments in the hub plane can be derived using equations 9.83–8.88, and equations 1.29–1.31. Assume uniform inflow, linear lift curve slope $c_l = a\alpha$, and a constant drag coefficient $c_d = c_{d0}$. Recall, that in forward flight we have

$$
\begin{aligned}
u_t &= x + \mu \sin\psi \\
u_p &= \lambda + x \overset{*}{\beta} + \beta\mu\cos\psi \\
\beta &= \beta_0 + \beta_{1c}\cos\psi + \beta_{1s}\sin\psi \\
\theta &= \theta_0 + x\theta_{tw} + \theta_{1c}\cos\psi + \theta_{1s}\sin\psi
\end{aligned}
$$

where

$$\lambda = \lambda_H = \lambda_{TPP} - \mu\beta_{1c}$$

and

$$
\begin{aligned}
\lambda_{TPP} &= \mu\tan\alpha + \frac{k_f C_T}{2\sqrt{\mu^2 + \lambda_{TPP}^2}} \\
\alpha &= \alpha_s + \beta_{1c} + \theta_{FP}
\end{aligned}
\tag{1.57}
$$

where α_s is the longitudinal shaft tilt angle with respect to the horizontal plane, θ_{FP} is the flight path angle positive for climb. The rotor thrust coefficient C_T, same in all planes for small angles, is given by

$$
\begin{aligned}
C_T &= \frac{T}{\rho A(\Omega R)^2} \\
&= \frac{\sigma a}{2}\frac{1}{2\pi}\int_0^{2\pi}\int_0^1 \left(u_t^2\theta - u_p u_t\right) dx\, d\psi \\
&= \frac{\sigma a}{2}\left[\frac{\theta_0}{3}\left(1 + \frac{3}{2}\mu^2\right) + \frac{\theta_{tw}}{4}\left(1+\mu^2\right) + \frac{\mu}{2}\theta_{1s} - \frac{\lambda}{2}\right]
\end{aligned}
\tag{1.58}
$$

If the twist is expressed as $\theta_{75} + (x - 3/4)\theta_{tw} + \theta_{1c}\cos\psi + \theta_{1s}\sin\psi$, then we have

$$C_T = \frac{\sigma a}{2}\left[\frac{\theta_{75}}{3}\left(1 + \frac{3}{2}\mu^2\right) + \frac{\theta_{tw}}{8}\mu^2 + \frac{\mu}{2}\theta_{1s} - \frac{\lambda}{2}\right]$$

The inflow can be expressed in NFP and TPP as follows.

$$C_T = \frac{\sigma a}{2}\left[\frac{\theta_0}{3}\left(1 + \frac{3}{2}\mu^2\right) + \frac{\theta_{tw}}{4}\left(1+\mu^2\right) - \frac{\lambda_{NFP}}{2}\right] \tag{1.59}$$

$$C_T = \frac{\sigma a}{2}\left[\frac{\theta_0}{3}\left(1 + \frac{3}{2}\mu^2\right) + \frac{\theta_{tw}}{4}\left(1+\mu^2\right) - \frac{\lambda_{TPP}}{2} + \frac{\mu}{2}(\beta_{1c} + \theta_{1s})\right] \tag{1.60}$$

The rotor drag force is given by

$$
\begin{aligned}
C_H =& \frac{H}{\rho A(\Omega R)^2} \\
=& \frac{\sigma a}{2}\frac{1}{2\pi}\int_0^{2\pi}\int_0^1 \left[\left(u_p u_t\theta - u_p^2 + \frac{c_{do}}{a}u_t^2\right)\sin\psi - \beta\cos\psi\left(u_t^2\theta - u_p u_t\right)\right] dx\, d\psi \\
=& \frac{\sigma a}{2}\left[\theta_0\left(-\frac{1}{3}\beta_{1c} + \frac{1}{2}\mu\lambda\right) + \theta_{tw}\left(-\frac{1}{4}\beta_{1c} + \frac{1}{4}\mu\lambda\right)\right. \\
&+ \theta_{1c}\left(-\frac{1}{6}\beta_0 - \frac{1}{8}\mu\beta_{1s}\right) + \theta_{1s}\left(-\frac{1}{4}\mu\beta_{1c} + \frac{1}{4}\lambda\right) \\
&\left.+ \frac{3}{4}\lambda\beta_{1c} + \frac{1}{6}\beta_0\beta_{1s} + \frac{1}{4}\mu\left(\beta_0^2 + \beta_{1c}^2\right) + \frac{C_{do}}{a}\left(\frac{\mu}{2}\right)\right]
\end{aligned}
\tag{1.61}
$$

Now use

$$C_{H_{TPP}} = C_H + \beta_{1c} C_T; \quad \lambda = \lambda_{TPP} - \mu\beta_{1c}$$

to obtain

$$\begin{aligned} C_{H_{TPP}} =& \frac{\sigma a}{2}\left[\theta_0\left(\frac{1}{2}\mu\lambda_{TPP}\right) + \theta_{tw}\left(\frac{1}{4}\mu\lambda_{TPP}\right) + \theta_{1c}\left(-\frac{1}{6}\beta_0 - \frac{1}{8}\mu\beta_{1s}\right) + \theta_{1s}\left(\frac{1}{4}\lambda_{TPP}\right)\right. \\ &\left. + \frac{1}{4}\lambda_{TPP}\beta_{1c} + \frac{1}{6}\beta_0\beta_{1s} + \frac{1}{4}\mu\beta_0^2\right] + \frac{\sigma C_{d_0}}{4}\mu \end{aligned} \tag{1.62}$$

The rotor side force is given by

$$\begin{aligned} C_Y =& \frac{Y}{\rho A(\Omega R)^2} \\ =& \frac{\sigma a}{2}\frac{1}{2\pi}\int_0^{2\pi}\int_0^1\left[-\left(u_p u_t\theta - u_p^2 + \frac{c_{do}}{a}u_t^2\right)\cos\psi - \beta\sin\psi\left(u_t^2\theta - u_p u_t\right)\right] dx\, d\psi \\ =& \frac{\sigma a}{2}\left[-\theta_0\left(\frac{3}{4}\mu\beta_0 + \frac{1}{3}\beta_{1s}\right) - \theta_{tw}\left(\frac{1}{4}\beta_{1s} + \frac{1}{2}\mu\beta_0\right) - \theta_{1c}\left(\frac{1}{4}\lambda + \frac{1}{4}\mu\beta_{1c}\right)\right. \\ &\left. -\theta_{1s}\left(\frac{1}{6}\beta_0 + \frac{1}{2}\mu\beta_{1s}\right) + \frac{3}{4}\lambda\beta_{1s} + \frac{3}{2}\mu\lambda\beta_0 - \frac{1}{6}\beta_0\beta_{1c} + \frac{1}{4}\mu\beta_{1c}\beta_{1s}\right] \end{aligned} \tag{1.63}$$

Now use

$$C_{Y_{TPP}} = C_Y + \beta_{1s} C_T; \quad \lambda = \lambda_{TPP} - \mu\beta_{1c}$$

to obtain

$$\begin{aligned} C_{Y_{TPP}} =& \frac{\sigma a}{2}\left[-\theta_0\left(\frac{3}{4}\mu\beta_0\right) - \theta_{tw}\left(\frac{1}{2}\mu\beta_0\right) - \theta_{1c}\left(\frac{1}{4}\lambda_{TPP}\right) - \theta_{1s}\left(\frac{1}{6}\beta_0\right)\right. \\ &\left. + \frac{1}{4}\lambda_{TPP}\beta_{1s} + \frac{3}{2}\mu\lambda_{TPP}\beta_0 - \frac{1}{6}\beta_0\beta_{1c}\right] \end{aligned} \tag{1.64}$$

The rotor torque is

$$\begin{aligned} C_Q =& \frac{Q}{\rho A(\Omega R)^2 R} \\ =& \frac{\sigma a}{2}\frac{1}{2\pi}\int_0^{2\pi}\int_0^1 x\left(u_p u_t\theta - u_p^2 + \frac{c_{do}}{a}u_t^2\right) dx \\ =& \frac{\sigma a}{2}\left[\lambda\left(\frac{\theta_0}{3} + \frac{\theta_{tw}}{4} + \frac{1}{4}\mu\theta_{1s} - \frac{1}{2}\mu\beta_{1c} - \frac{\lambda}{2}\right) + \mu\left(\frac{1}{6}\theta_{1c}\beta_0 - \frac{1}{3}\beta_0\beta_{1s}\right)\right. \\ &+ \mu^2\left(\frac{1}{16}\beta_{1s}\theta_{1c} + \frac{1}{16}\beta_{1c}\theta_{1s} - \frac{1}{4}\beta_0^2 - \frac{3}{16}\beta_{1c}^2 - \frac{1}{16}\beta_{1s}^2\right) \\ &\left. + \frac{1}{8}\theta_{1c}\beta_{1s} - \frac{1}{8}\theta_{1s}\beta_{1c} - \frac{1}{8}\left(\beta_{1c}^2 + \beta_{1s}^2\right)\right] + \frac{\sigma c d_0}{8}\left(1 + \mu^2\right) \end{aligned} \tag{1.65}$$

Replace λ with $\lambda_{TPP} - \mu\beta_{1c}$, in the first term of the above expression to produce

$$\begin{aligned} C_Q =& \frac{\sigma a}{2}\left[\lambda_{TPP}\left(\frac{\theta_0}{3} + \frac{\theta_{tw}}{4} - \frac{\lambda_{TPP}}{2} + \frac{1}{2}\mu\beta_{1c} + \frac{1}{4}\mu\theta_{1s}\right)\right. \\ &- \mu\left(\frac{1}{3}\beta_{1c}\theta_0 + \frac{1}{4}\beta_{1c}\theta_{tw} - \frac{1}{6}\theta_{1c}\beta_0 + \frac{1}{3}\beta_0\beta_{1s}\right) \\ &+ \mu^2\left(\frac{1}{16}\beta_{1s}\theta_{1c} + \frac{1}{16}\beta_{1c}\theta_{1s} - \frac{1}{4}\beta_0^2 - \frac{3}{16}\beta_{1c}^2 - \frac{1}{16}\beta_{1s}^2\right) \\ &\left. + \frac{1}{8}\theta_{1c}\beta_{1s} - \frac{1}{8}\theta_{1s}\beta_{1c} - \frac{1}{8}\left(\beta_{1c}^2 + \beta_{1s}^2\right)\right] + \frac{\sigma c d_0}{8}\left(1 + \mu^2\right) \end{aligned} \tag{1.66}$$

The expressions given above for torque are exact. It is important that all terms are retained for accurate predictions beyond advance ratio $\mu = 0.3$. The roll and pitch moment coefficients are derived from equations 1.47 and 1.50 as

$$\begin{aligned} C_{MX} &= \frac{\sigma a}{2\gamma}\left(\nu_\beta^2 - 1 - \frac{3}{2}\frac{e}{R}\right)\beta_{1s} + \frac{e}{R}\frac{\sigma a}{2}\frac{1}{2\pi}\int_0^{2\pi}\int_0^1 \left(u_t^2\theta - u_p u_t\right)\cos\psi dx\, d\psi \\ C_{MY} &= -\frac{\sigma a}{2\gamma}\left(\nu_\beta^2 - 1 - \frac{3}{2}\frac{e}{R}\right)\beta_{1c} + \frac{e}{R}\frac{\sigma a}{2}\frac{1}{2\pi}\int_0^{2\pi}\int_0^1 \left(u_t^2\theta - u_p u_t\right)\sin\psi dx\, d\psi \end{aligned}$$

Assume $e/R \cong 0$ for the following simple expressions.

$$\begin{aligned} C_{MX} &= \frac{\sigma a}{2\gamma}\left(\nu_\beta^2 - 1\right)\beta_{1s} \\ C_{MY} &= -\frac{\sigma a}{2\gamma}\left(\nu_\beta^2 - 1\right)\beta_{1c} \end{aligned} \tag{1.67}$$

1.6.2 Uncoupled trim

Uncoupled trim is a periodic blade response obtained for a given set of rotor control angles. The forward speed, shaft tilt angle, and flight path angle are prescribed. The following procedure can be used.

1. Start with $\lambda_{TPP} = \mu\tan(\alpha_s + \theta_{FP})$, $\beta_{1c} = \beta_{1s} = 0$, $\mu = V/(\Omega R)$.
2. Calculate β_0, β_{1c}, and β_{1s} from eqns. 1.33.
3. Update $\mu = V\cos(\alpha_s + \beta_{1c} + \theta_{FP})/(\Omega R)$.
4. Calculate C_T from eqn. 1.60.
5. Update λ_{TPP} from eqn. 1.57.

Iterate steps 2 to 5 till convergence.

Example 1.2: An articulated rotor model with 4% flap hinge offset is exposed to a wind speed of 200 ft/sec in the wind tunnel. If the blade tip speed is 600 ft/sec and the blades are set at collective pitch of 5°, calculate the tip path plane orientation with shaft angle, α_s, of 0°, 10° and −10°. Assume Lock number, $\gamma = 8$, solidity ratio, $\sigma = 0.05$ and lift curve slope, $a = 6$.

Use the above procedure to obtain the following results.

	$\alpha_s = 0°$	$\alpha_s = 10°$	$\alpha_s = -10°$
μ	0.3323	0.3303	0.3197
β_0	0.083	0.017	0.1418
β_{1c}	−4.52°	−2.32°	−6.44°
β_{1s}	−0.0303	−0.00489	−0.0536
C_T	0.00457	0.00066	0.00845
λ_{TPP}	−0.0194	0.0456	−0.0816
$(\alpha_s + \beta_{1c})$	−4.52°	7.68°	−16.44°

For a backward tilt of the shaft of 10°, the TPP is tilted back further by 16.44°. For a zero tilt of the shaft, TPP is tilted back by 4.52°. The change in TPP tilt is 11.92°. For a forward tilt of shaft of 10°, the TPP is tilted forward by 7.68°. The change in TPP tilt is 12.2°. This means that for a forward tilt of shaft, the TPP tilts forward at a faster rate. This results in an instability of rotor disk with respect to the angle of attack and is called the angle of attack of instability.

1.6.3 Coupled trim for an isolated rotor

In a coupled trim for an isolated rotor, the three control pitch angles are determined based on specific targets. The following two targets are useful.

1. Target rotor thrust and the hub roll and pitch moments.

2. Target rotor thrust and the first harmonic flapping β_{1c} and β_{1s}.

The first type produces similar airloads and structural loads on the rotor as in real flight. The second type produces similar wake geometries and acoustic characteristics.

The second type is used during wind tunnel tests. For a given longitudinal shaft tilt α_s, a popular set of targets are the thrust and zero first harmonic flapping angles. Such a condition can occur in free flight only if the aircraft center of gravity is located at the rotor hub.

The following procedure can be used for wind tunnel trim. Here, C_T, and β_{1c}, β_{1s} are the targets. θ_0, θ_{1s}, θ_{1c} are the unknowns. Initialize the unknowns to zero.
1 : Calculate λ_{TPP} from eqn. 1.57.
2 : Calculate θ_0, β_0, θ_{1s}, θ_{1c}.
From eqns. 1.33 we have

$$\beta_0 = \frac{\gamma}{v_\beta^2}\left[\frac{\theta_0}{8}(1+\mu^2) + \frac{\theta_{tw}}{10}\left(1+\frac{5}{6}\mu^2\right) + \frac{\mu}{6}(\theta_{1s}+\beta_{1c}) - \frac{\lambda_{TPP}}{6}\right] + \frac{\omega_{\beta 0}^2}{\Omega^2 v_\beta^2}\beta_p \tag{1.68}$$

$$\theta_{1c} = \beta_{1s} + \frac{1}{(1+\frac{1}{2}\mu^2)}\left[\frac{8}{\gamma}(v_\beta^2-1)\beta_{1c} + \frac{4}{3}\mu\beta_0\right] \tag{1.69}$$

$$\theta_{1s} = -\beta_{1c} + \frac{1}{(1+\frac{3}{2}\mu^2)}\left[-\frac{8}{3}\mu\left(\theta_0 + \frac{3}{4}\theta_{tw} - \frac{3}{4}\lambda_{TPP}\right) + \frac{8}{\gamma}(v_\beta^2-1)\beta_{1s}\right] \tag{1.70}$$

where λ has been replaced with $\lambda_{TPP} - \mu\beta_{1c}$.
Substituting $\beta_{1c} + \theta_{1s}$ from eqn. 1.70 into eqn. 1.60 we have

$$\theta_0 = \frac{\frac{6C_T}{\sigma a}(1+\frac{3}{2}\mu^2) - \frac{3}{4}\theta_{tw}(1-\frac{3}{2}\mu^2+\frac{3}{2}\mu^4) + \frac{3}{2}\lambda_{TPP}(1-\frac{1}{2}\mu^2) + \frac{12}{\gamma}\mu(v_\beta^2-1)\beta_{1s}}{1-\mu^2+\frac{9}{4}\mu^4} \tag{1.71}$$

Iterate step 2 till convergence.

A similar procedure can be used for moment trim. Here C_T, and C_{MX}, C_{MY} are the targets. θ_0, θ_{1s}, θ_{1c} are the unknowns. Initialize the unknowns to zero.
1 : Calculate λ_{TPP} from eqn. 1.57.
2 : Calculate β_{1c}, β_{1s} using the pitch and roll moment expressions, e.g., eqns. 1.67.
3 : Calculate θ_0 from eqn. 1.71, and β_0, θ_{1s}, θ_{1c} from eqns. 1.68, 1.70 and 1.69.
Iterate steps 2 and 3 till convergence.

1.6.4 Coupled trim for a full aircraft

The target is to achieve 3 force and 3 moment equilibriums. It is necessary to have 6 control variables.

The rotor control angles, which can be set by the pilot, are θ_0, θ_{1c}, and θ_{1s}. The yaw control is via the tail rotor collective θ_t. The two aircraft attitude angles, the longitudinal tilt α_s, and lateral tilt ϕ_s can be used as the two additional control variables. Note that the pilot does not have a direct control over these variables. The helicopter must be *flown into* these vehicle orientations to achieve trim.

Mathematically, the problem is formulated as follows. For a specified aircraft gross weight and forward speed, the trim solution evaluates rotor controls, θ_o, θ_{1c} and θ_{1s}, rotor dynamics e.g. flapping $\beta(\psi)$, the vehicle orientation, α_s and ϕ_s, tail rotor collective setting, and the rotor inflow, λ. The equations are the flap equation, inflow equation, and the 6 vehicle equilibrium equations. A popular approach is to neglect altogether the yawing moment equilibrium equation and thereby neglect the influence of the tail rotor on the solution. Thus we have 7 unknowns – 3 rotor controls, 2 fuselage attitudes, plus flapping and inflow, and 7 equations – 3 vehicle forces, 2 vehicle moments, plus flapping and inflow.

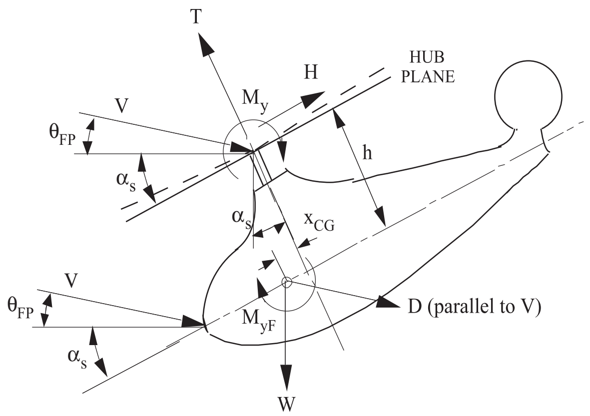
The flapping equation can be solved for any number of harmonics. Let us consider three harmonics here – β_0, β_{1c}, and β_{1s}.

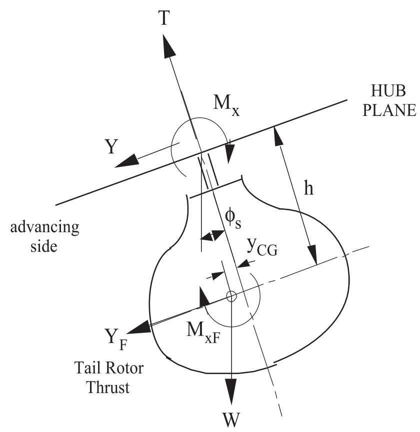
Aircraft Force and Moment Equilibrium Equations

Consider the left side view and front view of a helicopter in flight.

T = rotor thrust
H = rotor drag force
Y = rotor side force
W = weight
D = airframe drag in direction of V
Y_F = tail rotor thrust
M_X = rotor roll moment
M_y = rotor pitch moment
V = helicopter speed
M_{XF} = airframe roll moment
M_{YF} = airframe pitch moment
α_s = longitudinal shaft tilt with respect to vertical axis
ϕ_s = lateral shaft tilt with respect to vertical axis
X_{cg} = forward shift of cg from shaft axis
Y_{cg} = side shift of cg from shaft axis (positive right) towards advancing side
θ_{FP} = flight path angle

Note that the disk tilt, i.e. the TPP tilt $\alpha = \alpha_s + \beta_{1c} + \theta_{FP}$.





Vertical force equilibrium:

$$W - T\cos\alpha_s\cos\phi_s + D\sin\theta_{FP} - H\sin\alpha_s + Y\sin\phi_s + Y_F\sin\phi_s = 0 \tag{1.72}$$

Longitudinal force equilibrium:

$$D\cos\theta_{FB} + H\cos\alpha_s - T\sin\alpha_s\cos\phi_s = 0 \tag{1.73}$$

Lateral force equilibrium:

$$Y\cos\phi_s + Y_F\cos\phi_s + T\cos\alpha_s\sin\phi_s = 0 \tag{1.74}$$

Pitch moment equilibrium about hub:

$$M_y + M_{y_F} - W(X_{cg}\cos\alpha_s - h\sin\alpha_s) - D(x_{cg}\sin\alpha_s + h\cos\alpha_s) = 0 \tag{1.75}$$

Roll moment equilibrium about hub:

$$M_x + M_{x_F} + Y_F h + W(h\sin\phi_s - Y_{cg}\cos\phi_s) = 0 \tag{1.76}$$

Torque equilibrium about shaft:

$$Q - Y_F l_T = 0 \tag{1.77}$$

In addition to the six vehicle equilibrium equations we have an equation for the inflow and an equation for blade flapping. From the flapping equation, linear equations for the flap harmonics can be extracted, as many equations as the number of assumed harmonics. For example, eqns. 1.33 are three equations for three harmonics.

The trim equations can be simplified assuming: (1) small shaft tilt angles, and (2) zero flight path angle $\theta_{FP} = 0$. Additionally, the yaw degree of freedom can be ignored, i.e. remove the torque equation and the tail rotor collective θ_t as a trim variable.

$$W = T \tag{1.78}$$

$$D + H = T\alpha_s \tag{1.79}$$

$$Y + Y_F = -T\phi_s \tag{1.80}$$

$$M_y + M_{y_F} + W\left(h\alpha_s - X_{cg}\right) - hD = 0 \tag{1.81}$$

$$M_x + M_{x_F} + W\left(h\phi_s - Y_{cg}\right) + Y_F h = 0 \tag{1.82}$$

Non-dimensionalize all forces and moments by $\rho A\,(\Omega R)^2$ and $\rho A\,(\Omega R)^2\,R$ respectively. Define the fuselage drag D as

$$D = \frac{1}{2}\rho V^2 f \tag{1.83}$$

where f has units of area. It is the equivalent flat plate area of the hub, fuselage, landing gear etc. The drag coefficient then becomes

$$C_D = \frac{1}{2}\mu^2\,(f/A) \tag{1.84}$$

where A is the rotor disk area. Typically f/A varies from 1 to 3%. From eqn. 1.78

$$C_T = C_W \tag{1.85}$$

From eqns. 1.79 and 1.81 extract equations for α_s and β_{1c}. From eqn. 1.79

$$\begin{aligned}\alpha_s &= \frac{D}{W} + \frac{C_H}{C_T} = \frac{1}{2}\mu^2 \frac{f}{A}\frac{1}{C_T} + \frac{C_H}{C_T} \\ &= \frac{1}{2}\mu^2 \frac{f}{A}\frac{1}{C_T} + \frac{C_{HTPP}}{C_T} - \beta_{1c}\end{aligned} \tag{1.86}$$

From eqn. 1.81

$$\begin{aligned}\alpha_s &= \frac{X_{cg}}{h} + \frac{D}{W} - \frac{M_y}{hW} - \frac{M_{y_F}}{hW} \\ &= \frac{X_{cg}}{h} + \frac{1}{2}\mu^2 \frac{f}{A}\frac{1}{C_T} + \frac{\left(v_\beta^2-1\right)/\gamma}{\frac{h}{R}\frac{2C_T}{\sigma a}}\beta_{1c} - \frac{M_{y_F}}{hW}\end{aligned} \tag{1.87}$$

Equating the above two eqns. 1.86 and 1.87 obtain

$$\beta_{1c} = \frac{-\frac{X_{cg}}{h} + \frac{M_{y_F}}{hW} + C_{H_{TPP}}/C_T}{1 + \frac{(v_\beta^2-1)/\gamma}{\frac{h}{R}\frac{2C_T}{\sigma a}}} \tag{1.88}$$

Now use the above eqn. 1.88 in eqn. 1.86 to obtain

$$\alpha_s = \frac{\frac{x_{cg}}{h} - \frac{M_{y_F}}{hW} + \frac{(v_\beta^2-1)/\gamma}{\frac{h}{R}2\frac{C_T}{\sigma a}}\frac{C_{H_{TPP}}}{C_T}}{1 + \frac{(v_\beta^2-1)/\gamma}{\frac{h}{R}2\frac{C_T}{\sigma a}}} + \frac{1}{2}\frac{f}{A}\frac{\mu^2}{C_T} \tag{1.89}$$

Similarly use eqns. 1.80 and 1.82 to extract equations for ϕ_s and β_{1s}. From eqn. 1.80

$$\begin{aligned}\phi_s &= -\frac{Y_F}{W} - \frac{C_y}{C_T} \\ &= -\frac{Y_F}{W} - \frac{C_{y_{TPP}}}{C_T} + \beta_{1s}\end{aligned} \tag{1.90}$$

$Y_F/W = 0$ if the tail rotor in ignored. $Y_F/W = C_Q R/(C_T l_T)$ if the tail rotor is considered. l_T is the distance of the tail rotor thrust from the rotor hub. From eqn. 1.82

$$\begin{aligned}\phi_s &= \frac{y_{cg}}{h} - \frac{Y_F}{W} - \frac{M_x}{hW} - \frac{M_{x_F}}{hW} \\ &= \frac{y_{cg}}{h} - \frac{Y_F}{W} - \frac{\left(v_\beta^2-1\right)/\gamma}{h\frac{2C_T}{\sigma a}}\beta_{1s} - \frac{M_{x_F}}{hW}\end{aligned} \tag{1.91}$$

Equating the above two eqns. 1.90 and 1.91 obtain

$$\beta_{1s} = \frac{\frac{y_{cg}}{h} - \frac{M_{x_F}}{hW} + \frac{C_{y_{TPP}}}{C_T}}{1 + \frac{(v_\beta^2-1)/\gamma}{\frac{h}{R}\frac{2C_T}{\sigma a}}} \tag{1.92}$$

Now use the above eqn. 1.92 in eqn. 1.90 to obtain

$$\phi_s = \frac{\frac{y_{cg}}{h} - \frac{M_{x_F}}{hW} - \frac{(\nu_\beta^2-1)/\gamma}{\frac{h}{R}2\frac{C_T}{\sigma a}}\frac{C_{y_{TPP}}}{C_T}}{1 + \frac{(\nu_\beta^2-1)/\gamma}{\frac{h}{R}2\frac{C_T}{\sigma a}}} - \frac{C_Q}{C_T}\frac{R}{l_T} \tag{1.93}$$

Recall, that the inflow equation was (see eqn. 1.57)

$$\begin{aligned}\lambda_{TPP} &= \mu\tan(\alpha_s+\beta_{1c}+\theta_{FP}) + \frac{k_f C_T}{2\sqrt{\mu^2+\lambda_{TPP}^2}} \\ &\cong \mu(\alpha_s+\beta_{1c}+\theta_{FP}) + \frac{k_f C_T}{2\sqrt{\mu^2+\lambda_{TPP}^2}} \\ &= \mu\left(\frac{C_D+C_H}{C_T}\right) + \mu\beta_{1c} + \lambda_c + \frac{k_f C_T}{2\sqrt{\mu^2+\lambda_{TPP}^2}} \\ &= \mu\left(\frac{C_D+C_H+\beta_{1c}C_T}{C_T}\right) + \lambda_c + \frac{k_f C_T}{2\sqrt{\mu^2+\lambda_{TPP}^2}} \\ &= \frac{1}{2}\mu^3\left(\frac{f}{A}\right)\frac{1}{C_T} + \mu\frac{C_{HTPP}}{C_T} + \lambda_c + \frac{k_f C_T}{2\sqrt{\mu^2+\lambda_{TPP}^2}}\end{aligned} \tag{1.94}$$

The control angles θ_0, θ_{1c}, θ_{1s}, and coning β_0 can be calculated in the same manner as was done in coupled trim for an isolated rotor. The description is repeated here. From eqns. 1.33 we have

$$\beta_0 = \frac{\gamma}{v_\beta^2}\left[\frac{\theta_0}{8}(1+\mu^2) + \frac{\theta_{tw}}{10}\left(1+\frac{5}{6}\mu^2\right) + \frac{\mu}{6}(\theta_{1s}+\beta_{1c}) - \frac{\lambda_{TPP}}{6}\right] + \frac{\omega_{\beta 0}^2}{\Omega^2 v_\beta^2}\beta_p \tag{1.95}$$

$$\theta_{1c} = \beta_{1s} + \frac{1}{(1+\frac{1}{2}\mu^2)}\left[\frac{8}{\gamma}(v_\beta^2-1)\beta_{1c} + \frac{4}{3}\mu\beta_0\right] \tag{1.96}$$

$$\theta_{1s} = -\beta_{1c} + \frac{1}{(1+\frac{3}{2}\mu^2)}\left[-\frac{8}{3}\mu\left(\theta_0 + \frac{3}{4}\theta_{tw} - \frac{3}{4}\lambda_{TPP}\right) + \frac{8}{\gamma}(v_\beta^2-1)\beta_{1s}\right] \tag{1.97}$$

where λ has been replaced with $\lambda_{TPP} - \mu\beta_{1c}$. Substituting $\beta_{1c} + \theta_{1s}$ from the above equation into eqn. 1.60 for C_T, and solving for θ_0 we have

$$\theta_0 = \frac{\frac{6C_T}{\sigma a}(1+\frac{3}{2}\mu^2) - \frac{3}{4}\theta_{tw}(1-\frac{3}{2}\mu^2+\frac{3}{2}\mu^4) + \frac{3}{2}\lambda_{TPP}(1-\frac{1}{2}\mu^2) + \frac{12}{\gamma}\mu(v_\beta^2-1)\beta_{1s}}{1-\mu^2+\frac{9}{4}\mu^4} \tag{1.98}$$

The rotor drag and side forces are obtained from eqns. 1.62 and 1.64.

The above expressions can be used to calculate rotor trim iteratively using the following sequence.

1: Calculate C_T

$$C_T \cong C_W = \frac{W}{\rho\pi R^2(\Omega R)^2}$$

Initialize λ_{TPP}

$$\lambda_{TPP} = \kappa_f\frac{C_T}{2\mu} + \frac{1}{2}\left(\frac{f}{A}\right)\frac{\mu^3}{C_T}$$

Initialize $C_{H_{TPP}}$ and $C_{Y_{TPP}}$ to zero

Now iterate until λ_{TPP} converges as follows:

2 : Calculate $\beta_1 c$ using eqn. 1.88

3 : Calculate $\beta_1 s$ using eqn. 1.92

4 : Calculate α_s using eqn. 1.89

5 : Calculate ϕ_s using eqn. 1.93

6 : Calculate θ_0 using eqn. 1.98

7 : Calculate θ_{1s} using eqn. 1.97

8 : Calculate β_0 using eqn. 1.95
9 : Calculate θ_{1c} using eqn. 1.96
10 : Update λ_{TPP} using the last of eqns. 1.94
11 : Calculate $C_{H_{TPP}}$ using eqn. 1.62
12 : Calculate $C_{Y_{TPP}}$ using eqn. 1.64
Back to beginning of iteration.

In case of hover, λ_{TPP} remains fixed to the uniform inflow value. Any one of the other variables can be iterated over.

The rotor power can be calculated using eqn. 1.66. A simpler alternative expression is given in the next section. When yaw equilibrium is considered, then for a conventional configuration, the tail rotor collective is a trim variable. The yaw equilibrium equation is given by eqn. 1.77. In non-dimensional form

$$C_Q - \frac{l_T}{R} C_{YF} \frac{(\Omega_T R_T)^2}{(\Omega R)^2} \frac{A_T}{A} = 0$$

where $(\Omega_T R_T)^2/(\Omega R)^2$ is the tip speed ratio of the tail rotor to the main rotor, $C_{YF} = Y_F/\rho A_T(\Omega_T R_T)^2$ is the tail rotor thrust coefficient, and A_T/A is the ratio of tail rotor disk area to main rotor disk area. The tail rotor collective is then related to the tail rotor thrust by

$$\theta_{75T} = \frac{6C_{YF}}{\sigma_T a_T} + \frac{3}{2}\kappa_h \sqrt{\frac{C_{YF}}{2}}$$

with assumption of uniform inflow and linear tail rotor twist. σ_T and a_T are the tail rotor solidity and blade element lift curve slopes.

1.6.5 Rotor Power and Lift to Drag Ratio

Consider the non-dimensional torque C_Q, as in eqn. 1.66. Recall, that the non-dimensional power is equal to the non-dimensional torque C_Q. The expression was of the following form

$$C_Q = \frac{\sigma a}{2}\left[\lambda_{TPP}\left(\frac{\theta_0}{3} + \frac{\theta_{tw}}{4} - \frac{\lambda_{TPP}}{2} + \frac{\mu\beta_{1c}}{2} + \frac{\mu\theta_{1s}}{4}\right)\right] + \frac{\sigma a}{2}\frac{1}{4}\frac{C_{do}}{a}\left(1+\mu^2\right) + \ldots \text{ other terms} \tag{1.99}$$

where the '... other terms' are terms that are independant of inflow λ_{TPP} and profile drag c_{do}, and are functions of only the blade flapping angle and the control angles. From the expression of thrust in the tip path plane (eqn. 1.60) we have

$$\frac{\sigma a}{2}\left(\frac{\theta_0}{3} + \frac{\theta_{tw}}{4} - \frac{\lambda_{TPP}}{2} + \frac{\mu\beta_{1c}}{2} + \frac{\mu\theta_{1s}}{4}\right) = C_T - \frac{\sigma a}{2}\left(\frac{\theta_0\mu^2}{2} + \frac{\theta_{tw}\mu^2}{4} + \frac{\theta_{1s}\mu}{4}\right) \tag{1.100}$$

Using the above expression we have

$$C_Q = \lambda_{TPP}C_T - \lambda_{TPP}\frac{\sigma a}{2}\left(\frac{\theta_0\mu^2}{2} + \frac{\theta_{tw}\mu^2}{4} + \frac{\theta_{1s}\mu}{4}\right) + \frac{\sigma a}{2}\frac{1}{4}\frac{C_{do}}{a}\left(1+\mu^2\right) + \ldots \text{ other terms} \tag{1.101}$$

Now from eqns. 1.94 we have

$$\lambda_{TPP} = \mu\left(\frac{C_D + C_H}{C_T}\right) + \mu\beta_{1c} + \lambda_c + \lambda_i \tag{1.102}$$

Hence

$$\begin{aligned} \lambda_{TPP}C_T &= \mu C_D + \mu\left(C_H + \beta_{1c}C_T\right)\lambda_c C_T + \lambda_i C_T \\ &= \mu C_D + \mu C_{H_{TPP}} + \lambda_c C_T + \lambda_i C_T \end{aligned} \tag{1.103}$$

Substitute the above expression of $\lambda_{TPP}C_T$ in the expression for C_Q

$$\begin{aligned} C_Q =& \mu C_D + \mu C_{H_{TPP}} + \lambda_c C_T + \lambda_i C_T \\ & - \lambda_{TPP}\frac{\sigma a}{2}\left(\frac{\theta_0\mu^2}{2} + \frac{\theta_{tw}\mu^2}{4} + \frac{\theta_{1s}\mu}{4}\right) + \frac{\sigma a}{2}\frac{1}{4}\frac{C_{do}}{a}\left(1+\mu^2\right) + \ldots \text{ other terms} \end{aligned} \tag{1.104}$$

Now, $\mu C_{H_{TPP}}$ can be calculated from eqn. 1.62 as

$$\mu C_{H_{TPP}} = \lambda_{TPP}\frac{\sigma a}{2}\left(\frac{\theta_0\mu^2}{2} + \frac{\theta_{tw}\mu^2}{4} + \frac{\theta_{1s}\mu}{4}\right) + \frac{\sigma a}{2}\frac{1}{4}\frac{C_{do}}{a}2\mu^2 \quad + (\ldots\ldots) \tag{1.105}$$

It can be shown that the terms $(\ldots\ldots)$ cancel with those described earlier as ... other terms. Also, recall that

$$\mu C_D = \frac{1}{2}\mu^3\left(\frac{f}{A}\right)$$

Thus the final expression of non-dimensional power (or torque) in forward flight takes the following form

$$\begin{aligned} C_P &= \lambda_i C_T + \frac{\sigma C_{d0}}{8}\left(1+3\mu^2\right) + \frac{1}{2}\mu^3\left(\frac{f}{A}\right) + \lambda_c C_T \\ &= \frac{\kappa_f C_T}{2\sqrt{\lambda_{TPP}^2 + \mu^2}} + \frac{\sigma C_{d0}}{8}\left(1+3\mu^2\right) + \frac{1}{2}\mu^3\left(\frac{f}{A}\right) + \lambda_c C_T \end{aligned} \tag{1.106}$$

λ_i is the induced inflow perpendicular to the tip path plane. The above is the familiar form used in a simple momentum theory analysis of a rotor in forward flight using uniform inflow.

$$C_P = C_{Pi} + C_{Po} + C_{Pp} + C_{Pc}$$

C_{Pi} = rotor induced power required to produce thrust
C_{Po} = rotor profile power required to overcome rotor drag (turn in real fluid)
C_{Pp} = parasite power required to overcome airframe drag
C_{Pc} = rotor climb power required to increase gravitational potential.
The induced power is given by

$$C_{Pi} = \frac{\kappa_f C_T}{2\sqrt{\lambda_{TPP}^2 + \mu^2}} \quad \approx \quad \kappa_f\frac{C_T^2}{2\mu} \quad \text{for} \quad \mu > 0.15 \tag{1.107}$$

The profile power is often modified empirically to include radial flow and reversed flow effects

$$C_{Po} = \frac{\sigma C_{d_o}}{8}(1 + 4.6\mu^2) \tag{1.108}$$

The parasite power is

$$C_{p_p} = \frac{1}{2}\mu^3\left(\frac{f}{A}\right)$$

The climb power is given by

$$C_{Pc} = \lambda_c C_T \qquad \left(\lambda_c = \frac{V_c}{\Omega R}\right)$$

where V_c is the climb velocity. Thus, the climb velocity can be calculated from the available power and level flight power as

$$V_c = \frac{P_a - (P_i + P_o + P_p)}{T} = \frac{\Delta P}{W}$$

Note that, while using blade element theory, the required rotor power is calculated directly from eqn. 1.66. This expression includes all components of power and is difficult to extract the individual components. The analytical extraction is given above to identify the different components and have a physical feel regarding the growth and decay of each with forward speed. The induced power decreases with forward speed. The profile power increases as square of forward speed. The parasite power increases as cube of forward speed. The reduction of induced power with forward speed is due to the uniform inflow assumption. In real flight the induced power increases, gradually above $\mu > 0.25$ due to nonuniform inflow. Either of the expressions, eqn. 1.66 or eqn. 1.106, can be used. Both produce the same result. If eqn. 1.106 is used, often the radial flow corrected expression of profile power (eqn. 1.108) is used.

The power to generate thrust (induced power) and to overcome rotor drag (profile power) together can be associated with an effective drag of a rotor C_{DE}.

$$C_{Pi} + C_{Po} = \mu C_{DE}$$

That is,

$$\mu C_{DE} = C_P - (C_{Pp} + C_{Pc}) = C_P - (\mu C_D + \lambda_c C_T)$$

In level flight then,

$$C_{DE} = \frac{C_P}{\mu} - C_D$$

Under trim condition the net rotor propulsive force C_X must equal the airframe drag C_D, hence the above expression is also written as

$$C_{DE} = \frac{C_P}{\mu} - C_X$$

where $X = T \sin\alpha_s \cos\phi_s - H\cos\alpha_s = D$. The rotor lift-to-drag ratio is given by the ratio between lift and effective drag

$$(L/D_E) = \frac{C_L}{C_P/\mu - C_D} \approx \frac{C_T}{C_P/\mu - C_D}$$

Just as Figure of Merit is the measure of rotor efficiency in hover, L/DE is the measure of rotor efficiency in forward flight. Note that during autorotation, $C_P = 0$, and the rotor effective drag equals the airframe drag (or propulsive force).

$$C_{DE} = -C_X \quad \text{in autorotation}$$

Example 1.3:

Numerical results are calculated for a rotor with the following characteristics. Yaw equilibrium is ignored.

Rotor

$N_b = 4$	$R = 25ft$	$c = 1.5ft$	$\Omega R = 700ft/s$
$v_\beta = 1.05/rev$	$\gamma = 8.0$	$C_{la} = 5.73$	$C_{d0} = 0.01$

Vehicle

$W = 15,000 lbs$	$h = 6.0 ft$	$l_T = 32 ft$	$f = 20 ft^2$
$x_{cg} = -2 ft$	$y_{cg} = 0 ft$	Engine $= 2000 HP$	$\rho = 0.002377$slugs$/ft^3$
$M_{xF} = 0 ft - lbs$	$M_{yF} = 0 ft - lbs$	$\kappa_{\text{h}} = 1.15$	$\kappa_f = 1.00$

(a) Hover at sea level

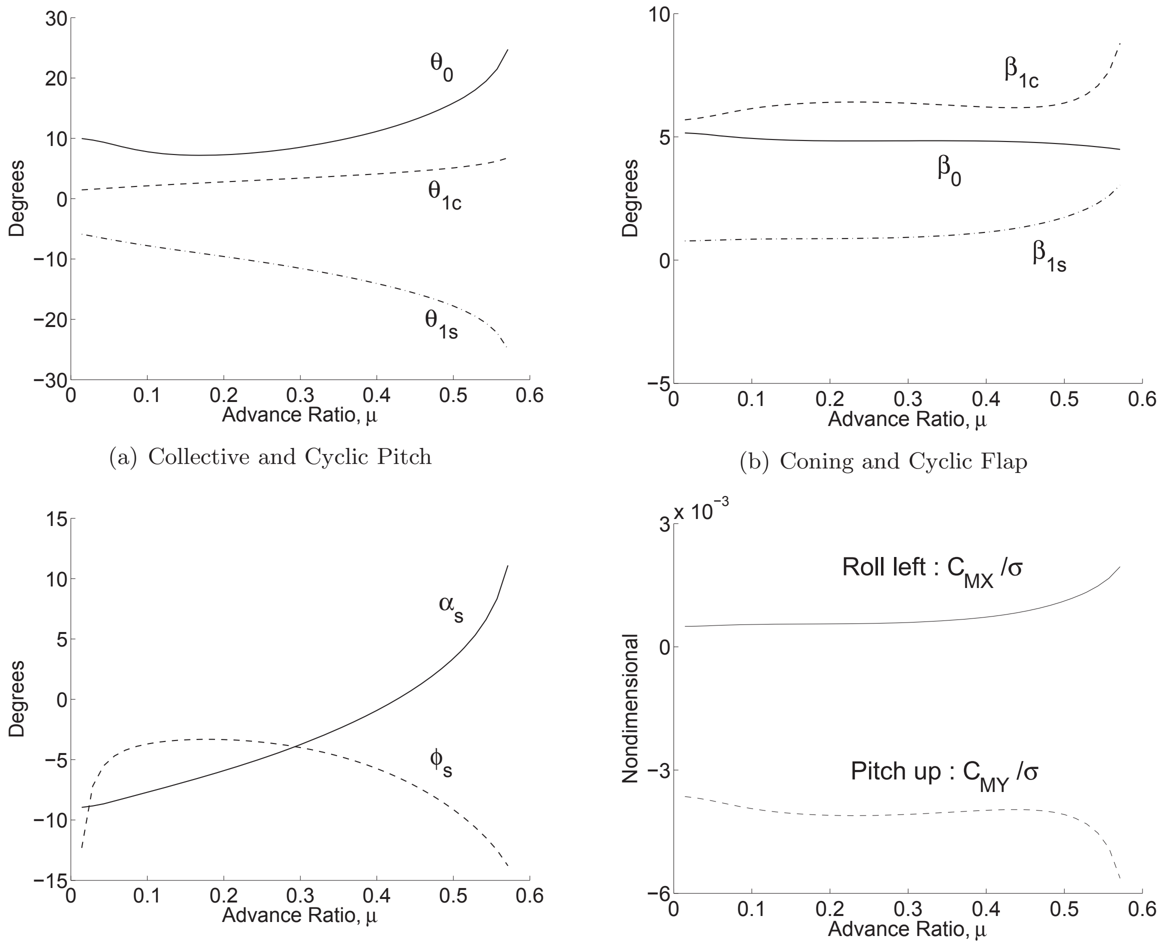
shaft HP required = 1535 HP
$\theta_0 = 10.81°$ $\theta_{1c} = 1.31°$ $\theta_{1s} = -5.47°$
$\alpha_s = -9.22°$ $\alpha_s = -4.92°$
$\beta_0 = 5.24°$ $\beta_{1c} = 5.55°$ $\beta_{1s} = 0.74°$
Maximum climb velocity v_c= 34.08 ft/sec

(b) Forward flight of 200 ft/sec at sea level

shaft HP required = 947 HP
$\mu = 0.2857$ $C_T = 0.006559$ $\lambda_{TPP} = 0.02284$
$C_P = 0.000325$
$\theta_0 = 8.25°$ $\theta_{1c} = 3.31°$ $\theta_{1s} = -11.24°$
$\beta_0 = 4.84°$ $\beta_{1c} = 6.38°$ $\beta_{1s} = 0.91°$
$\alpha_s = -4.10°$ $\phi_s = -3.84°$
Maximum climb velocity v_c = 38.6 ft/sec
The variation of trim parameters with advance ratio are shown in figure 1.16

Example 1.4:
The rotor and vehicle characteristics are given below.
Rotor
4-bladed, radius = 27 ft, chord = 1.75 ft
Tip speed ΩR = 700 ft/sec, Lock number $\gamma = 8$
Hingeless blades with flap frequency = 1.08 /rev
Airfoil $C_{la} = 6$, $C_{d0} = 0.01$
Vehicle
Weight = 16,000 lbs h/R = 0.2R
Assume $M_{x_F} = M_{y_F} = Y_F = 0$
f/A = 0.1 (flat plate area/Disk Area)
$x_{cg} = 0.01R$ (forward of shaft axis), $y_{cg} = 0$
Engine Shaft Power = 2000 HP
Assume uniform inflow in hover and forward flight ($\kappa_h = \kappa_f = 1.15$).
Calculate for hover
(a) shaft HP needed
(b) control settings
(c) maximum climb velocity
(d) flap response
Calculate for a forward flight of 280 ft/sec
(e) shaft HP needed
(f) control settings
(g) maximum climb velocity
(h) flap response
Ignore yaw equilibrium.

Hover



(a) Collective and Cyclic Pitch

(b) Coning and Cyclic Flap

(c) Shaft angles w.r.t. vertical axis

(d) Hub roll and pitch moments

Figure 1.16: **Variation of aircraft trim angles with forward flight speed (Example 1.3)**

$$C_T = \frac{W}{\rho \pi R^2 (\Omega R)^2} = \frac{1600}{(0.002378)\pi(27)^2(700)^2} = 0.006$$

$$\sigma = \frac{N_b c}{\pi R} = \frac{4(1.75)}{\pi(27)} = 0.0825$$

$$\lambda = \kappa \sqrt{\frac{C_T}{2}} = 0.063$$

$$C_P = \lambda C_T + \frac{\sigma C_{do}}{8} = 0.063(0.006) + \frac{0.0825(0.01)}{8} = 0.00048$$

(a) The shaft HP is given by

$$\text{P} = C_p \pi R^2 \rho (\Omega R)^3 = 0.00048\pi(27)^2(0.002378)(700)^3 = 8.988 \times 10^5 \text{ ft-lb/sec}$$

$$= \frac{8.988 \times 10^5}{550} = 1634 \text{ HP}$$

(b) Using the iterative procedure with $\mu = 0$

$$\theta_0 = 9.57^\circ$$

$\theta_{1c} = -0.109^\circ$

$\theta_{1s} = 0.469^\circ$

$\alpha_s = 0.166^\circ$

$\phi_s = -0.026^\circ$

(c) Maximum climb velocity

$$V_c = \frac{2\Delta P}{W} = \frac{2(2000 - 1634)}{16000} 550 = 25.2 \text{ ft/sec}$$

(d) Flap Response

$\beta_0 = 4.09^\circ$

$\beta_{1c} = -0.48^\circ$

$\beta_{1s} = -0.03^\circ$

Forward Flight

(e) Using the iterative procedure, the Shaft HP is

$\mu = 0.3947$

$C_T = 0.0060$

$\lambda = 0.05877 \quad \alpha = 9.559, \quad \phi_s = -0.1291^\circ$

$C_P = 0.0005369 \quad$ P = 1823HP

$C_{H_{TPP}} = 0.0002034 \quad C_{Y_{TPP}} = 0.0000167$

(f) Control angles

$\theta_0 = 10.9^\circ$

$\theta_{1c} = 1.823^\circ$

$\theta_{1s} = -9.963^\circ$

(g) Maximum climb velocity

$$V = \frac{\Delta P}{W} = 6.08 \text{ ft/sec}$$

(h) Flap response

$\beta_0 = 3.728^\circ$

$\beta_{1c} - 0.174^\circ$

$\beta_{1s} = -0.0301^\circ$

1.6.6 The Jacobian Method for Trim

The method described earlier, using analytical expressions for rotor forces and moments, was a point iteration procedure, also called Picard's iterations. In this procedure, the general approach to solving a set of nonlinear equations

$$f_1(x_1, x_2, \ldots, x_n) = 0$$

$$f_2(x_1, x_2, \ldots, x_n) = 0$$

$$\ldots$$

$$f_n(x_1, x_2, \ldots, x_n) = 0$$

or in vector notation

$$\mathbf{f}(\mathbf{x}) = \mathbf{0}$$

is to re-express $\mathbf{f}(\mathbf{x})$ as $\mathbf{g}(\mathbf{x}) - \mathbf{x}$ so that the equation takes the following form

$$\mathbf{x} = \mathbf{g}(\mathbf{x})$$

The solution procedure is then simply to iterate

$$\mathbf{x}^{k+1} = \mathbf{g}(\mathbf{x}^k); \quad k = 0, 1, 2, \ldots,$$

The procedure is useful for simple models and initial design calculations – even though convergence is not gauranteed. For non-uniform inflow, higher frequencies of blade dynamics, unsteady aerodynamics, and for the nonlinear trim equations, there will not be analytical expressions. The rotor forces and moments are then obtained numerically by integrating the blade element forces. The non-linear trim equations are then solved using the Newton-Raphson procedure.

The Newton-Raphson procedure is based on the calculation of trim Jacobian. Start from an initial estimate of the six trim variables $\mathbf{x}^0 = x_1^0, x_2^0, \ldots, x_6^0$. Calculate the rotor forces and moments using these initial estimates. Initial estimates are often obtained using the simple model given in the previous section. Now substitute in the vehicle equilibrium eqns. 1.72 – 1.77. These equations have the general form $\mathbf{f}(\mathbf{x}) = 0$. Upon substitution, the right hand side of the equations will not be zero but have non-zero residuals, since obviously $\mathbf{f}(\mathbf{x}^0) \neq 0$. The objective is to determine an increment $\Delta\mathbf{x}$ such that

$$\mathbf{f}(\mathbf{x}^0 + \Delta\mathbf{x}) = \mathbf{0}$$

A Taylor expansion of the above leads to

$$\begin{aligned}
&f_1 + \frac{\partial f_1}{\partial x_1}\Delta x_1 + \frac{\partial f_1}{\partial x_2}\Delta x_2 + ... + \frac{\partial f_1}{\partial x_6}\Delta x_6 + \text{higher order terms} = 0 \\
&f_2 + \frac{\partial f_2}{\partial x_1}\Delta x_1 + \frac{\partial f_2}{\partial x_2}\Delta x_2 + ... + \frac{\partial f_2}{\partial x_6}\Delta x_6 + \text{higher order terms} = 0 \\
&... \\
&f_6 + \frac{\partial f_6}{\partial x_1}\Delta x_1 + \frac{\partial f_6}{\partial x_2}\Delta x_2 + ... + \frac{\partial f_6}{\partial x_6}\Delta x_6 + \text{higher order terms} = 0
\end{aligned} \tag{1.109}$$

where the derivatives and functions are evaluated about the solution $\mathbf{x}^0$. Dropping the higher order terms we have the requirement

$$\mathbf{f}^0 + \mathbf{J}\Delta\mathbf{x} = \mathbf{0}$$