Mathematical Preliminaries

Introduction to Electromagnetism with Practice

Theory & Applications

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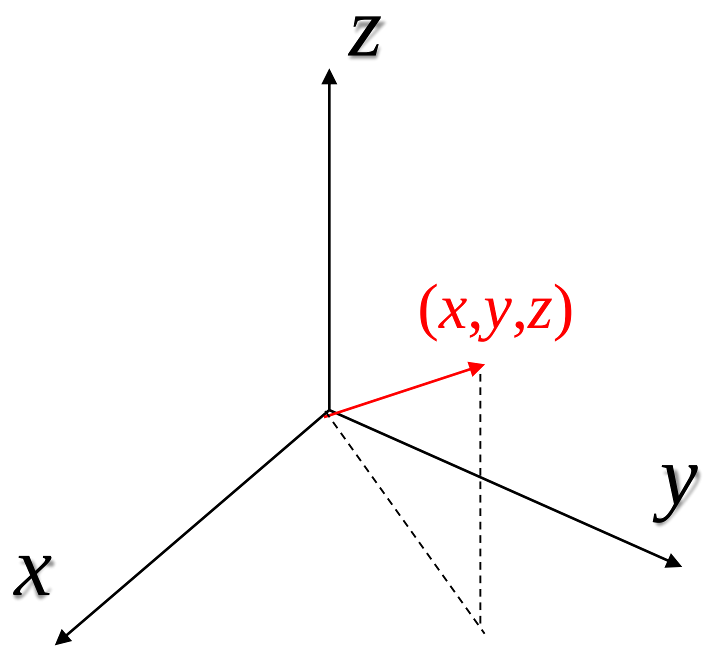
Curvilinear Coordinates

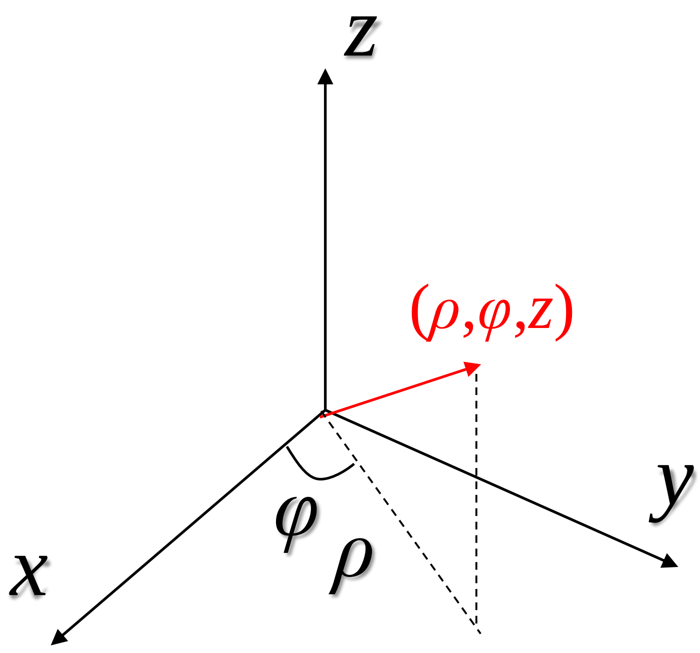
Remind: Cartesian Coordinates

3 Lengths



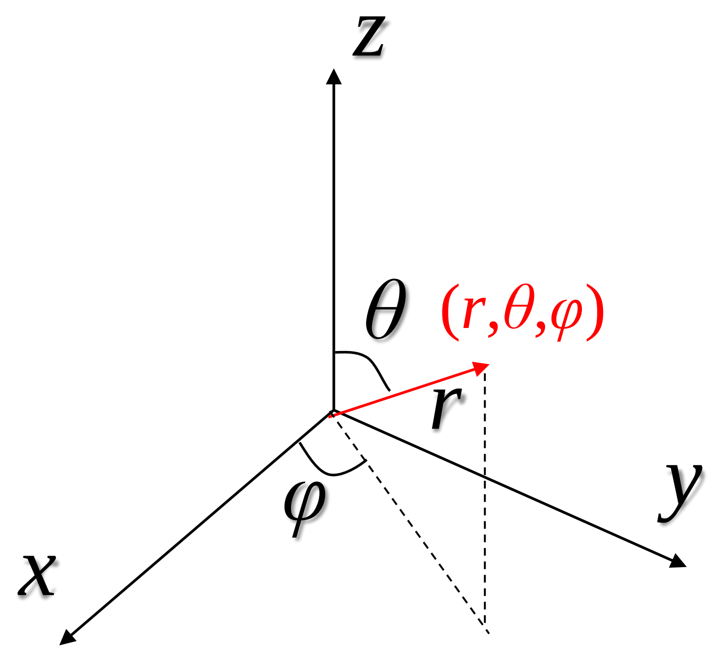
Remind: Cylindrical Coordinates

2 Lengths + 1 Angle



Remind: Spherical Coordinates

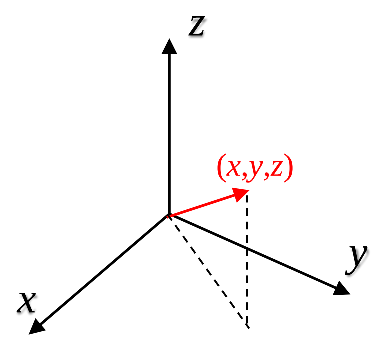
1 Length + 2 Angles

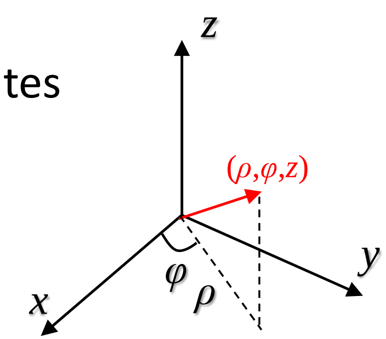


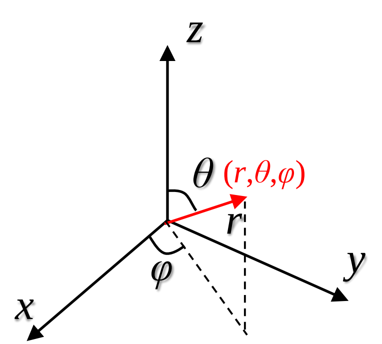
Remind: Selecting Proper Coordinates for Geometry

Standard for selecting coordinates

"Symmetry"







Orthogonal Curvilinear Coordinates

Coordinates: u_1, u_2, u_3

Distances "or" Angles

Unit Vectors: $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$

"Directions" of
Positive Displacements of $u_{1\text{-}3}$

Assume that $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ are orthogonal at each point

Orthogonal Curvilinear Coordinates

$$\mathbf{e}_i \cdot \mathbf{e}_j = \delta_{ij} \qquad (i, j = 1, 2, 3)$$

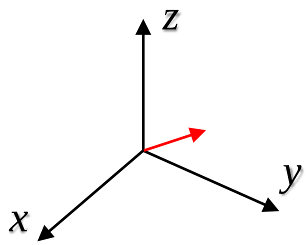
What we have to learn…

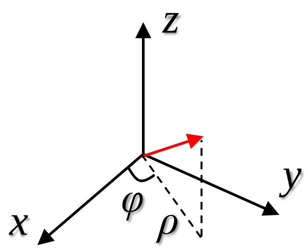
I. Transformations between Different Coordinates

II. Differential Operators in Curvilinear Coordinates

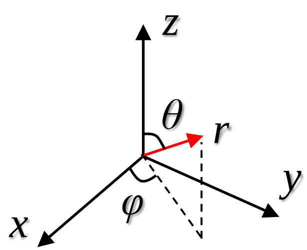
Transformation between Coordinates

Transformation of Positions





$$x = \rho \cos \varphi$$
$$y = \rho \sin \varphi$$
$$z = z$$



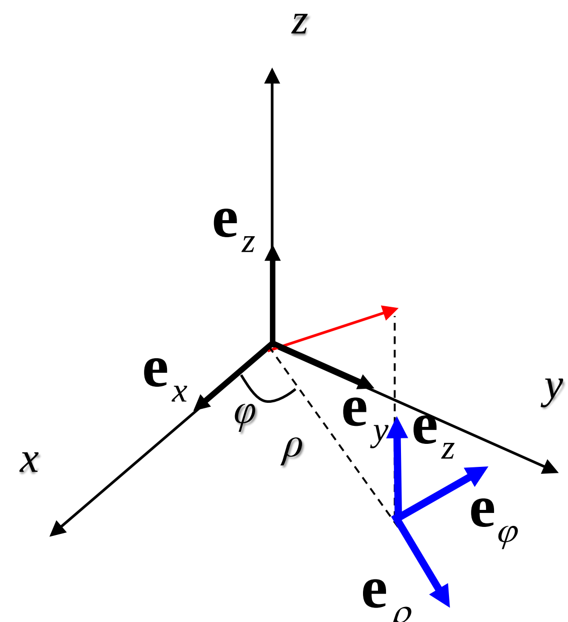
$$x = r \sin \theta \cos \varphi$$
$$y = r \sin \theta \sin \varphi$$
$$z = r \cos \theta$$

Transformation of Vectors: Cylindrical Coordinates

$$\mathbf{E} = E_x \mathbf{e}_x + E_y \mathbf{e}_y + E_z \mathbf{e}_z$$
$$= E_\rho \mathbf{e}_\rho + E_\varphi \mathbf{e}_\varphi + E_z \mathbf{e}_z$$

$$E_\rho = \mathbf{e}_\rho \cdot \mathbf{E} = E_x \mathbf{e}_\rho \cdot \mathbf{e}_x + E_y \mathbf{e}_\rho \cdot \mathbf{e}_y + E_z \mathbf{e}_\rho \cdot \mathbf{e}_z$$
$$= E_x \cos\varphi + E_y \sin\varphi$$

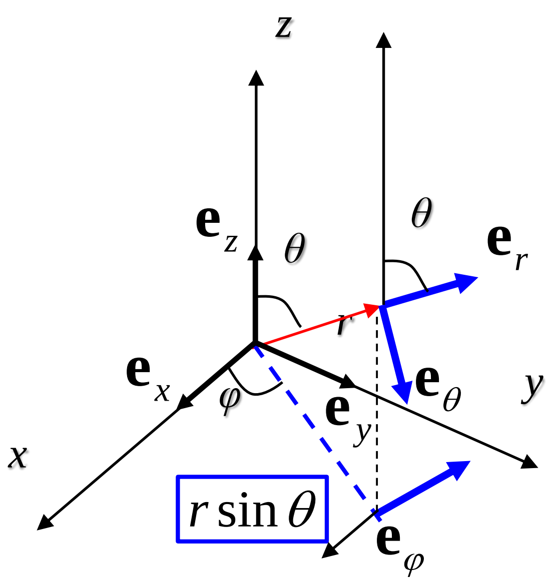
$$E_\varphi = \mathbf{e}_\varphi \cdot \mathbf{E} = E_x \mathbf{e}_\varphi \cdot \mathbf{e}_x + E_y \mathbf{e}_\varphi \cdot \mathbf{e}_y + E_z \mathbf{e}_\varphi \cdot \mathbf{e}_z$$
$$= -E_x \sin\varphi + E_y \cos\varphi$$



$$\boxed{\begin{bmatrix} E_\rho \\ E_\varphi \\ E_z \end{bmatrix} = \begin{bmatrix} \cos\varphi & \sin\varphi & 0 \\ -\sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} E_x \\ E_y \\ E_z \end{bmatrix}}$$

Transformation of Vectors: Spherical Coordinates

$$\mathbf{E} = E_x\mathbf{e}_x + E_y\mathbf{e}_y + E_z\mathbf{e}_z$$
$$= E_r\mathbf{e}_r + E_\theta\mathbf{e}_\theta + E_\varphi\mathbf{e}_\varphi$$



$$\boxed{\begin{bmatrix} E_r \\ E_\theta \\ E_\varphi \end{bmatrix} = \begin{bmatrix} \sin\theta\cos\varphi & \sin\theta\sin\varphi & \cos\theta \\ \cos\theta\cos\varphi & \cos\theta\sin\varphi & -\sin\theta \\ -\sin\varphi & \cos\varphi & 0 \end{bmatrix} \begin{bmatrix} E_x \\ E_y \\ E_z \end{bmatrix}}$$

Transformation of Vectors: Cylindrical to Spherical

$$\begin{bmatrix} E_\rho \\ E_\varphi \\ E_z \end{bmatrix} = \begin{bmatrix} \cos\varphi & \sin\varphi & 0 \\ -\sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} E_x \\ E_y \\ E_z \end{bmatrix}$$

$$\begin{bmatrix} E_r \\ E_\theta \\ E_\varphi \end{bmatrix} = \begin{bmatrix} \sin\theta\cos\varphi & \sin\theta\sin\varphi & \cos\theta \\ \cos\theta\cos\varphi & \cos\theta\sin\varphi & -\sin\theta \\ -\sin\varphi & \cos\varphi & 0 \end{bmatrix} \begin{bmatrix} E_x \\ E_y \\ E_z \end{bmatrix}$$

$$\boxed{\begin{bmatrix} E_r \\ E_\theta \\ E_\varphi \end{bmatrix} = \begin{bmatrix} \sin\theta\cos\varphi & \sin\theta\sin\varphi & \cos\theta \\ \cos\theta\cos\varphi & \cos\theta\sin\varphi & -\sin\theta \\ -\sin\varphi & \cos\varphi & 0 \end{bmatrix} \begin{bmatrix} \cos\varphi & \sin\varphi & 0 \\ -\sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 1 \end{bmatrix}^{-1} \begin{bmatrix} E_\rho \\ E_\varphi \\ E_z \end{bmatrix}}$$

Differential Operators – Curvilinear Coordinates

Infinitesimal Displacement

Coordinates: u_1, u_2, u_3

Distances "or" Angles

Unit Vectors: $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$

"Directions" of *Positive* Displacements of $u_{1\text{-}3}$

Assume that $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ are orthogonal at each point

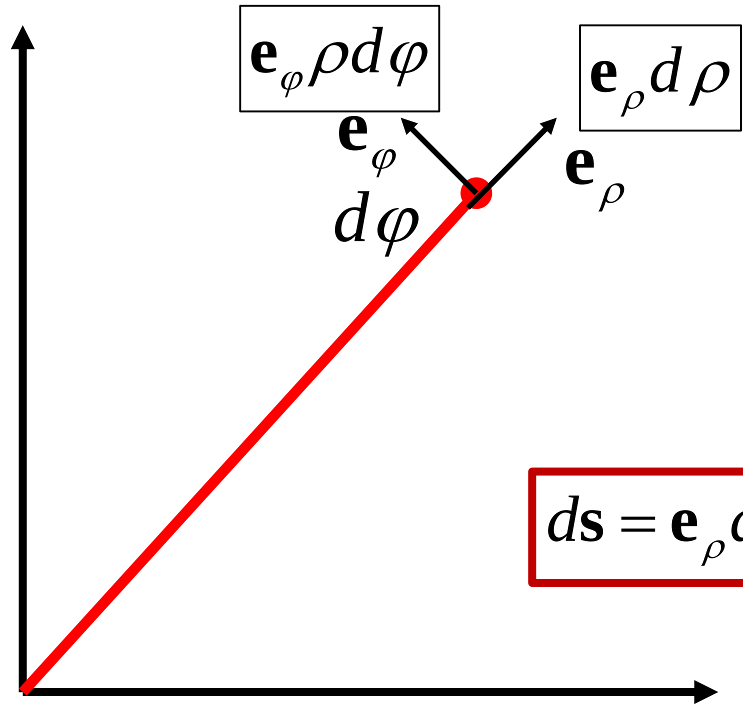
Orthogonal Curvilinear Coordinates

Infinitesimal Displacement Vector

$$\boxed{d\mathbf{s} = \mathbf{e}_1 h_1 du_1 + \mathbf{e}_2 h_2 du_2 + \mathbf{e}_3 h_3 du_3 = \sum_{k=1}^{3} \mathbf{e}_k h_k du_k}$$

h_k: Metric Coefficients: connecting coordinates to distance

Infinitesimal Displacement: Polar Coordinates



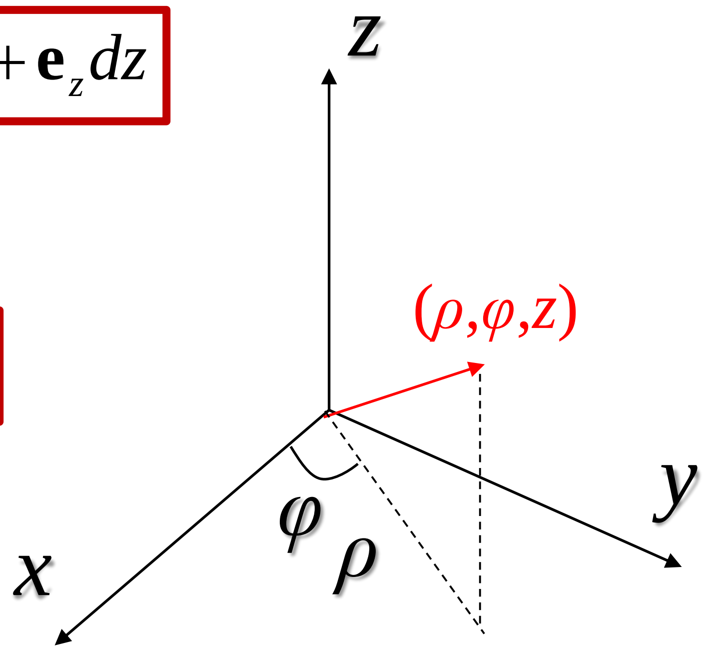
$$\boxed{d\mathbf{s} = \mathbf{e}_\rho d\rho + \mathbf{e}_\varphi \rho d\varphi}$$

$$\boxed{h_\rho = 1, h_\varphi = \rho}$$

Infinitesimal Displacement: Cylindrical Coordinates

$$\boxed{d\mathbf{s} = \mathbf{e}_\rho d\rho + \mathbf{e}_\varphi \rho d\varphi + \mathbf{e}_z dz}$$

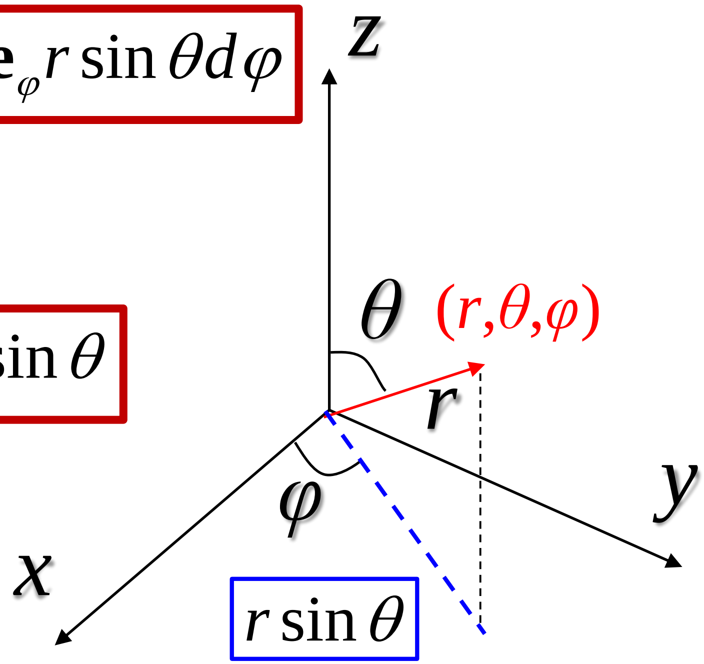
$$\boxed{h_\rho = 1, h_\varphi = \rho, h_z = 1}$$

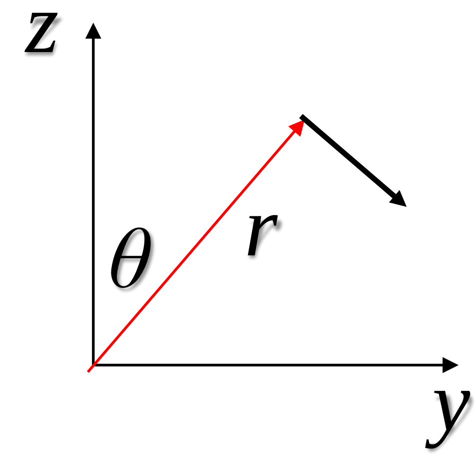


Infinitesimal Displacement: Spherical Coordinates

$$\boxed{d\mathbf{s} = \mathbf{e}_r dr + \mathbf{e}_\theta r d\theta + \mathbf{e}_\varphi r \sin\theta d\varphi}$$

$$\boxed{h_r = 1, h_\theta = r, h_\varphi = r \sin\theta}$$





Gradient Revisited

Definition: The gradient of a function drives the *infinitesimal change* of the function *along the displacement vector*

$$\boxed{dV = \nabla V \cdot d\mathbf{s}}$$

We know that

$$dV = \frac{\partial V}{\partial u_1} du_1 + \frac{\partial V}{\partial u_2} du_2 + \frac{\partial V}{\partial u_3} du_3$$

$$= \nabla V \cdot d\mathbf{s} = \nabla V \cdot \left(\mathbf{e}_1 h_1 du_1 + \mathbf{e}_2 h_2 du_2 + \mathbf{e}_3 h_3 du_3 \right)$$

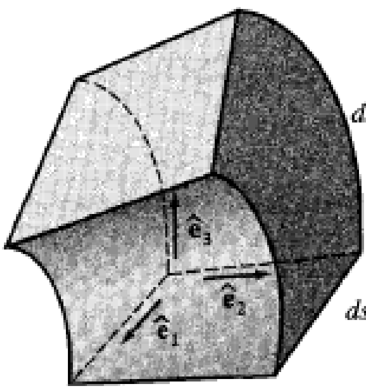
$$\boxed{\nabla V = \frac{1}{h_1} \frac{\partial V}{\partial u_1} \mathbf{e}_1 + \frac{1}{h_2} \frac{\partial V}{\partial u_2} \mathbf{e}_2 + \frac{1}{h_3} \frac{\partial V}{\partial u_3} \mathbf{e}_3 = \sum_{k=1}^{3} \frac{1}{h_k} \frac{\partial V}{\partial u_k} \mathbf{e}_k}$$

Divergence Revisited

Definition: The divergence of a vector field characterizes the *source/sink* of the vector *in the infinitesimal volume*

= The flux per unit volume through an infinitesimal closed surface

$$\boxed{\nabla\cdot\mathbf{E}(u_1,u_2,u_3)=\lim_{V\to 0}\frac{1}{V}\oint_S \mathbf{E}(u_1,u_2,u_3)\cdot d\mathbf{S}}$$



$$\begin{aligned}\nabla\cdot\mathbf{E}(u_1,u_2,u_3)dV &= \left[E_1h_2h_3\Big|_{u_1+du_1} - E_1h_2h_3\Big|_{u_1}\right]du_2du_3\\ &+\left[E_2h_3h_1\Big|_{u_2+du_2} - E_2h_3h_1\Big|_{u_2}\right]du_3du_1\\ &+\left[E_3h_1h_2\Big|_{u_3+du_3} - E_3h_1h_2\Big|_{u_3}\right]du_1du_2\end{aligned}$$

E_1 (along $\mathbf{e}_1$) penetrates the infinitesimal surface $ds_2ds_3 = h_2du_2h_3du_3$

Divergence Revisited

$$dV = h_1 h_2 h_3 du_1 du_2 du_3$$

$$\begin{aligned}\nabla\cdot\mathbf{E}(u_1,u_2,u_3)h_1h_2h_3du_1du_2du_3 &= \left[E_1h_2h_3\Big|_{u_1+du_1} - E_1h_2h_3\Big|_{u_1} \right]du_2du_3 \\ &+ \left[E_2h_3h_1\Big|_{u_2+du_2} - E_2h_3h_1\Big|_{u_2} \right]du_3du_1 \\ &+ \left[E_3h_1h_2\Big|_{u_3+du_3} - E_3h_1h_2\Big|_{u_3} \right]du_1du_2\end{aligned}$$

$$\begin{aligned}\nabla\cdot\mathbf{E}(u_1,u_2,u_3)h_1h_2h_3 &= \frac{1}{du_1}\left[E_1h_2h_3\Big|_{u_1+du_1} - E_1h_2h_3\Big|_{u_1} \right] \\ &+ \frac{1}{du_2}\left[E_2h_3h_1\Big|_{u_2+du_2} - E_2h_3h_1\Big|_{u_2} \right] \\ &+ \frac{1}{du_3}\left[E_3h_1h_2\Big|_{u_3+du_3} - E_3h_1h_2\Big|_{u_3} \right]\end{aligned}$$

Divergence Revisited

$$\nabla \cdot \mathbf{E}(u_1,u_2,u_3)h_1h_2h_3 = \frac{1}{du_1}\left[E_1h_2h_3\Big|_{u_1+du_1} - E_1h_2h_3\Big|_{u_1} \right]$$

$$+\frac{1}{du_2}\left[E_2h_3h_1\Big|_{u_2+du_2} - E_2h_3h_1\Big|_{u_2} \right]$$

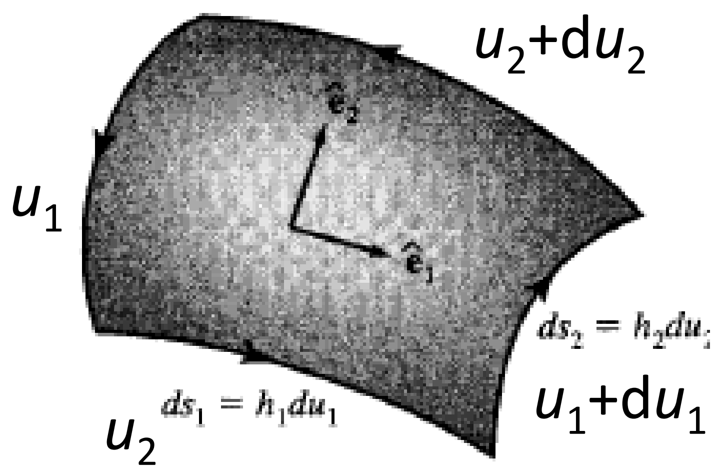
$$+\frac{1}{du_3}\left[E_3h_1h_2\Big|_{u_3+du_3} - E_3h_1h_2\Big|_{u_3} \right]$$

$$\boxed{\nabla \cdot \mathbf{E} = \frac{1}{h_1h_2h_3}\left[\frac{\partial\left(E_1h_2h_3\right)}{\partial u_1} + \frac{\partial\left(E_2h_3h_1\right)}{\partial u_2} + \frac{\partial\left(E_3h_1h_2\right)}{\partial u_3} \right]}$$

Curl Revisited

Definition: The curl of a vector field characterizes the *circulation* of the vector *in the infinitesimal volume*
= The circulation per unit area around an infinitesimal loop

$$\mathbf{n}\cdot\left(\nabla\times\mathbf{E}(u_1,u_2,u_3)\right)=\lim_{S\to 0}\frac{1}{S}\oint_C \mathbf{E}(u_1,u_2,u_3)\cdot d\mathbf{l}$$



$$\mathbf{e}_3\cdot\left(\nabla\times\mathbf{E}\right)dS_3=\left[E_2h_2\Big|_{u_1+du_1}-E_2h_2\Big|_{u_1}\right]du_2$$
$$-\left[E_1h_1\Big|_{u_2+du_2}-E_1h_1\Big|_{u_2}\right]du_1$$

E_1 (along $\mathbf{e}_1$) circulates along the infinitesimal line $ds_1 = h_1du_1$

Curl Revisited

$$dS_3 = h_1 h_2 du_1 du_2$$

$$\mathbf{e}_3 \cdot (\nabla \times \mathbf{E}) h_1 h_2 = \frac{1}{du_1}\left[E_2 h_2 \Big|_{u_1+du_1} - E_2 h_2 \Big|_{u_1} \right] - \frac{1}{du_2}\left[E_1 h_1 \Big|_{u_2+du_2} - E_1 h_1 \Big|_{u_2} \right]$$

$$\mathbf{e}_3 \cdot (\nabla \times \mathbf{E}) = \frac{h_3}{h_1 h_2 h_3}\left[\frac{\partial (E_2 h_2)}{\partial u_1} - \frac{\partial (E_1 h_1)}{\partial u_2} \right]$$

$$\mathbf{e}_1 \cdot (\nabla \times \mathbf{E}) = \frac{h_1}{h_1 h_2 h_3}\left[\frac{\partial (E_3 h_3)}{\partial u_2} - \frac{\partial (E_2 h_2)}{\partial u_3} \right]$$

$$\mathbf{e}_2 \cdot (\nabla \times \mathbf{E}) = \frac{h_2}{h_1 h_2 h_3}\left[\frac{\partial (E_1 h_1)}{\partial u_3} - \frac{\partial (E_3 h_3)}{\partial u_1} \right]$$

$$\boxed{\nabla \times \mathbf{E} = \frac{1}{h_1 h_2 h_3}\begin{vmatrix} \mathbf{e}_1 h_1 & \mathbf{e}_2 h_2 & \mathbf{e}_3 h_3 \\ \dfrac{\partial}{\partial u_1} & \dfrac{\partial}{\partial u_2} & \dfrac{\partial}{\partial u_3} \\ h_1 E_1 & h_2 E_2 & h_3 E_3 \end{vmatrix}}$$

Scalar Laplacian Revisited

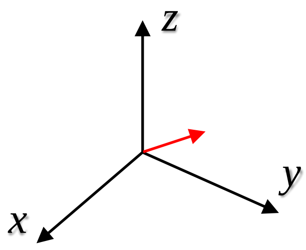
$$\boxed{\nabla^2 V = \nabla \cdot \left(\nabla V \right)}$$

$$\nabla V = \frac{1}{h_1}\frac{\partial V}{\partial u_1}\mathbf{e}_1 + \frac{1}{h_2}\frac{\partial V}{\partial u_2}\mathbf{e}_2 + \frac{1}{h_3}\frac{\partial V}{\partial u_3}\mathbf{e}_3 = \sum_{k=1}^{3}\frac{1}{h_k}\frac{\partial V}{\partial u_k}\mathbf{e}_k$$

$$\nabla \cdot \mathbf{E} = \frac{1}{h_1 h_2 h_3}\left[\frac{\partial\left(E_1 h_2 h_3\right)}{\partial u_1} + \frac{\partial\left(E_2 h_3 h_1\right)}{\partial u_2} + \frac{\partial\left(E_3 h_1 h_2\right)}{\partial u_3}\right]$$

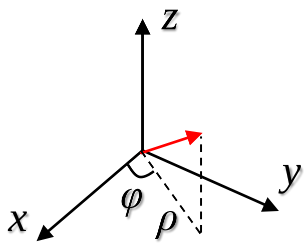
$$\boxed{\nabla^2 V = \frac{1}{h_1 h_2 h_3}\left[\frac{\partial\left(\frac{h_2 h_3}{h_1}\frac{\partial V}{\partial u_1}\right)}{\partial u_1} + \frac{\partial\left(\frac{h_3 h_1}{h_2}\frac{\partial V}{\partial u_2}\right)}{\partial u_2} + \frac{\partial\left(\frac{h_1 h_2}{h_3}\frac{\partial V}{\partial u_3}\right)}{\partial u_3}\right]}$$

Scalar Laplacian Revisited



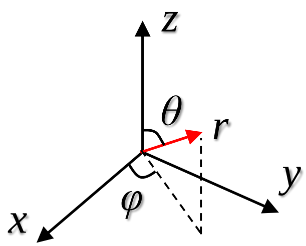
Cartesian Coordinate

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$



Cylindrical Coordinate

$$\nabla^2 = \frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho \frac{\partial}{\partial \rho}\right) + \frac{1}{\rho^2}\frac{\partial^2}{\partial \varphi^2} + \frac{\partial^2}{\partial z^2}$$



Spherical Coordinate

$$\nabla^2 = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial}{\partial r}\right) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial}{\partial \theta}\right) + \frac{1}{r^2 \sin^2\theta}\frac{\partial^2}{\partial \varphi^2}$$

Vector Laplacian Revisited

$$\nabla \times (\nabla \times \mathbf{A}) = \nabla(\nabla \cdot \mathbf{A}) - \nabla^2 \mathbf{A}$$

$$\Rightarrow \boxed{\nabla^2 \mathbf{A} = \nabla(\nabla \cdot \mathbf{A}) - \nabla \times (\nabla \times \mathbf{A})}$$

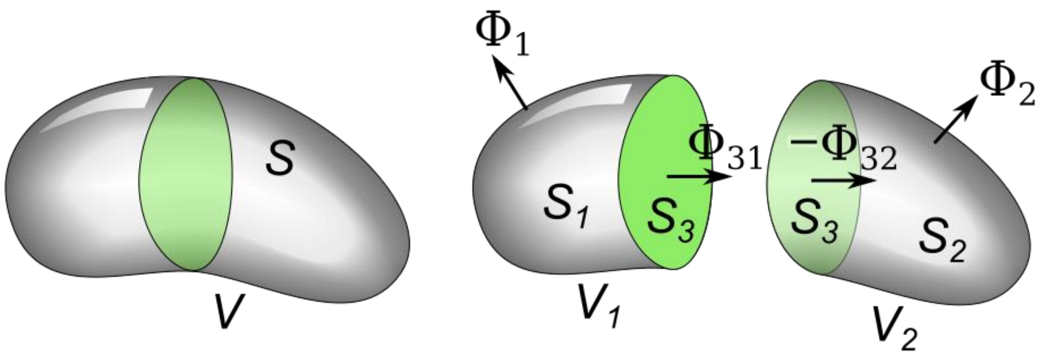
Cylindrical Coordinates

$$\nabla^2 \mathbf{v} = \begin{bmatrix} \dfrac{\partial^2 v_r}{\partial r^2} + \dfrac{1}{r^2}\dfrac{\partial^2 v_r}{\partial \phi^2} + \dfrac{\partial^2 v_r}{\partial z^2} + \dfrac{1}{r}\dfrac{\partial v_r}{\partial r} - \dfrac{2}{r^2}\dfrac{\partial v_\phi}{\partial \phi} - \dfrac{v_r}{r^2} \\ \dfrac{\partial^2 v_\phi}{\partial r^2} + \dfrac{1}{r^2}\dfrac{\partial^2 v_\phi}{\partial \phi^2} + \dfrac{\partial^2 v_\phi}{\partial z^2} + \dfrac{1}{r}\dfrac{\partial v_\phi}{\partial r} + \dfrac{2}{r^2}\dfrac{\partial v_r}{\partial \phi} - \dfrac{v_\phi}{r^2} \\ \dfrac{\partial^2 v_z}{\partial r^2} + \dfrac{1}{r^2}\dfrac{\partial^2 v_z}{\partial \phi^2} + \dfrac{\partial^2 v_z}{\partial z^2} + \dfrac{1}{r}\dfrac{\partial v_z}{\partial r} \end{bmatrix}.$$

Gauss's Theorem: Conserved for Curvilinear Coordinates

Divergence Theorem

$$\boxed{\int_V (\nabla \cdot \mathbf{u})\, d^3x = \oint_S \mathbf{u} \cdot d\mathbf{S}}$$



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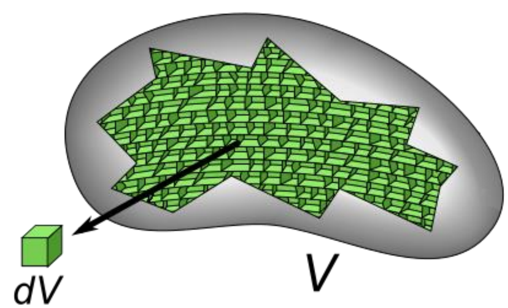
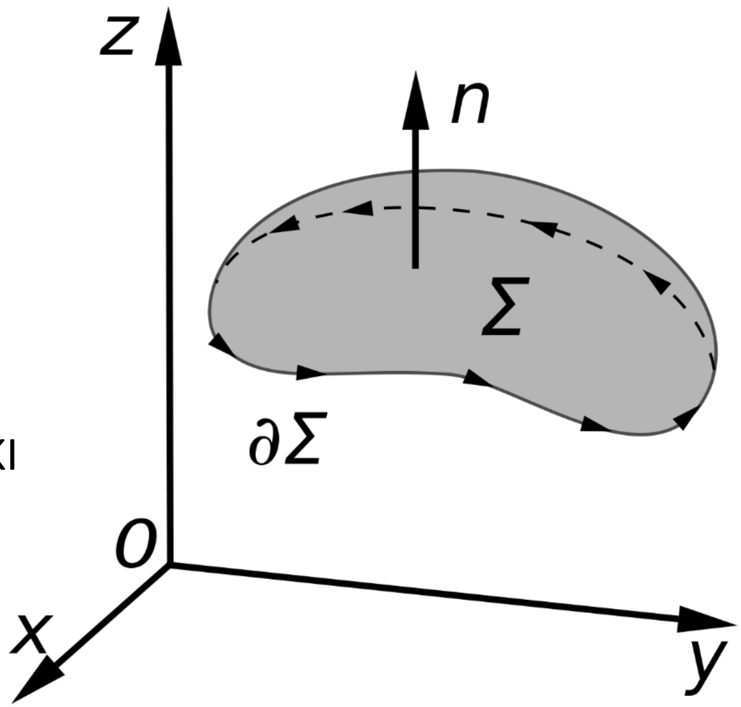


Stokes's Theorem: Conserved for Curvilinear Coordinates

Curl Theorem

$$\boxed{\int_S (\nabla \times \mathbf{u}) \cdot d\mathbf{S} = \oint_C \mathbf{u} \cdot d\mathbf{l}}$$



WIKI

Important Relations

Gradient 1/*r*

$$\nabla \frac{1}{r} = \left(\mathbf{e}_r \frac{\partial}{\partial r} + \mathbf{e}_\theta \frac{1}{r} \frac{\partial}{\partial \theta} + \mathbf{e}_\varphi \frac{1}{r \sin\theta} \frac{\partial}{\partial \varphi} \right) \left(\frac{1}{r} \right) = -\mathbf{e}_r \frac{1}{r^2} = -\frac{\mathbf{r}}{r^3}$$

$$\nabla \frac{1}{|\mathbf{x}-\mathbf{x}'|} = \left(\mathbf{e}_x \frac{\partial}{\partial x} + \ldots \right) \left(\frac{1}{\sqrt{(x-x')^2 + (y-y')^2 + (z-z')^2}} \right) = -\frac{\mathbf{x}-\mathbf{x}'}{|\mathbf{x}-\mathbf{x}'|^3}$$

$$\nabla' \frac{1}{|\mathbf{x}-\mathbf{x}'|} = \left(\mathbf{e}_x \frac{\partial}{\partial x'} + \ldots \right) \left(\frac{1}{\sqrt{(x-x')^2 + (y-y')^2 + (z-z')^2}} \right) = \frac{\mathbf{x}-\mathbf{x}'}{|\mathbf{x}-\mathbf{x}'|^3}$$

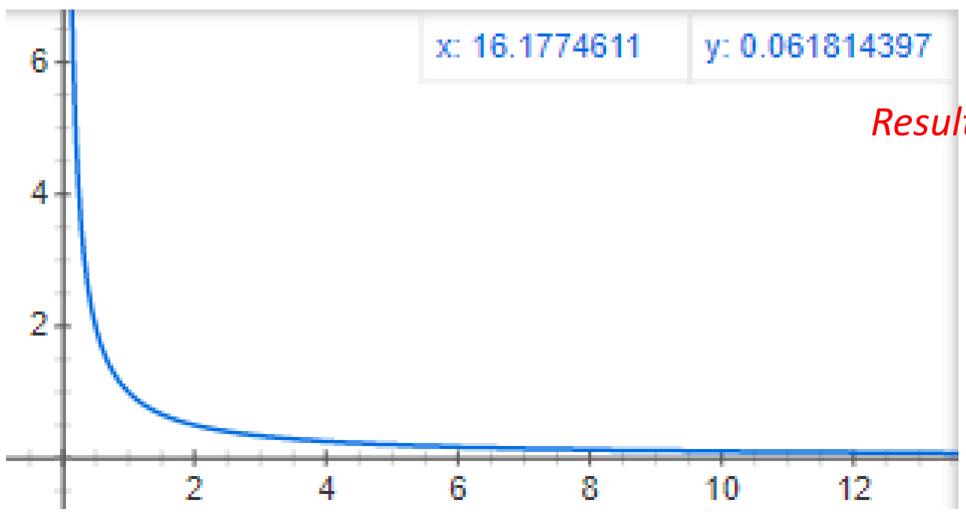
$$\boxed{\nabla \frac{1}{r} = -\frac{\mathbf{r}}{r^3}} \qquad \boxed{\nabla \frac{1}{|\mathbf{x}-\mathbf{x}'|} = -\nabla' \frac{1}{|\mathbf{x}-\mathbf{x}'|} = -\frac{\mathbf{x}-\mathbf{x}'}{|\mathbf{x}-\mathbf{x}'|^3}}$$

3D Laplacian: 1/*r*

$$\nabla^2 \frac{1}{r} = \left(\frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial}{\partial r} \right) + \frac{1}{r^2 \sin\theta} \frac{\partial}{\partial \theta} \left(\sin\theta \frac{\partial}{\partial \theta} \right) + \frac{1}{r^2 \sin^2\theta} \frac{\partial^2}{\partial \varphi^2} \right) \frac{1}{r}$$

$$= \frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial}{\partial r} \frac{1}{r} \right) = \frac{1}{r^2} \frac{\partial}{\partial r} (-1) = 0\,?$$

In contrast to the first-order derivative, the second-order derivative is not well defined near 0



Resulting in an exotic behavior!

3D Laplacian: 1/*r*

Let's apply a tricky method!

$$\frac{1}{r} = \lim_{a\to 0}\frac{1}{\sqrt{r^2+a^2}}$$

$$\nabla^2\frac{1}{\sqrt{r^2+a^2}} = \left(\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}\right)\frac{1}{\sqrt{r^2+a^2}}$$

$$= \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\frac{1}{\sqrt{r^2+a^2}}\right) = -\frac{3a^2}{\left(r^2+a^2\right)^{5/2}}$$

Integration over all the space!

$$\int_{V_\infty}\nabla^2\frac{1}{\sqrt{r^2+a^2}}dv = -\int_{V_\infty}\frac{3a^2}{\left(r^2+a^2\right)^{5/2}}dv = -\int_0^{2\pi}\int_0^{\pi}\int_0^{\infty}\frac{3a^2}{\left(r^2+a^2\right)^{5/2}}r^2\sin\theta\, dr\, d\theta\, d\varphi = -4\pi$$

3D Laplacian: 1/r

a-independent

$$\int_{V_\infty} \nabla^2 \frac{1}{\sqrt{r^2+a^2}}\, dv = -4\pi \quad \Rightarrow \quad \int_{V_\infty} \nabla^2 \frac{1}{r}\, dv = -4\pi$$

$$\nabla^2 \frac{1}{r} = 0 \qquad (r \neq 0)$$

Definition of the ***Delta Function***

$$\delta^3(\mathbf{r}) = 0 \qquad (\mathbf{r} \neq \mathbf{O}) \qquad\qquad \int_{V_\infty} \delta^3(\mathbf{r})\, dv = 1$$

$$\boxed{\nabla^2 \frac{1}{r} = -4\pi\delta^3(\mathbf{r})} \qquad\qquad \boxed{\nabla^2 \frac{1}{|\mathbf{x}-\mathbf{x}'|} = -4\pi\delta^3(\mathbf{x}-\mathbf{x}')}$$

2D Laplacian: lnρ

$$\boxed{\nabla_T{}^2 \ln \rho = 2\pi\delta^2(\mathbf{r})}$$

$$\boxed{\nabla_T{}^2 \ln\left|\mathbf{x}-\mathbf{x}'\right| = 2\pi\delta^2(\mathbf{x}-\mathbf{x}')}$$

2D & 3D Laplacian: ln(1/ρ) & 1/r

$$\nabla_T{}^2 \ln \rho = 2\pi\delta^2(\mathbf{r}) \qquad \nabla_T{}^2 \ln|\mathbf{x}-\mathbf{x}'| = 2\pi\delta^2(\mathbf{x}-\mathbf{x}')$$

2D

$$\boxed{\nabla_T{}^2 \ln \frac{1}{\rho} = -2\pi\delta^2(\mathbf{r})} \qquad \boxed{\nabla_T{}^2 \ln \frac{1}{|\mathbf{x}-\mathbf{x}'|} = -2\pi\delta^2(\mathbf{x}-\mathbf{x}')}$$

3D

$$\boxed{\nabla^2 \frac{1}{r} = -4\pi\delta^3(\mathbf{r})} \qquad \boxed{\nabla^2 \frac{1}{|\mathbf{x}-\mathbf{x}'|} = -4\pi\delta^3(\mathbf{x}-\mathbf{x}')}$$

Static Electric Fields

Introduction to Electromagnetism with Practice

Theory & Applications

Sunkyu Yu

Dept. of Electrical and Computer Engineering

Seoul National University

Maxwell’s Equations

General Form: Light in the Vacuum

$$\nabla \times \mathbf{E} = -\mu_0 \frac{\partial \mathbf{H}}{\partial t}$$

$$\nabla \times \mathbf{H} = \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t}$$

$$\nabla \cdot \mathbf{E} = 0$$

$$\nabla \cdot \mathbf{H} = 0$$

Maxwell's Equations
in the ***Vacuum***

$\mu_0 = 4\pi \times 10^{-7}$ (H/m)
$\varepsilon_0 = 8.854 \times 10^{-12}$ (F/m)

General Form: Light in Media

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$

$$\nabla \times \mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t}$$

$$\nabla \cdot \mathbf{D} = \rho$$

$$\nabla \cdot \mathbf{B} = 0$$

Maxwell's Equations
in ***General Media***

$$\mathbf{D} = \mathbf{D}(\mathbf{E}, \mathbf{H})$$

$$\mathbf{B} = \mathbf{B}(\mathbf{E}, \mathbf{H})$$

Also, functions of $\mathbf{x}, t, \mathbf{k}, \omega$

Approximated Form: Light in Simpler Media

$$\mathbf{D} = \mathbf{D}(\mathbf{E}, \mathbf{H})$$
$$\mathbf{B} = \mathbf{B}(\mathbf{E}, \mathbf{H})$$

This approx. loop is not unique!

Linear, Local (**k**-independent)

functions of **x**,*t*,*ω*

$$\mathbf{D} = \varepsilon_0 \overline{\boldsymbol{\varepsilon}}\mathbf{E} + \overline{\boldsymbol{\chi}}_{\mathrm{EH}}\mathbf{H}$$
$$\mathbf{B} = \overline{\boldsymbol{\chi}}_{\mathrm{HE}}\mathbf{E} + \mu_0 \overline{\boldsymbol{\mu}}\mathbf{H}$$

Without Bi-Isotropy/Bi-Anisotropy

$$\mathbf{D} = \varepsilon_0 \overline{\boldsymbol{\varepsilon}}\mathbf{E}$$
$$\mathbf{B} = \mu_0 \overline{\boldsymbol{\mu}}\mathbf{H}$$

Isotropic

$$\mathbf{D} = \varepsilon_0 \varepsilon_{\mathrm{r}}(\mathbf{x}, t, \omega)\mathbf{E}$$
$$\mathbf{B} = \mu_0 \mu_{\mathrm{r}}(\mathbf{x}, t, \omega)\mathbf{H}$$

Static Materials

functions of **x**,*ω*

$$\mathbf{D} = \varepsilon_0 \varepsilon_{\mathrm{r}}(\mathbf{x}, \omega)\mathbf{E}$$
$$\mathbf{B} = \mu_0 \mu_{\mathrm{r}}(\mathbf{x}, \omega)\mathbf{H}$$

Homogeneous

functions of *ω*

$$\mathbf{D} = \varepsilon_0 \varepsilon_{\mathrm{r}}(\omega)\mathbf{E}$$
$$\mathbf{B} = \mu_0 \mu_{\mathrm{r}}(\omega)\mathbf{H}$$

Linear, Local, Isotropic, Static, Homogeneous Media

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$

$$\nabla \times \mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t}$$

$$\nabla \cdot \mathbf{D} = \rho$$

$$\nabla \cdot \mathbf{B} = 0$$

Maxwell's Equations
in ***Simple Media***

$$\mathbf{D} = \varepsilon_0 \varepsilon_\mathrm{r}(\omega)\mathbf{E}$$

$$\mathbf{B} = \mu_0 \mu_\mathrm{r}(\omega)\mathbf{H}$$

Simple + Source-Free Media

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$

$$\nabla \times \mathbf{H} = \frac{\partial \mathbf{D}}{\partial t}$$

$$\nabla \cdot \mathbf{D} = 0$$

$$\nabla \cdot \mathbf{B} = 0$$

Maxwell's Equations
in ***Simple***
+ Source-Free Media

$$\mathbf{J} = \mathbf{O}, \quad \rho = 0$$

$$\mathbf{D} = \varepsilon_0 \varepsilon_\mathrm{r}(\omega)\mathbf{E}$$

$$\mathbf{B} = \mu_0 \mu_\mathrm{r}(\omega)\mathbf{H}$$

In this Lecture…Focusing on Static Fields, not on Light

very, very slow variation of light fields
~ very small energy of photons

$$\nabla \times \mathbf{E} = -\cancel{\frac{\partial \mathbf{B}}{\partial t}}$$

$$\nabla \times \mathbf{H} = \mathbf{J} + \cancel{\frac{\partial \mathbf{D}}{\partial t}}$$

$$\nabla \cdot \mathbf{D} = \rho$$

$$\nabla \cdot \mathbf{B} = 0$$

Maxwell's Equations
for Static Fields

$$\mathbf{D} = \varepsilon_0 \varepsilon_\mathrm{r} \mathbf{E}$$

$$\mathbf{B} = \mu_0 \mu_\mathrm{r} \mathbf{H}$$

In this Chapter... Static Electric Fields

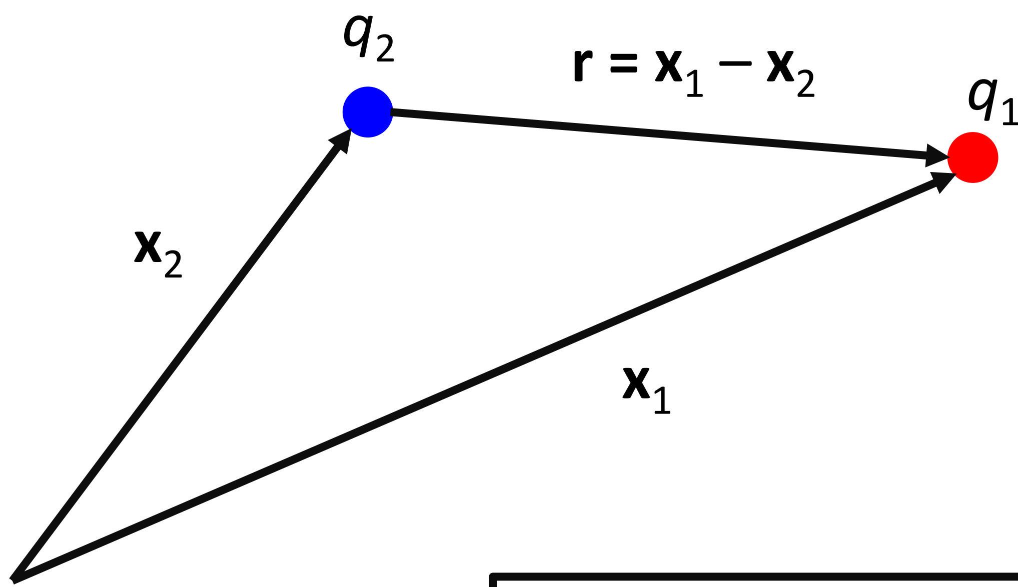
$$\nabla \times \mathbf{E} = \mathbf{0}$$

$$\nabla \cdot \mathbf{D} = \rho$$

$$\mathbf{D} = \varepsilon_0 \varepsilon_\mathrm{r} \mathbf{E}$$

Electrostatics – Coulomb's Law

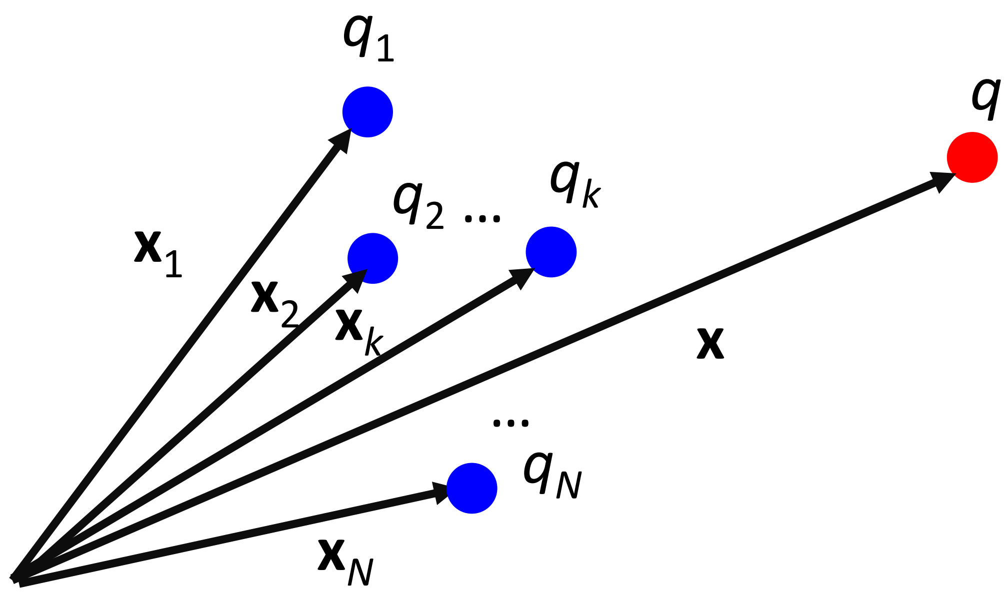
Coulomb's Law



$$\mathbf{F}_1 = \frac{1}{4\pi\varepsilon_0} q_1 q_2 \frac{\mathbf{x}_1 - \mathbf{x}_2}{\left|\mathbf{x}_1 - \mathbf{x}_2\right|^3}$$

$$= \frac{1}{4\pi\varepsilon_0} q_1 q_2 \frac{\mathbf{r}}{\left|\mathbf{r}\right|^3} = \frac{1}{4\pi\varepsilon_0} q_1 q_2 \frac{\hat{\mathbf{r}}}{\left|\mathbf{r}\right|^2}$$

Coulomb's Law: Superposition Principle



$$\mathbf{F} = \frac{1}{4\pi\varepsilon_0} q \sum_{k=1}^{N} q_k \frac{\mathbf{x} - \mathbf{x}_k}{\left|\mathbf{x} - \mathbf{x}_k\right|^3}$$

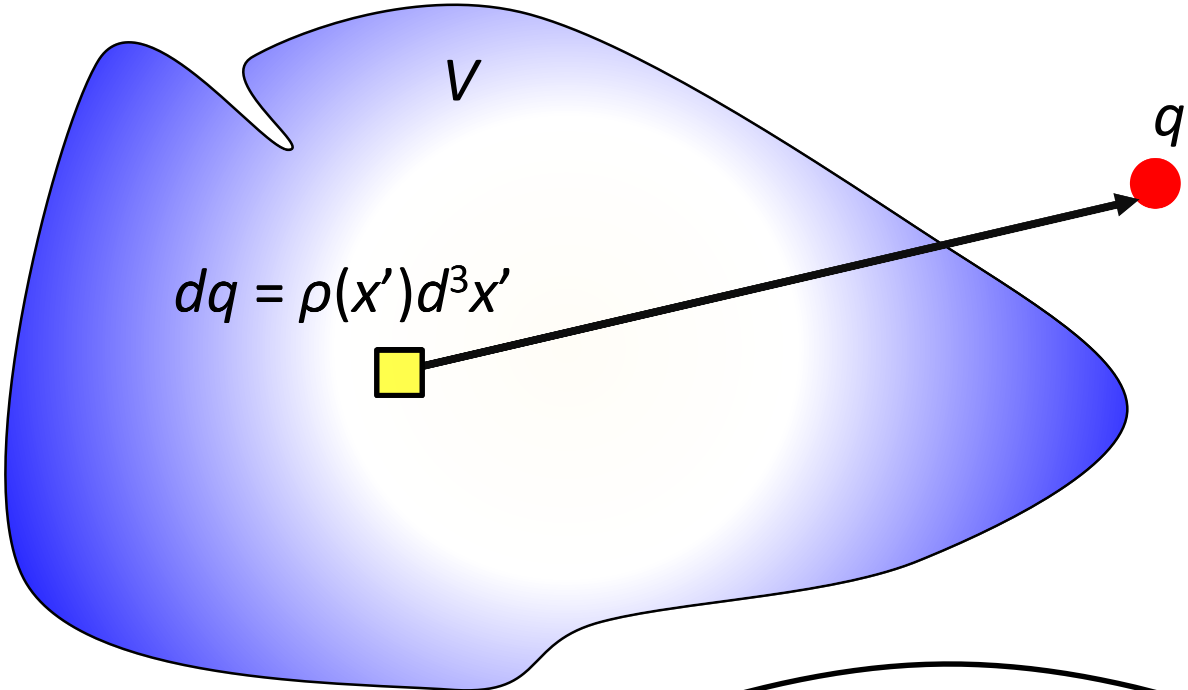
Remind: Gradient 1/*r*

$$\boxed{\nabla \frac{1}{r} = -\frac{\mathbf{r}}{r^3}} \quad \boxed{\nabla \frac{1}{|\mathbf{x}-\mathbf{x}'|} = -\nabla' \frac{1}{|\mathbf{x}-\mathbf{x}'|} = -\frac{\mathbf{x}-\mathbf{x}'}{|\mathbf{x}-\mathbf{x}'|^3}}$$

$$\mathbf{F} = \frac{1}{4\pi\varepsilon_0} q \sum_{k=1}^{N} q_k \frac{\mathbf{x}-\mathbf{x}_k}{|\mathbf{x}-\mathbf{x}_k|^3}$$

$$\mathbf{F} \sim \sum_{k=1}^{N} q_k \left(-\nabla \frac{1}{|\mathbf{x}-\mathbf{x}_k|} \right)$$

Coulomb's Law: Superposition Principle – Continuum



$$\mathbf{F} = \frac{1}{4\pi\varepsilon_0} q \sum_{k=1}^{N} q_k \frac{\mathbf{x}-\mathbf{x}_k}{\left|\mathbf{x}-\mathbf{x}_k\right|^3}$$

$$\mathbf{F} = \frac{1}{4\pi\varepsilon_0} q \int_V \frac{\mathbf{x}-\mathbf{x}'}{\left|\mathbf{x}-\mathbf{x}'\right|^3} \rho(\mathbf{x}')d^3x'$$

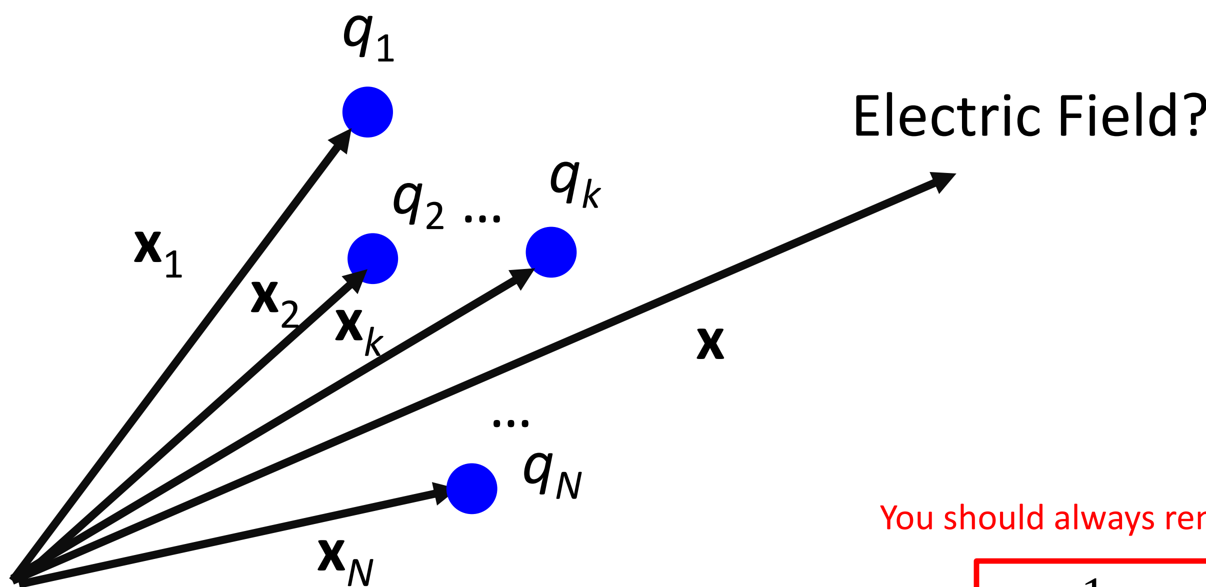
Electric Field

Electric Field
= **The force per unit charge** exerted on a small "test charge" q in the limit $q \to 0$

$$\boxed{\mathbf{E}(\mathbf{x}) = \lim_{q \to 0} \frac{\mathbf{F}}{q}}$$

Why $q \to 0$? To prohibit the effect on the other charges and existing fields

Electric Field from Discrete Charges



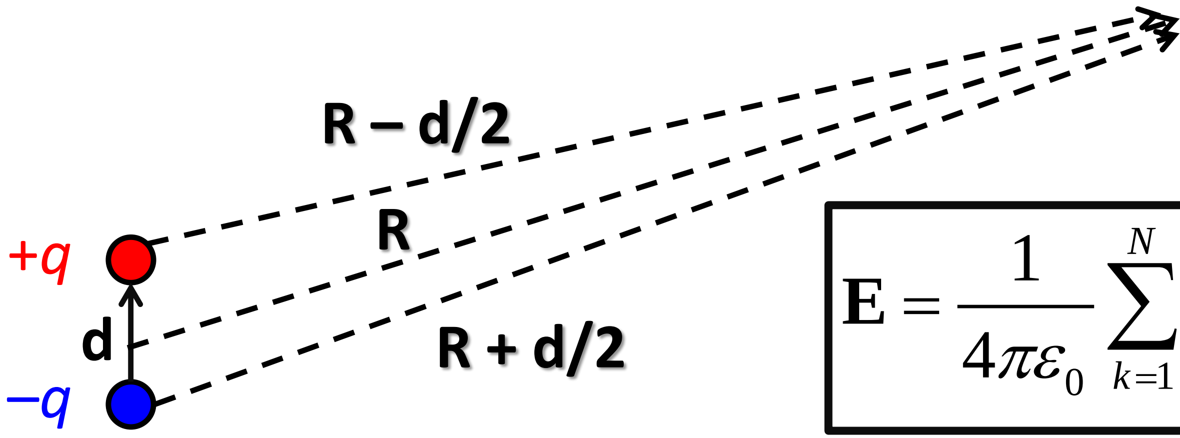
You should always remember this!

$$\boxed{\nabla \frac{1}{|\mathbf{x}-\mathbf{x}'|} = -\frac{\mathbf{x}-\mathbf{x}'}{|\mathbf{x}-\mathbf{x}'|^3}}$$

$$\mathbf{F} = \frac{1}{4\pi\varepsilon_0} q \sum_{k=1}^{N} q_k \frac{\mathbf{x}-\mathbf{x}_k}{|\mathbf{x}-\mathbf{x}_k|^3} \Rightarrow \boxed{\mathbf{E} = \frac{1}{4\pi\varepsilon_0} \sum_{k=1}^{N} q_k \frac{\mathbf{x}-\mathbf{x}_k}{|\mathbf{x}-\mathbf{x}_k|^3}}$$

Example 001

***Electric Dipole*: Estimating an electric field far from the dipole?**



$$\mathbf{E}=\frac{1}{4\pi\varepsilon_0}\sum_{k=1}^{N}q_k\frac{\mathbf{x}-\mathbf{x}_k}{\left|\mathbf{x}-\mathbf{x}_k\right|^3}$$

$$\mathbf{E}=\frac{1}{4\pi\varepsilon_0}\left(q\frac{\mathbf{R}-\frac{\mathbf{d}}{2}}{\left|\mathbf{R}-\frac{\mathbf{d}}{2}\right|^3}-q\frac{\mathbf{R}+\frac{\mathbf{d}}{2}}{\left|\mathbf{R}+\frac{\mathbf{d}}{2}\right|^3}\right)$$

We cannot learn a lot from accurate but too complex equations!

Example 001

Electric Dipole: Estimating an electric field far from the dipole?

$$\left|\mathbf{R}-\frac{\mathbf{d}}{2}\right|^{-3}=\left|\left(\mathbf{R}-\frac{\mathbf{d}}{2}\right)\cdot\left(\mathbf{R}-\frac{\mathbf{d}}{2}\right)\right|^{-\frac{3}{2}}=\left(R^2+\frac{d^2}{4}-\mathbf{R}\cdot\mathbf{d}\right)^{-\frac{3}{2}}=\left[R^2\left(1-\frac{\mathbf{R}\cdot\mathbf{d}}{R^2}+\frac{d^2}{4R^2}\right)\right]^{-\frac{3}{2}}$$

$$=R^{-3}\left(1-\frac{\mathbf{R}\cdot\mathbf{d}}{R^2}+\frac{d^2}{4R^2}\right)^{-\frac{3}{2}}\sim R^{-3}\left(1+\frac{3}{2}\frac{\mathbf{R}\cdot\mathbf{d}}{R^2}\right)$$

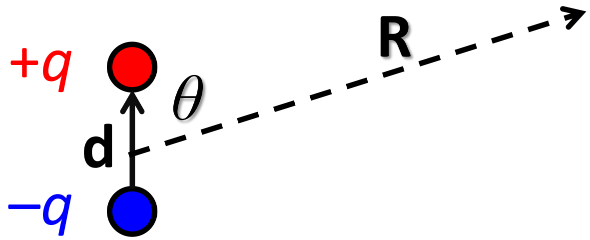
$$\mathbf{E}\sim\frac{q}{4\pi\varepsilon_0}\left[\left(\mathbf{R}-\frac{\mathbf{d}}{2}\right)R^{-3}\left(1+\frac{3}{2}\frac{\mathbf{R}\cdot\mathbf{d}}{R^2}\right)-\left(\mathbf{R}+\frac{\mathbf{d}}{2}\right)R^{-3}\left(1-\frac{3}{2}\frac{\mathbf{R}\cdot\mathbf{d}}{R^2}\right)\right]$$

$$=\frac{q}{4\pi\varepsilon_0 R^3}\left[\left(\mathbf{R}-\frac{\mathbf{d}}{2}\right)\left(1+\frac{3}{2}\frac{\mathbf{R}\cdot\mathbf{d}}{R^2}\right)-\left(\mathbf{R}+\frac{\mathbf{d}}{2}\right)\left(1-\frac{3}{2}\frac{\mathbf{R}\cdot\mathbf{d}}{R^2}\right)\right]$$

$$=\frac{q}{4\pi\varepsilon_0 R^3}\left[3\frac{\mathbf{R}\cdot\mathbf{d}}{R^2}\mathbf{R}-\mathbf{d}\right]$$

Example 001

Electric Dipole: Estimating an electric field far from the dipole?



Electric Dipole Moment

$$\mathbf{p} = q\mathbf{d}$$

$$\mathbf{E} \sim \frac{q}{4\pi\varepsilon_0 R^3}\left[3\frac{\mathbf{R}\cdot\mathbf{d}}{R^2}\mathbf{R}-\mathbf{d}\right]$$

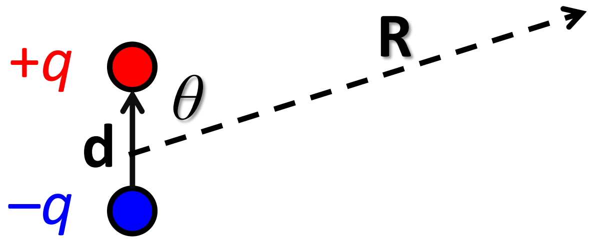
$$=\frac{1}{4\pi\varepsilon_0 R^3}\left[3\frac{\mathbf{R}\cdot\mathbf{p}}{R^2}\mathbf{R}-\mathbf{p}\right]=\frac{1}{4\pi\varepsilon_0 R^3}\left[3\frac{Rp\cos\theta}{R^2}R\mathbf{e}_r-\mathbf{e}_z p\right]$$

$$=\frac{1}{4\pi\varepsilon_0 R^3}\left[3\frac{Rp\cos\theta}{R^2}R\mathbf{e}_r-\left(\mathbf{e}_r\cos\theta-\mathbf{e}_\theta\sin\theta\right)p\right]$$

$$=\frac{p}{4\pi\varepsilon_0 R^3}\left(\mathbf{e}_r 2\cos\theta+\mathbf{e}_\theta\sin\theta\right)$$

Example 001

Electric Dipole: Estimating an electric field far from the dipole?



Electric Dipole Moment

$$\mathbf{p} = q\mathbf{d}$$

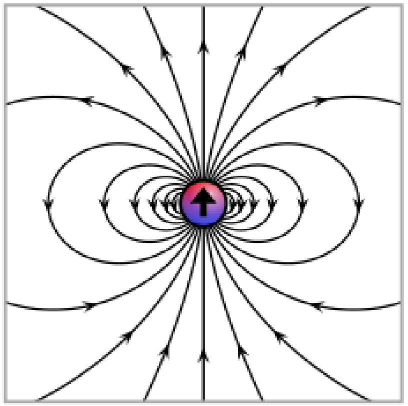
$$\boxed{\mathbf{E} \sim \frac{1}{4\pi\varepsilon_0 R^3}\left[3\frac{\mathbf{R}\cdot\mathbf{p}}{R^2}\mathbf{R} - \mathbf{p}\right] = \frac{p}{4\pi\varepsilon_0 R^3}\left(\mathbf{e}_r\, 2\cos\theta + \mathbf{e}_\theta \sin\theta\right)}$$

$$\theta = 0 \qquad \mathbf{E} \sim \frac{2p}{4\pi\varepsilon_0 R^3}\mathbf{e}_r$$

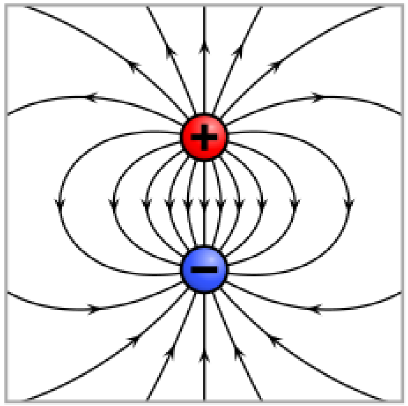
$$\theta = \frac{\pi}{2} \qquad \mathbf{E} \sim \frac{p}{4\pi\varepsilon_0 R^3}\mathbf{e}_\theta$$

$$\theta = \pi \qquad \mathbf{E} \sim -\frac{2p}{4\pi\varepsilon_0 R^3}\mathbf{e}_r$$

dipole

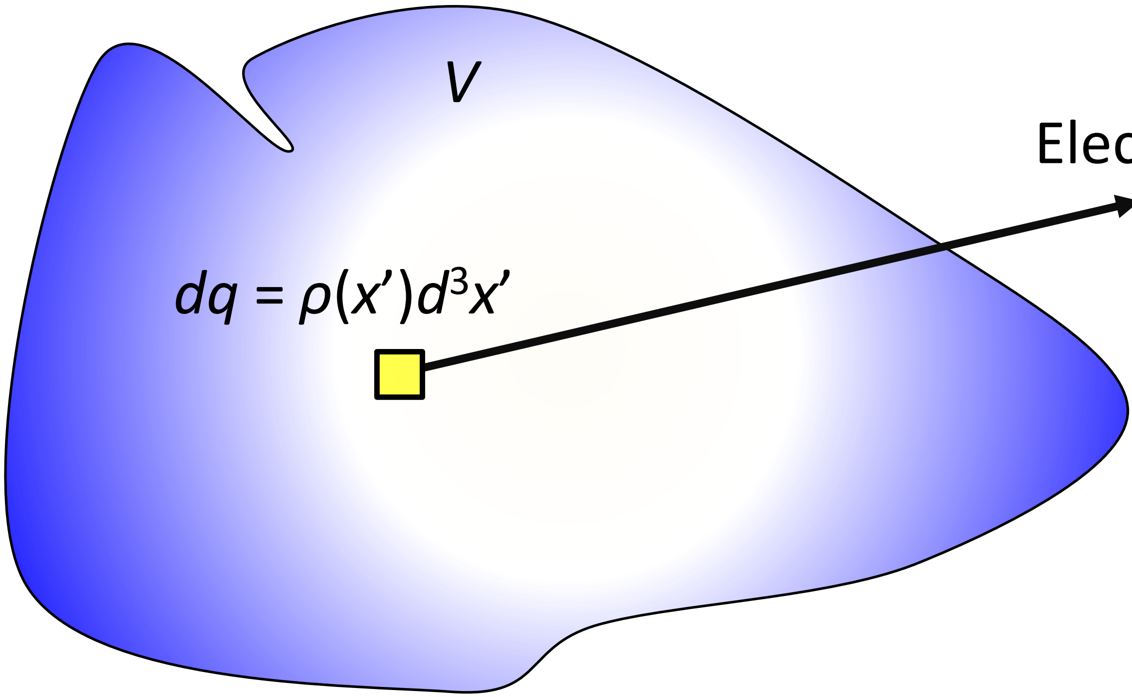


charges



WIKI

Electric Field from a Charge Continuum



You should always remember this!

$$\nabla \frac{1}{|\mathbf{x}-\mathbf{x}'|} = -\frac{\mathbf{x}-\mathbf{x}'}{|\mathbf{x}-\mathbf{x}'|^3}$$

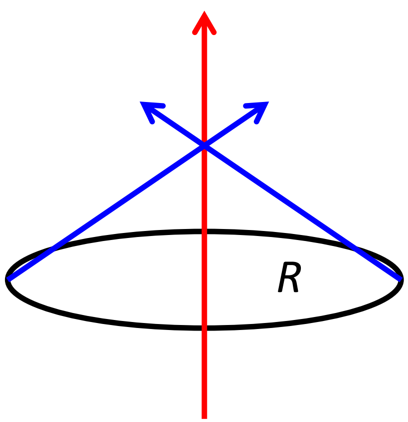
$$\mathbf{F} = \frac{1}{4\pi\varepsilon_0} q \int_V \frac{\mathbf{x}-\mathbf{x}'}{|\mathbf{x}-\mathbf{x}'|^3} \rho(\mathbf{x}') d^3x' \Rightarrow \boxed{\mathbf{E} = \frac{1}{4\pi\varepsilon_0} \int_V \frac{\mathbf{x}-\mathbf{x}'}{|\mathbf{x}-\mathbf{x}'|^3} \rho(\mathbf{x}') d^3x'}$$

Example 002

Example 3.1 (a) Find **E** on the symmetry axis of a ring with radius R and uniform charge per unit length λ. (b) Use the results of part (a) to find **E** on the symmetry axis of a disk with radius R and uniform charge per unit area σ. (c) Use the results of part (b) to find **E** for an infinite sheet with uniform charge density σ. Discuss the matching condition at $z = 0$.

(a) $$\mathbf{E} = \frac{1}{4\pi\varepsilon_0}\int_V \frac{\mathbf{x}-\mathbf{x}'}{|\mathbf{x}-\mathbf{x}'|^3}\rho(\mathbf{x}')d^3x'$$

In 1D: $$\boxed{\mathbf{E} = \frac{1}{4\pi\varepsilon_0}\int_C \frac{\mathbf{x}-\mathbf{x}'}{|\mathbf{x}-\mathbf{x}'|^3}\rho(\mathbf{x}')dx'}$$



z-component

$$\mathbf{E} = \frac{1}{4\pi\varepsilon_0}\int_0^{2\pi}\mathbf{e}_z \frac{\sqrt{z^2+R^2}\,\frac{z}{\sqrt{z^2+R^2}}}{\left(\sqrt{z^2+R^2}\right)^3}\lambda R d\varphi$$

$$= \frac{2\pi R\lambda}{4\pi\varepsilon_0}\mathbf{e}_z \frac{z}{\left(\sqrt{z^2+R^2}\right)^3} = \frac{\lambda R}{2\varepsilon_0}\frac{z}{\left(z^2+R^2\right)^{\frac{3}{2}}}\mathbf{e}_z$$

$$\boxed{\mathbf{E} = \frac{\lambda R}{2\varepsilon_0}\frac{z}{\left(z^2+R^2\right)^{\frac{3}{2}}}\mathbf{e}_z}$$

Example 002

Example 3.1 (a) Find $\mathbf{E}$ on the symmetry axis of a ring with radius R and uniform charge per unit length λ. (b) Use the results of part (a) to find $\mathbf{E}$ on the symmetry axis of a disk with radius R and uniform charge per unit area σ. (c) Use the results of part (b) to find $\mathbf{E}$ for an infinite sheet with uniform charge density σ. Discuss the matching condition at $z = 0$.

(b)

Electric Field induced by $q = 2\pi R\lambda$

Ring: $$\mathbf{E} = \frac{\lambda R}{2\varepsilon_0}\frac{z}{\left(z^2+R^2\right)^{\frac{3}{2}}}\mathbf{e}_z \Longrightarrow$$

Electric Field induced by the unit charge

$$\mathbf{E} = \frac{1}{4\pi\varepsilon_0}\frac{z}{\left(z^2+R^2\right)^{\frac{3}{2}}}\mathbf{e}_z$$

annual ring

Disk: Integration of rings → *Superposition Principle*

$$\mathbf{E} = \frac{1}{4\pi\varepsilon_0}\int_0^R \frac{2\pi r\sigma z}{\left(z^2+r^2\right)^{\frac{3}{2}}}\mathbf{e}_z dr = \frac{\sigma z}{2\varepsilon_0}\mathbf{e}_z\int_0^R \frac{r}{\left(z^2+r^2\right)^{\frac{3}{2}}}dr$$

$$= \frac{\sigma z}{2\varepsilon_0}\mathbf{e}_z\int_0^R \frac{r}{\left(z^2+r^2\right)^{\frac{3}{2}}}dr = \frac{\sigma}{2\varepsilon_0}\mathbf{e}_z\left(1-\frac{z}{\sqrt{z^2+R^2}}\right),$$

$$\boxed{\mathbf{E} = \frac{\sigma}{2\varepsilon_0}\mathbf{e}_z\left(1-\frac{z}{\sqrt{z^2+R^2}}\right)}$$

Example 002

Example 3.1 (a) Find $\mathbf{E}$ on the symmetry axis of a ring with radius R and uniform charge per unit length λ. (b) Use the results of part (a) to find $\mathbf{E}$ on the symmetry axis of a disk with radius R and uniform charge per unit area σ. (c) Use the results of part (b) to find $\mathbf{E}$ for an infinite sheet with uniform charge density σ. Discuss the matching condition at $z = 0$.

(c)

Consider some interesting cases… ($z > 0$)

Ring

$$\boxed{\mathbf{E} = \frac{\lambda R}{2\varepsilon_0}\frac{z}{\left(z^2+R^2\right)^{\frac{3}{2}}}\mathbf{e}_z}$$

Disk

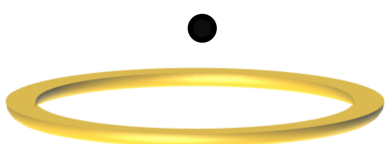
$$\boxed{\mathbf{E} = \frac{\sigma}{2\varepsilon_0}\mathbf{e}_z\left(1-\frac{z}{\sqrt{z^2+R^2}}\right)}$$

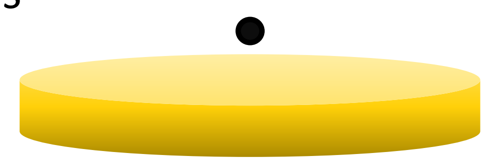
$R >> z$

$$\mathbf{E} \sim \frac{\lambda}{2\varepsilon_0}\frac{z}{R^2}\mathbf{e}_z = \frac{Q}{4\pi\varepsilon_0}\frac{z}{R^3}\mathbf{e}_z$$

$$\mathbf{E} \sim \frac{\sigma}{2\varepsilon_0}\mathbf{e}_z\left(1-\frac{z}{R}\right)$$

Near the structures





Example 002

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(c)

Consider some interesting cases… (z > 0)

Ring

$$\boxed{\mathbf{E} = \frac{\lambda R}{2\varepsilon_0}\frac{z}{\left(z^2+R^2\right)^{\frac{3}{2}}}\mathbf{e}_z}$$

Disk

$$\boxed{\mathbf{E} = \frac{\sigma}{2\varepsilon_0}\mathbf{e}_z\left(1-\frac{z}{\sqrt{z^2+R^2}}\right)}$$

Discontinuity! (Discussed later)

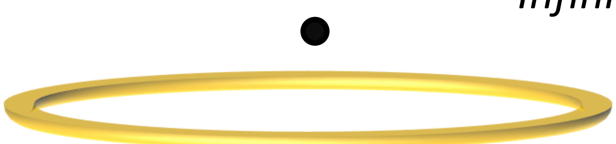
$R \to \infty$

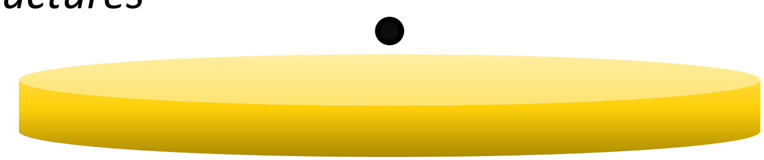
$$\mathbf{E} \sim 0$$

$$\mathbf{E} \sim +\frac{\sigma}{2\varepsilon_0}\mathbf{e}_z \quad (z>0), \qquad \mathbf{E} \sim -\frac{\sigma}{2\varepsilon_0}\mathbf{e}_z \quad (z<0)$$

Uniform

Infinite structures





Example 002

Example 3.1 (a) Find **E** on the symmetry axis of a ring with radius R and uniform charge per unit length λ. (b) Use the results of part (a) to find **E** on the symmetry axis of a disk with radius R and uniform charge per unit area σ. (c) Use the results of part (b) to find **E** for an infinite sheet with uniform charge density σ. Discuss the matching condition at $z = 0$.

(c)

Consider some interesting cases… (z > 0)

Ring

$$\boxed{\mathbf{E} = \frac{\lambda R}{2\varepsilon_0}\frac{z}{\left(z^2+R^2\right)^{\frac{3}{2}}}\mathbf{e}_z}$$

Disk

$$\boxed{\mathbf{E} = \frac{\sigma}{2\varepsilon_0}\mathbf{e}_z\left(1-\frac{z}{\sqrt{z^2+R^2}}\right)}$$

$$\sim 1-\frac{1}{2}\left(\frac{R}{z}\right)^2$$

Taylor Expansion $\frac{R}{z}$

$R << z$

$$\mathbf{E} \sim \frac{\lambda}{2\varepsilon_0}\frac{R}{z^2}\mathbf{e}_z = \frac{Q}{4\pi\varepsilon_0}\frac{1}{z^2}\mathbf{e}_z$$

$$\mathbf{E} \sim \frac{Q}{4\pi\varepsilon_0}\frac{1}{z^2}\mathbf{e}_z$$

● *Far from the structures ~ Point-like behaviors!* ●

Electrostatics – Maxwell's Equations

Postulates: Differential Form

$\varepsilon_{\mathrm{r}} = 1$ in the Vacuum

$$\nabla \times \mathbf{E} = \mathbf{O}$$

$$\nabla \cdot \mathbf{D} = \rho$$

$$\mathbf{D} = \varepsilon_0 \varepsilon_r \mathbf{E}$$

$\Rightarrow$

$$\boxed{\begin{aligned} \nabla \times \mathbf{E} &= \mathbf{O} \\ \nabla \cdot \mathbf{E} &= \frac{\rho}{\varepsilon_0} \end{aligned}}$$

Postulates: Integral Form

$\varepsilon_r = 1$ in the Vacuum

$$\nabla \times \mathbf{E} = \mathbf{0}$$

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\varepsilon_0}$$

Gauss & Stokes

$$\oint_C \mathbf{E} \cdot d\mathbf{l} = 0$$

$$\oint_S \mathbf{E} \cdot d\mathbf{S} = \frac{Q}{\varepsilon_0}$$