Therefore the update is given by

$$\Delta\mathbf{x} = \mathbf{J}^{-1}(\mathbf{0} - \mathbf{f}^0)$$

or in general

$$\Delta\mathbf{x} = \mathbf{J}^{-1}(\mathbf{f}^T - \mathbf{f}^0)$$

where the targets are not zero but $\mathbf{f}^T$. For the complete aircraft, of course, the targets are zero. $\mathbf{J}$ is called the Jacobian matrix.

For an isolated rotor, moment trim, the updates are based on the following Jacobian.

$$\mathbf{J}\left\{\begin{array}{c}\Delta\theta_0\\ \Delta\theta_{1c}\\ \Delta\theta_{1s}\end{array}\right\} = \left[\begin{array}{ccc}\frac{\partial T}{\partial\theta_0} & \frac{\partial T}{\partial\theta_{1c}} & \frac{\partial T}{\partial\theta_{1s}}\\ \frac{\partial M_X}{\partial\theta_0} & \frac{\partial M_X}{\partial\theta_{1c}} & \frac{\partial M_X}{\partial\theta_{1s}}\\ \frac{\partial M_Y}{\partial\theta_0} & \frac{\partial M_Y}{\partial\theta_{1c}} & \frac{\partial M_Y}{\partial\theta_{1s}}\end{array}\right]\left\{\begin{array}{c}\Delta\theta_0\\ \Delta\theta_{1c}\\ \Delta\theta_{1s}\end{array}\right\} = \left\{\begin{array}{c}T_0 - T\\ M_{X0} - M_X\\ M_{Y0} - M_Y\end{array}\right\} \quad (1.110)$$

where T_0, M_{X0} and M_{Y0} are the trim targets.

For an isolated rotor wind tunnel trim the trim variable updates are based on the following Jacobian.

$$\mathbf{J}\left\{\begin{array}{c}\Delta\theta_0\\ \Delta\theta_{1c}\\ \Delta\theta_{1s}\end{array}\right\} = \left[\begin{array}{ccc}\frac{\partial T}{\partial\theta_0} & \frac{\partial T}{\partial\theta_{1c}} & \frac{\partial T}{\partial\theta_{1s}}\\ \frac{\partial\beta_{1c}}{\partial\theta_0} & \frac{\partial\beta_{1c}}{\partial\theta_{1c}} & \frac{\partial\beta_{1c}}{\partial\theta_{1s}}\\ \frac{\partial\beta_{1s}}{\partial\theta_0} & \frac{\partial\beta_{1s}}{\partial\theta_{1c}} & \frac{\partial\beta_{1s}}{\partial\theta_{1s}}\end{array}\right]\left\{\begin{array}{c}\Delta\theta_0\\ \Delta\theta_{1c}\\ \Delta\theta_{1s}\end{array}\right\} = \left\{\begin{array}{c}T_0 - T\\ \beta_{1c0} - \beta_{1c}\\ \beta_{1s0} - \beta_{1s}\end{array}\right\} \quad (1.111)$$

where T_0, β_{1c0} and β_{1s0} are the trim targets.

The Jacobians are calculated by perturbing the initial estimates of each trim variable by 5%–10%, and using the finite differences of the perturbed trim targets. If the trim targets are linear functions of the trim variables the solution is obained in one iteration. Generally they are nonlinear functions and several iterations are neccessary. After each iteration, the Jacobian must be re-calculated, about the current trim variables. For rotors, except in the case of severe stall, this is often not necessary. The Jacobian is often calculated only once, before the trim iterations begin, and stored for all subsequent iterations. This procedure is called the *modified Newton* procedure.

Questions

1. What are the advantages and the disadvantages of the tractor and pusher type tail rotors?
2. Which one of the following rotors need tail rotors for hovering?

   i) Coaxial rotor (ABC-Sikorsky)

   ii) Circulation controlled rotor (X-wing-NSRDC)

   iii) Tilt rotor (XV-15-Bell)

   iv) Tandem rotor (Chinook-Boeing Vertol)

   v) Tip jet rotor

3. Justify the following:

- The helicopters with conventional rotors are limited to a forward speed of about 170 knots.
- In hovering flight, the rotor disk follows the shaft (in about 3 revs).
- A rotation of the tail boom in the opposite direction of the blades rotation can be troublesome.
- For a rotor with hinge offset, the phase lag of the flapping motion, with respect to the pitch motion is not 90°.
- For a fixed wing, control surfaces such as flaps and ailerons are used to control the lift, but that is not the case with rotor blades.
- It is quite common that a small precone of 2 to 3 degrees is given to hingeless blades.
- For a flapping rotor with no cyclic pitch (tail rotor), the hub and control planes are equivalent.
- For a feathering rotor with non flapping (propeller with cyclic pitch), the hub plane and TPP are identical.
- The rotor behaves as a gyro, maintaining its orientation relative to the inertial space in vacuum.
- A teetering rotor perhaps is not practical for large helicopters.
- An optimum rotor is a hypothetical rotor that is efficient in hover for one thrust level.
- The induced rotor power is the largest in hover.

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2. Gessow, A., and Meyers, G.C., Aerodynamics of the Helicopter, Frederick Ungar Publishing Co., (1952) Ch. 3, 4, 7, 8 and 9.

Chapter 2

Flap Dynamics

Blade flapping is the motion of the blade normal to the rotation plane. In this chapter, the natural vibration characteristics and response due to external force such as an aerodynamic force is examined for a flapping blade. Initially, a simplified model is used, where the blade is assumed rigid with a hinge offset. Later on, a more refined model is used, where the blade is represented as an elastic beam. The primary objective of this chapter is to grasp various mathematical tools as applicable to rotor analyses through an application to isolated flap mode dynamics. It is very important to understand the need and usage of these tools for a simple case of flapping blade, such that these could be extended to more complex coupled blade dynamics in later chapters.

It should be kept in mind that the dynamics of flap mode is by itself an important step towards the understanding of coupled rotor dynamics. The knowledge of natural vibration characteristics of isolated flap motion is important for vibration, loads, blade stresses and aeromechanical stability. In fact, the fundamental rotating flap frequency is a key physical parameter and has a direct influence on vehicle performance, flight stability and rotor dynamics. Typically for an articulated blade, this frequency varies from 1.03 to 1.05 times the rotational speed, whereas for hingeless blades, the flap frequency varies from 1.08 to 1.15 times the rotational speed. For an articulated rotor, the maximum bending stress takes place about mid-span of the blade where as for a hingeless rotor, it takes place near the root. We shall see later that higher the flap frequency, the larger will be the bending stresses in the blade.

2.1 Rigid Blade Model

The rotor blade is assumed to be rigid undergoing a single degree of motion, i.e., flapping motion about a hinge. A real rotor can have : (1) a mechanical hinge, or (2) a virtual hinge. Articulated rotors have mechanical hinges. Hingeless rotors, even though they are called hingeless, have virtual hinges. Virtual hinges are created by flexible structures near the blade root which are softer than the rest of the blade. Thus a hingeless rotor can also be modeled as a rigid rotor flapping about a hinge. Except that in this case, the hinge location or offset is an equivalent one determined from experiment, or a flexible blade analysis.

2.1.1 Hinged Blade with zero offset

Consider a rigid blade hinged at the rotation axis. See Fig. 2.2 (a). This simple configuration represents an articulated rotor blade. The blade undergoes a single degree of motion, i.e. flapping. Assume that there is no spring restraint and the flap hinge offset is set to zero. The forces acting on an element of length dr, and their moment arms, are listed below. The flapping angle, β is assumed to be small. The inertial force (IF) on the element is defined as the mass of the element

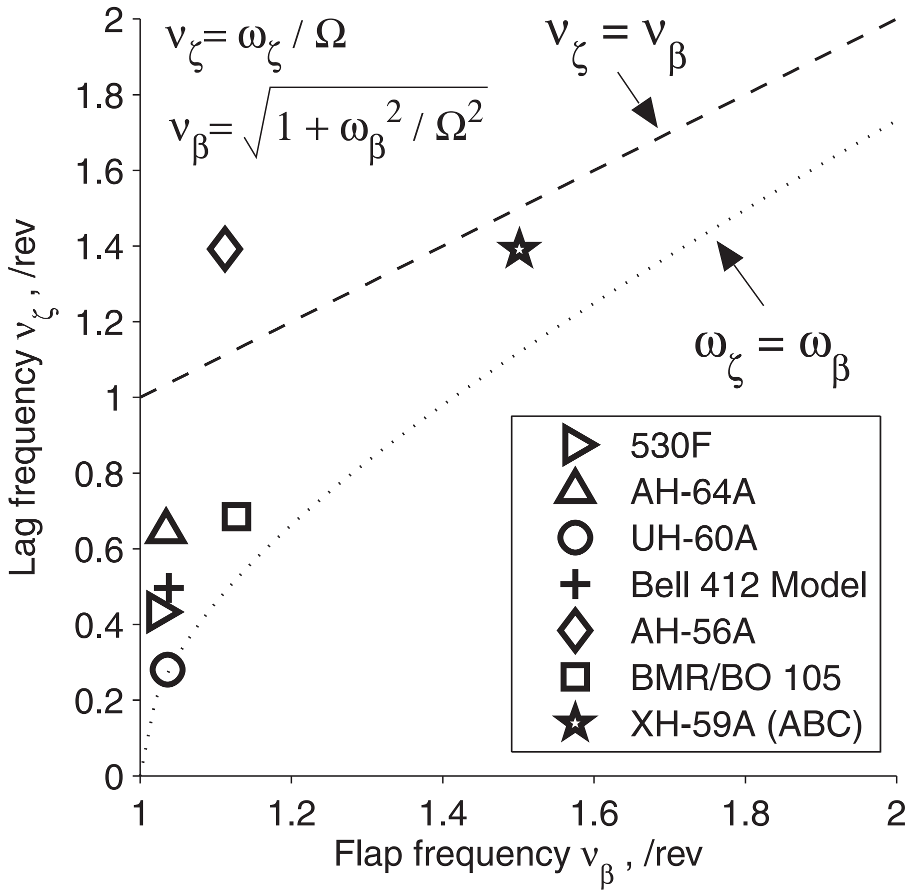


Figure 2.1: **Flap and lag frequencies of rotor blades**

multiplied with the flapping acceleration of the element, acting opposite to the direction of the flapping accleration (See Chapter 1, Section 1.2.3).

a) inertia force (IF): $m\,dr\,r\ddot{\beta}$ arm r with respect to rotation axis

b) centrifugal force (CF): $m\,dr\,\Omega^2 r$ arm $z = r\beta$

c) aerodynamic force (AF): $F_z\,dr$ arm r

Taking moment about flap hinge

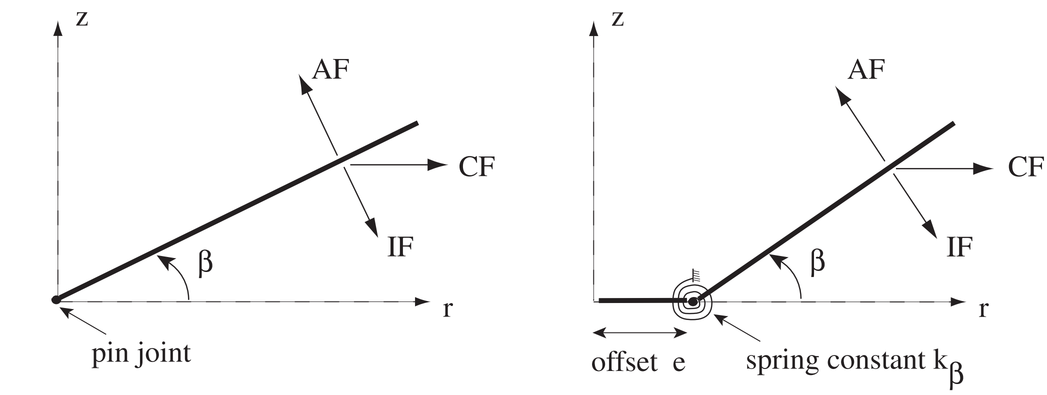
$$\int_0^R (mr\ddot{\beta}\,dr)r + \int_0^R (m\Omega^2 r\,dr)r\beta - \int_0^R (F_z dr)r = 0$$

$$\left(\int_0^R mr^2 dr\right)(\ddot{\beta} + \Omega^2\beta) = \int_0^R rF_z dr$$

$\int_0^R mr^2 dr =$ mass moment of inertia about flap hinge $= I_b$, with units$(lb - in - sec)^2$ or $kg - m^2$

For a uniform blade $I_b = \frac{mR^3}{3}$ where m is the mass per unit length (lb - sec^2/in^2 or kg/m). The above expression leads to the flap equation

$$\ddot{\beta} + \Omega^2\beta = \frac{1}{I_b}\int_0^R rF_z dr \tag{2.1}$$



(a) flapping blade with zero hinge offset (b) flapping blade with hinge offset e

Figure 2.2: **Rigid blade flapping model**

Now express time in terms of azimuth angle

$$\psi = \Omega t$$

where Ω is the rotational speed, rad/sec. The derivatives are

$$\dot{\beta} = \frac{\partial \beta}{\partial t} = \Omega \frac{\partial \beta}{\partial \psi} = \Omega \overset{*}{\beta} \qquad \ddot{\beta} = \Omega^2 \overset{**}{\beta}$$

$\dot{\beta}$ was the rate of change of flapping with time (rad/sec), whereas $\overset{*}{\beta}$ is the rate of change of flapping with azimuth of rotation (rad/rad). The flap equation then becomes

$$\frac{\partial^2 \beta}{\partial \psi^2} + \beta = \frac{1}{I_b \Omega^2} \int_0^R r F_z dr$$

or

$$\overset{**}{\beta} + \beta = \gamma \overline{M}_\beta \tag{2.2}$$

where

$$\overset{**}{\beta} = \frac{\partial^2 \beta}{\partial \psi^2} \qquad \gamma = \frac{\rho a c R^4}{I_b} \qquad \overline{M}_\beta = \frac{1}{\rho a c R^4 \Omega^2} \int_0^R r F_z dr$$

γ is called the Lock number, ρ is the air density, a is the aerodynamic lift curve slope, c is the chord and R is the rotor radius. The Lock number represents the ratio of aerodynamic force and inertia force. Typically, the value of γ varies from 5 to 10, the smaller number for a heavy blade whereas the larger value for a light blade.

The flapping equation can be imagined to represent a single degree of freedom spring-mass system The natural frequency of the system, from eqn. 2.1, is Ω rad/sec.

i. e., $\omega_\beta = \Omega$ rad/sec

The natural frequency of the system, from eqn. 2.2, is 1 rad/rad.

i. e., $\nu_\beta = 1$ rad/sec

where

$$\nu_\beta = \frac{\omega_\beta}{\Omega}$$

The unit of rad/rad is also defined as per rev (/rev or p). Thus, 1 rad/rad is 1/rev (or 1p), 2 rad/rad is 2/rev (or 2p) and so on. For this configuration, the spring stiffness is a result of centrifugal force. To visualize this spring, consider the simple example of a stone tied to a thread and rotated. Very soon the thread is taut, and the stone stretches out due to centrifugal force. The natural frequency of this system will be the rotational frequency itself. The aerodynamic force $\overline{M}_\beta$ can be motion dependent and will be discussed in later sections.

2.1.2 Hinged Blade with Offset

Consider a rigid blade hinged at a distance e from the rotation axis. It is assumed that there is a flap bending spring at the hinge. See Fig. 2.2(b). This configuration represents an articulated blade with a hinge offset. It can also be a simple approximation for a hingeless blade (as shown in the next section).

$$h(q_1, q_2, q_3, \ldots, t) = 0 \tag{2.3}$$

$$\ddot{h} + \alpha\dot{h} + \beta h = 0 \tag{2.4}$$

Force acting on an element dr are

a) inertia force (IF): $m\,dr(r-e)\ddot{\beta}$ arm$(r-e)$ about hinge

b) centrifugal force (CF): $m\,dr\,\Omega^2 r$ arm$(r-e)\beta$

c) aerodynamic force (AF): $F_z dr$ arm$(r-e)$

d) spring moment (SF): $k_\beta(\beta-\beta_p)$ about hinge

β_p is a precone angle. Taking moment about flap hinge,

$$\int_e^R m(r-e)^2 dr\,\ddot{\beta} + \int_e^R m\Omega^2 r(r-e)dr\,\beta - \int_e^R F_z(r-e)dr + k_\beta(\beta-\beta_p) = 0$$

or

$$k_\beta(\beta-\beta_p) = \int_e^R F_z(r-e)dr - \int_e^R m\Omega^2 r(r-e)dr\,\beta - \int_e^R m(r-e)^2 dr\,\ddot{\beta} \tag{2.5}$$

The spring moment $k_\beta(\beta-\beta_p)$ is the flapping moment, or the flap bending moment, at the hinge. It is a dynamic load, i.e., the balance of external forcing, minus centrifugal forcing, minus the part used up by blade acceleration.

$$I_\beta = \int_e^R m(r-e)^2 dr, \qquad \text{mass moment of inertia about flap hinge}$$

$$\int_e^R mr(r-e)dr = \int_e^R m(r-e)^2 dr + \int_e^R me(r-e)dr$$

$$= I_\beta\left(1 + \frac{e\int_e^R m(r-e)dr}{\int_e^R m(r-e)^2 dr}\right)$$

Flap equation

$$I_\beta \left\{ \ddot{\beta} + \Omega^2 \left(1 + \frac{e\int_e^R m(r-e)dr}{I_\beta}\right)\beta + \frac{k_\beta}{I_\beta}(\beta - \beta_p)\right\} = \int_e^R F_z(r-e)dr$$

Writing in non-dimensional form by dividing with $I_b\Omega^2$.

$$\overset{*}{I_\beta} (\overset{**}{\beta} + \nu_\beta^2 \beta) = \frac{k_\beta}{I_b}\frac{1}{\Omega^2}\beta_p + \gamma \overline{M}_\beta \qquad \overset{*}{I_\beta} = \frac{I_\beta}{I_b}$$

ν_β is the non-dimensional flap frequency

$$\nu_\beta^2 = 1 + \frac{e\int_e^R m(r-e)dr}{I_b} + \frac{k_\beta}{I_b\Omega^2} \qquad \text{/rev}$$

For uniform blade, the second term is

$$\frac{e}{I_\beta}\int_e^R m(r-e)dr = \frac{3e}{2(R-e)} \simeq \frac{3}{2}\frac{e}{R}$$

The third term represents the non-rotating natural frequency made dimensionless using rotational frequency

$$\frac{k_\beta}{I_b} = \omega_{\beta 0}^2 \qquad \text{rad/sec}$$

The term $I_\beta^* = I_\beta / I_b$ is nearly equal to unity. Thus the flap equation can be written as

$$\overset{**}{\beta} + \nu_\beta^2 \beta = \frac{\omega_0^2}{\Omega^2}\beta_p + \gamma \overline{M}_\beta$$

The β_p is the precone angle given to the blade to reduce the steady flap moment about the hinge and typically its value is about 2 to 3°. Again this represents a single degree spring-mass system, as shown in Fig. 2.3.

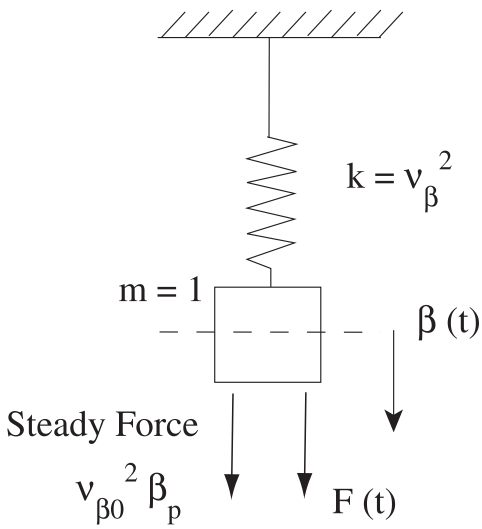


Figure 2.3: **The rigid blade flapping equation represents a single degree of freedom spring-mass system**

The natural frequency of the system

$$\begin{aligned} \frac{\omega_\beta}{\Omega} &= \nu_\beta && \text{/rev} \\ \omega_\beta &= \nu_\beta \Omega && \text{rad/sec} \\ &= \left(1 + \frac{3}{2}\frac{e}{R} + \frac{\omega_{\beta 0}^2}{\Omega^2}\right)^{1/2} \Omega && \text{for a uniform blade} \end{aligned}$$

where $\omega_{\beta 0}$ is the non-rotating natural frequency, rad/sec. For zero spring case

$$\omega_n = \left(1 + \frac{3}{2}\frac{e}{R}\right)^{1/2} \Omega \quad \text{rad/sec}$$

Typically e varies between 4 to 6 % of rotor radius for an articulated blade. The variation of flap frequency with hinge offset is given in Fig. 2.4

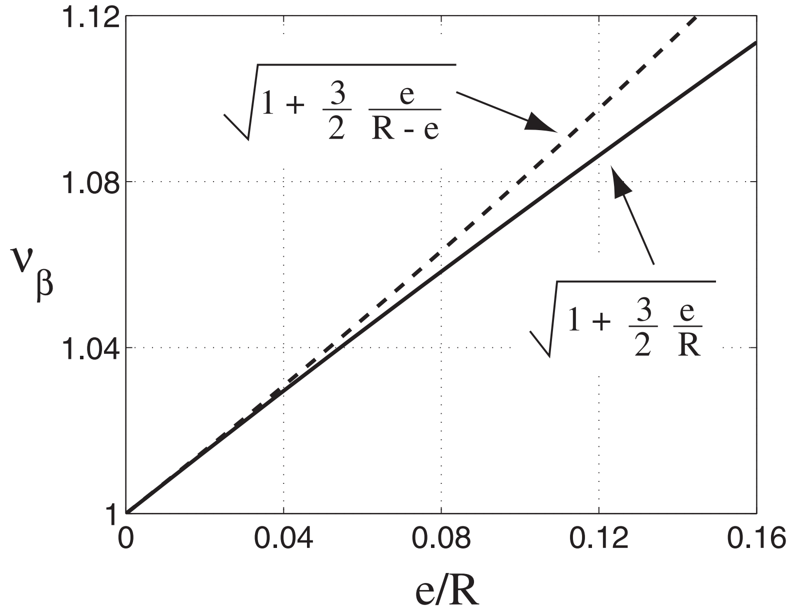


Figure 2.4: **Variation of flapping frequency with hinge offset; Spring constant**$k_\beta = 0$

Example 2.1:

Calculate the rotating flap frequency of an articulated rotor blade with hinge offset of 1 ft. from the rotation axis. Given are the rotor radius = 20 feet and the RPM = 360.

$$\frac{e}{R} = \frac{1}{20} = 0.05$$

$$1 + \frac{3}{2}\frac{e}{R} = 1.075$$

$$\begin{aligned} \text{Flap frequency} &= \left(1 + \frac{3}{2}\frac{e}{R}\right)^{1/2} = 1.037 && \text{per rev} \\ &= 1.037\Omega && \text{rad/sec} \\ &= 1.037 \times \frac{360}{60} \times 2\pi && \\ &= 39.1 && \text{rad/sec} \end{aligned}$$

2.1.3 Hingeless Blade with Equivalent Hinge Offset

To simplify analysis, a hingeless blade can be idealized into a rigid blade with an offset hinge and a bending spring at the hinge. This representation can be useful for the calculation of flight dynamic and aeroelastic stability because the global characteristics are well represented with this simple model. It is assumed that the fundamental mode shape and the fundamental frequency is available for the hingeless blade, either using a flexible blade model (described later) or determined through an experiment. To obtain an equivalent simplified configuration, two constants are to be determined, e and k_β. Two simple methods to calculate these constants are as follows.

One method is to compare the nonrotating and rotating natural frequencies.

$$\nu_\beta = \frac{\omega_\beta}{\Omega} = \left(1 + \frac{3}{2}\frac{e}{R} + \frac{\omega_{\beta 0}^2}{\Omega^2}\right)^{1/2}$$

where ω_{β_0} and ω_β are the non-rotating and rotating flap frequencies for basic hingeless blade. Then,

$$k_\beta = \omega_{b0}^2 I_b$$

$$e = \frac{2}{3}R\left(\nu_\beta^2 - \frac{\omega_{b_0}^2}{\Omega^2} - 1\right)$$

where I_b is the flap mass moment of inertia. A second method is to compare the rotating bending

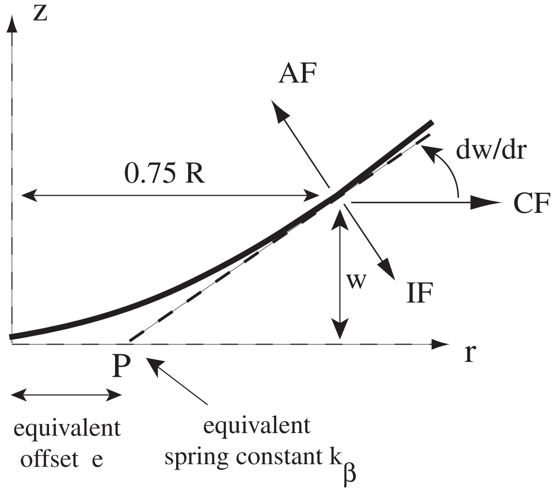


Figure 2.5: **Equivalent rigid blade flapping model for a hingeless blade**

slope and displacement at a reference station 75% R (see Fig. 2.5) and the rotating frequency. Extend the slope line to find point P on the undeflected blade axis. Then,

$$e = R - \frac{w}{\left(\frac{dw}{dr}\right)}$$

This spring stiffness can be calculated comparing the rotating flap frequency.

$$k_\beta = I_\beta \Omega^2 \left(\nu_\beta^2 - 1 - \frac{3}{2}\frac{e}{R}\right)$$

2.2 Flexible Beam Model

A better representation for a rotor blade is to assume it is an elastic beam restrained at the root. The blade undergoes bending deflection distribution under loading. The assumption of treating blade as a slender beam is quite appropriate because the cross sectional dimensions are much smaller than the length. In this chapter, only flap bending (out of plane) is considered. The lead-lag bending (in-plane) and torsion will be introduced in the next chapter. The physics of a rotating beam in bending differs from that of a non-rotating beam because of nonlinear coupling with axial elongation. To understand this coupling, consider first the dynamics of axial elongation alone for a rotating beam.

2.2.1 Axial Deformation

A beam element located at a distance r from the rotation axis before deformation is at a location $r+u$ after deformation. The centrifugal force acting on the element is then $mdr\Omega^2(r+u)$. A force balance on the element gives

$$\left(T+T'dr\right)-T+mdr\Omega^2\left(r+u\right)-mdr\ddot{u}+f_h dr=0$$

from which the gradient of tensile force follows

$$T'=m\ddot{u}-m\Omega^2\left(r+u\right)-f_h \tag{2.6}$$

The tensile force is related to the axial elongation as

$$T=EAu'$$

Substituting in eqn. 2.6 we obtain the governing equation for axial elongation

$$m\ddot{u}-m\Omega^2 u-\left(EAu'\right)'=f_h+m\Omega^2 r \tag{2.7}$$

Note that a force balance at a section gives the following expression for the tensile force

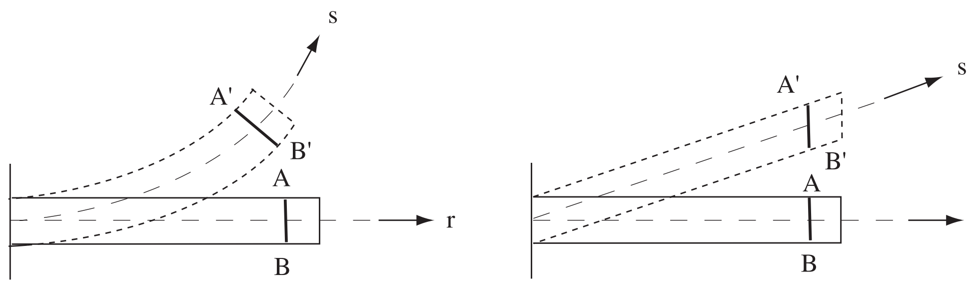
$$T=\int_r^R\left(-m\ddot{u}+m\Omega^2 u+m\Omega^2\rho+f_h\right)d\rho \tag{2.8}$$

which when differentiated once using Leibnitz theorem gives back eqn. 2.6.

2.2.2 Euler-Bernoulli Theory of Bending

The Euler-Bernoulli assumption states that a plane section normal to the beam centerline remains plane and normal after deformation (see Fig. 2.6). This is a valid assumption when the shear deflection is negligible. The assumption helps to uncouple bending and shear deflections. The assumption lets one express the rotation of a section solely in terms of its translation. If the out of plane bending deflection is w then the bending slope is simply the derivative of the deflection, i.e. $w^+=\frac{dw}{ds}$, where w^+ is the derivative with respect to the span coordinate along the deformed beam s. For small deflections, the derivative can be taken with respect to the undeformed beam coordinate r. Thus $w^+\simeq\frac{dw}{dr}=w'$. The bending curvature κ, and the radius of curvature ρ, are then related to the deflection by the following kinematic relation

$$\kappa=\frac{1}{\rho}=\frac{w^{++}}{[1+w^{+\,2}]^{\frac{3}{2}}}\simeq\frac{w''}{[1+w'^2]^{\frac{3}{2}}} \tag{2.9}$$



(a) Pure bending; Plane section A-B remains plane and perpendicular to beam centerline after deformation A'-B'

(b) Pure shearing; Plane section A-B remains plane but not perpendicular to beam centerline after deformation A'-B'

Figure 2.6: **Pure bending and pure shearing of beams**

The strain due to bending ϵ_{rr} at a distance z from the beam centerline is related to the curvature by $\epsilon_{rr} = z/\rho$. The strain is then related to the stress σ_{rr} by the constitutive relation

$$\sigma_{rr} = E\epsilon_{rr} = E\frac{z}{\rho} \tag{2.10}$$

The bending moment at any section is then given by the resultant

$$M(r) = \int_{\text{Area}} \sigma_{rr} z dA = \int_{\text{Area}} E\frac{z^2}{\rho} dA = \frac{EI}{\rho} = EI\frac{w''}{[1+w'^2]^{\frac{3}{2}}} \simeq EIw'' \tag{2.11}$$

The kinematic relation 2.9, the constitutive relation 2.10, and the resultant relation 2.11 together form the Euler-Bernoulli beam theory. Note that the constitutive relation 2.10 and the resultant 2.11 generates the well-known identity

$$\frac{M}{I} = \frac{E}{\rho} = \frac{\sigma_{rr}}{z} \tag{2.12}$$

2.2.3 Flap Bending Equation using Newtons's Laws

Let us derive a general equation of motion of a beam under external loading. It is assumed small deflections as compared to its dimensions. Also, it is assumed that the rotation of the element is small as compared to the vertical displacement. Thus, the rotary inertia effects are neglected in the derivation. For convenience, structural damping is neglected. The geometry of the deformed beam is shown in Fig. 2.7. where $f_z(r,t)$ is the vertical load per unit span (kg/m, lb/in) and $w(r,t)$ is the vertical deflection at station r (m, in). Consider an element dr of mass $m\,dr$, where m is mass per unit span, with units kg/m or lb/in. The forces acting on it are the inertia force $m\ddot{w}\,dr$, the external vertical force f_z, the external axial force f_H, and the internal tensile, shear, and bending loads T (N, lbf), S (N, lbf), and M (N-m, lbf-in). S is positive when it acts in the positive direction z (i.e., upward) on a negative x plane (i.e., left face of element). M is positive when top fiber under compression. T is positive in tension. The bending slope $w' = dw/dr$ is assumed to be small, i.e. w'^2 will be neglected with respect to unity. Thus $\cos w' \simeq 1$ and $\sin w' \simeq w'$.

Consider the equilibrium of forces and moment on the element. Force equilibrium in the z-direction gives

$$f_z dr + S - S - \frac{dS}{dr}dr - m\ddot{w}\,dr - T\frac{dw}{dr} + \left(T + \frac{dT}{dr}\right)\left(\frac{dw}{dr} + \frac{d^2w}{dr^2}dr\right) = 0$$

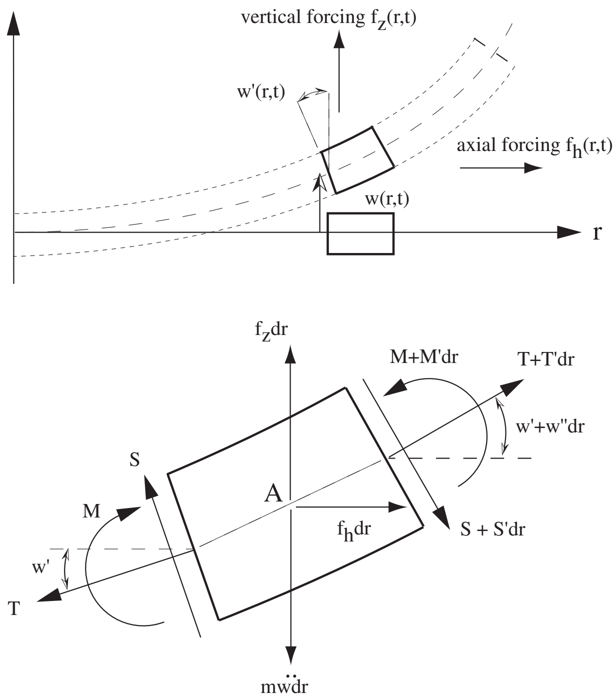


Figure 2.7: **Pure bending of a rotating beam and free body diagram of a beam segment**

or

$$f_z - \frac{dS}{dr} - m\ddot{w} + T\frac{d^2w}{dr^2} + \frac{dT}{dr}\frac{dw}{dr} = 0$$

or

$$\frac{dS}{dr} = f_z - m\ddot{w} + \frac{d}{dr}\left(T\frac{dw}{dr}\right) \tag{2.13}$$

The above expression states that the spatial derivative of shear is equal to the sectional loading distribution. The expression leads to the governing partial differential equation (PDE) for deflection w. To this end, S and T must be expressed in terms of either w or other known quantities. S can be expressed in terms of w by considering the moment equilibrium about the center of the element, point A. Moment about A gives

$$M + S\,dr - M - \frac{dM}{dr}dr = 0$$

or

$$\frac{dM}{dr} = S \tag{2.14}$$

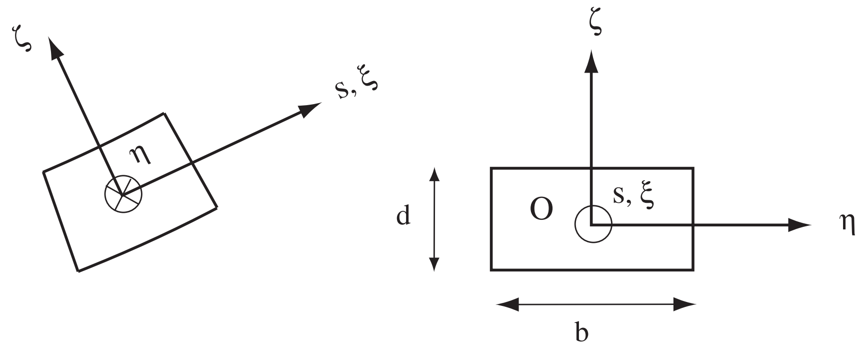
The above expression states that the spatial derivative of bending moment is the shear distribution. Now use the Euler-Bernoulli beam theory result to obtain the beam model. Consider a rectangular beam cross-section as shown in Fig. 2.8. Let O be the shear center of the section. The shear center, by definition, is such point where a force applied vertically creates a pure bending deformation with no accompanying twist. For a rectangular closed section this point is the area centroid. Let (ξ, η) be the principle axes at this section, i.e. $I_{\xi\eta} = 0$. The Euler-Bernoulli beam theory gives

$$M = EI\frac{d^2w}{dr^2} \tag{2.15}$$

where I equals $I_{\eta\eta}$, the area moment of inertia about the principle axis η, with units m^2 or in^2. For a rectangular section

$$I_{\eta\eta} = \frac{bd^3}{12} \quad \text{m}^4 \text{or in}^4$$

E is the Young's Modulus of the beam material, with units N/m^2 or lbf/in^2. EI is the flexural stiffness about the principle axis η, with units N-m^2 or lbf-in^2. Thus eqn.2.14 becomes



(a) Side view of the deformed element (b) Cross-section of the deformed element

Figure 2.8: **Rectangular cross-section of a beam with centroid 0**

$$S = \frac{d}{dr}\left(EI\frac{d^2w}{dr^2}\right)$$

Substitute the above expression in eqn.2.13 to obtain

$$\frac{d^2}{dr^2}\left(EI\frac{d^2w}{dr^2}\right) + m\frac{\partial^2 w}{\partial t^2} - \frac{d}{dr}\left(T\frac{dw}{dr}\right) = f_z(r,t) \tag{2.16}$$

T is related to axial elongation via eqn. 2.6. Thus the governing equation in bending takes the following form.

$$m\ddot{w} + \left(EIw''\right)'' - \left(EAu'w'\right)' = f_z(r,t) \tag{2.17}$$

The difference between the above equation and the non-rotating beam equation is the nonlinear coupling term $EAu'w'$. The coupling term can also be expressed in a different manner. Recall that T is also be expressed as eqn. 2.8 Thus the governing equation can also be written in the following form.

$$m\ddot{w} + \left(EIw''\right)'' - \left[w'\int_x^R \left(-m\ddot{u} + m\Omega^2 u + m\Omega^2\rho + f_h\right) d\rho\right]' = f_z(r,t) \tag{2.18}$$

The above form is useful for a simple flapping blade analysis without axial dynamics. Without axial dynamics, i.e. $u = 0$, eqn. 2.17 reduces to the non-rotating beam equation. The form given in eqn. 2.18, however, can be used to retain the centrifugal term $m\Omega^2\rho$. Physically, the centrifugal term affects u which affects w via the nonlinear coupling term $EAu'w'$, but eqn. 2.18 helps us recast the fundamentally non-linear problem into a linear form by ignoring the axial elongation but retaining the effect of centrifugal stiffness.

Let us study this centrifugal stiffness term further. Using force equilibrium in axial direction, i.e. eqn. 2.6, ignoring axial elongation, and assuming no external forcing in the axial direction, we have

$$\frac{dT}{dr} + m\Omega^2 r = 0$$

or

$$\frac{dT}{dr} = -m\Omega^2 r$$

For a uniform beam, integration of the above expression yields

$$T = -\frac{1}{2}m\Omega^2 r^2 + C$$

$$\text{At r = R, T = 0} \quad \text{therefore} \quad C = \frac{1}{2}m\Omega^2 R^2$$

$$T = \frac{m\Omega^2}{2}(R^2 - r^2)$$

where Ω is the rotation speed, rad/sec. In general for a non-uniform blade we have

$$T = \int_r^R m\Omega^2 \rho d\rho \tag{2.19}$$

which is same as eqn. 2.8 with axial deformation u and axial forcing f_h ignored.

To summarize, under static conditions, $T = 0$ and $\ddot{w} = 0$, the shear force and bending moment at any spanwise station can be calculated directly using eqns. 2.13 and 2.14 using the external loading f_z. Under dynamic conditions, but non-rotating, the inertial term in eqn. 2.13 depends on the vertical displacement w. Therefore it is necessary to combine eqns. 2.13 and 2.14 and solve for the vertical displacement w. Rotation of the beam adds the tensile force term on the right hand side of eqn. 2.13. The tensile force term depends on the axial elongation u via eqn. 2.8. Therefore now it is necessary to combine eqns. 2.13 and 2.14 with eqn. 2.8. This combination is nonlinear in nature. But in its simplest form, ignoring axial dynamics but still retaining the centrifugal effect of rotation, it can be expressed in the following form.

$$\begin{aligned} &m\ddot{w} + \left(EIw''\right)'' - \left(Tw'\right)' = f_z(r,t) \\ &T = \int_r^R m\Omega^2 \rho d\rho \end{aligned} \tag{2.20}$$

2.2.4 Second Order Nonlinear Coupled Axial Elongation-Flap Bending

In the previous section, the governing equation for a rotating flapping blade was derived as follows

$$m\ddot{w} + \left(EIw''\right)'' - \left(EAu'w'\right)' = f_z(r,t) \quad : \text{Flap}$$

$$m\ddot{u} - m\Omega^2 u - \left(EAu'\right)' = f_h + m\Omega^2 r \quad : \text{Axial}$$

The equations were then reduced to the simplest linear form as given in eqn. 2.20. The above non-linear equations assumed small deformations. For large deformations, non-linearities upto second order can be retained. It is important to note that these equations are of little engineering value by themselves. In fact, even the non-linear equations given above for small deformations provide no significant improvement in the prediction of w compared to the simpler formulation given in eqn. 2.20; unless offcourse the axial deformation is desired. For rotor dynamics however, axial deformations by themselves are of lesser engineering value. The non-linear couplings due to large deformations will be critical later, while analyzing the coupled flap-lag-torsion dynamics of real rotor blades. There, several sources of nonlinear structural couplings will be encountered - geometric, coriolis, and centrifugal. The purpose in this section is to provide a simple illustration of the physical source, and a second order treatment, of one such nonlinear coupling: the elongation-flap bending coupling that occurs for large deformation in the presence of centrifugal force.

Large deformations require two changes in the previous formulation.

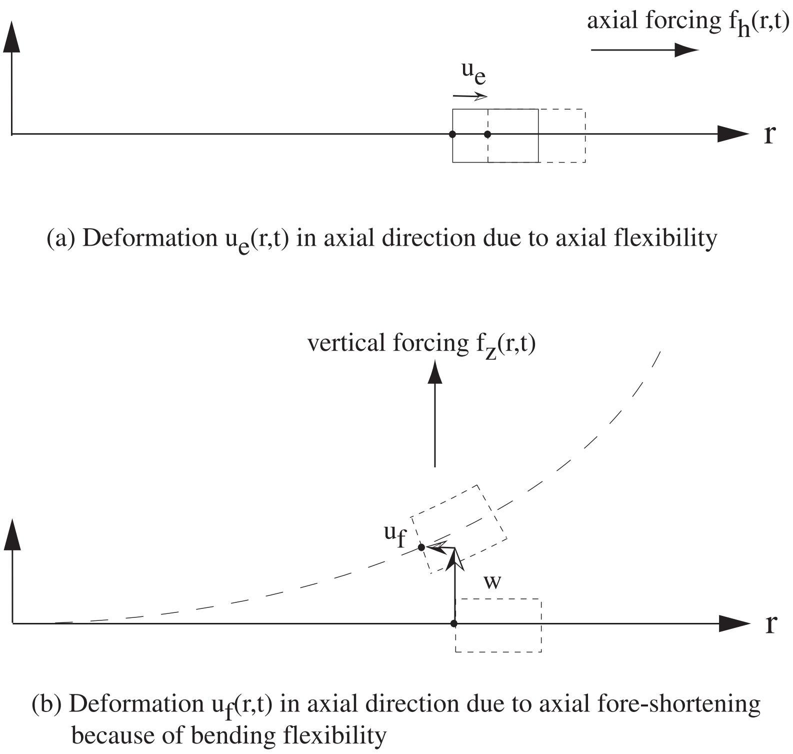


Figure 2.9: **Axial dynamics of a beam**

1. The axial force balance term: $(T + T'dr) - T$, is now modified to $(T + T'dr)\cos(w' + w''dr) - T\cos w'$.

2. The tensile force T is no longer expressible as EAu'. It must be replaced with EAu'_e where u_e is the axial elongation of the beam. Note that u is the deformation in the axial direction. For large deformations u and u_e are not the same.

Let us first understand the axial elongation u_e and the axial deformation u physically. u is not simply the component of u_e in the radial direction, or vice versa. The bending deflection w introduces a axial deformation simply by virtue of the fact that the length of the beam must remain the same after bending. This is an axial foreshortening effect, u_f. The total axial deformation u is the sum of these two effects (see Fig. 2.9).

$$u = u_e + u_f$$

The axial foreshortening u_f is caused by the bending of the beam and can therefore be expressed as a function of w. Figure 2.10 shows a beam element dr in the undeformed and deformed positions.

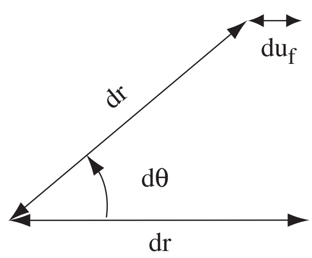


Figure 2.10: **Axial foreshortening of a beam due to bending**

Because the length of the element remains constant during pure bending the rotation creates an axial foreshortening du_f.

$$\begin{aligned} du_f &= \text{axial length of } dr \text{ after deformation} - \text{axial length of } dr \text{ before deformation} \\ &= dr\cos d\theta - dr \\ &= -2\sin^2(d\theta/2)dr \\ &= -(1/2)w'^2 dr \quad \text{using } \mathrm{d}\theta = w', \text{ and } \mathrm{d}\theta \text{ is small} \end{aligned}$$

Thus at any station r

$$\begin{aligned} u_f(r) &= -\frac{1}{2}\int_0^r w'^2 dr \\ \dot{u}_f(r) &= -\int_0^r w'\dot{w}' dr \\ \ddot{u}_f(r) &= -\int_0^r (\dot{w}'^2 + w'\ddot{w}')dr \end{aligned} \tag{2.21}$$

The equation for axial equilibrium now contains the following expression

$$\left(T + T'dr\right)\cos\left(w' + w''dr\right) - T\cos w'$$

Noting that upto second order, $\cos\theta = 1 - \theta^2/2$, we have

$$\cos\left(w' + w''dr\right) = 1 - \frac{1}{2}w'^2 - w'w''$$

$$\cos w' = 1 - \frac{1}{2}w'^2$$

It follows

$$\begin{aligned} \left(T + T'dr\right)\cos\left(w' + w''dr\right) - T\cos w' =& T'\left(1 - \frac{1}{2}w'^2\right) - Tw'w'' \\ =& \left[T\left(1 - \frac{1}{2}w'^2\right)\right]' \end{aligned} \tag{2.22}$$

Thus the former eqn. 2.6 now takes the following form

$$\left[T\left(1 - \frac{1}{2}w'^2\right)\right]' = m\ddot{u} - m\Omega^2\left(r + u\right) - f_h \tag{2.23}$$

T remains to be replaced in terms of u. $T = EAu_e'$, thus u_e needs to be expressed in terms of u. To this end, note that

$$\begin{aligned} du =& (dr + du_e)\cos d\theta - dx \\ =& dr\left(-\frac{1}{2}w'^2\right) + du_e\left(1 - \frac{1}{2}w'^2\right) \end{aligned} \tag{2.24}$$

It follows

$$u_e'\left(1 - \frac{1}{2}w'^2\right) = u' + \frac{1}{2}w'^2$$

Thus

$$T\left(1 - \frac{1}{2}w'^2\right) = EAu_e'\left(1 - \frac{1}{2}w'^2\right) = EA\left(u' + \frac{1}{2}w'^2\right)$$

The new axial dynamics equation becomes

$$\left[EA\left(u' + \frac{1}{2}w'^2\right)\right]' = m\ddot{u} - m\Omega^2\left(r+u\right) - f_h \tag{2.25}$$

The flap equation requires the term $(Tw')'$. Note that we have, to second order

$$\begin{aligned} u_e' &\approx \left(u' + \frac{1}{2}w'^2\right)\left(1 + \frac{1}{2}w'^2\right) \\ &\approx u' + \frac{1}{2}w'^2 \end{aligned} \tag{2.26}$$

Thus

$$Tw' = EAu_e'w' \approx EA\left(\approx u' + \frac{1}{2}w'^2\right)w'$$

which to second order remains same as before, i.e.

$$Tw' = EAu'w'$$

Thus the flap equation remains the same as before, eqn. 2.17. The final equations for this case are therefore

$$\begin{aligned} &m\ddot{w} + \left(EIw''\right)'' - \left(EAu'w'\right)' = f_z(r,t) \\ &\left[EA\left(u' + \frac{1}{2}w'^2\right)\right]' = m\ddot{u} - m\Omega^2\left(r+u\right) - f_h \end{aligned} \tag{2.27}$$

The time varying aerodynamic forcing in the axial direction is an order of magnitude smaller than the lift. Both the axial flexibility u_e and the axial forshortening u_f are in general an order of magnitude smaller than the bending displacement w. The natural frequency in axial flexibility u_e is usually greater than 10/rev. The axial dynamics is therefore often neglected during a simple analysis, and only eqn. 2.17 is considered.

However, it is important to understand that the nonlinear term $(EAu'w')'$ in the bending equation cannot be dropped. For a rotating beam this term introduces the centrifugal stiffening. Thus care must be taken while linearizing the beam bending equation. It must be replaced with $(Tw')'$, where

$$T = \int_r^R m\Omega^2 \rho d\rho$$

The axial dynamics can then be ignored. The axial forshoterning can be found from eqn. 2.21 once w is known. The steady axial deflection can be found simply from

$$T = EAu_e'$$

While this procedure was well-understood by rotorcraft designers, it created a hurdle for space-craft dynamicists during the development of flexible multibody analysis. Dropping the nonlinear term led to what was known as the 'spin-up' problem, an erroneous dynamic softening of the rotor beam during spin up which led to unbounded tip deflection. This was because the mutlibody analyses were made for general purpose structures which could not incorporate specific linearization methods depending on the topology of the problem. The problem was subsequently rectified by adding the necessary corrections termed 'geometric stiffness due to operating loads' [10]. Thus, to summarize, if axial dynamics is neglected in the analysis, eqn. 2.20 must be used. If axial dynamics is included, eqn. 2.27 must be used.

2.2.5 Axial Elongation as a Quasi-coordinate

The axial elongation u_e presented above is called a quasi-coordinate. A coordinate is called a quasi-coordinate when it is related to physical displacements and angles through integrals that cannot be evaluated in closed form. The integrals cannot be evaluated because they involve velocities or angular velocities (or their kinetic analogues in terms of gradients and curvature) of the physical displacements. For example, the quasi-coordinate u_e is related to the axial displacement u in the following manner

$$u = u_e - \frac{1}{2}\int_0^r w'^2 dr + O(\epsilon^4)$$

Similarly, in the presence of both flap and lag bending deflections, we have

$$u = u_e - \frac{1}{2}\int_0^r \left(v'^2 + w'^2\right) dr + O(\epsilon^4)$$

where w and v are the flap and lag deflections.

Torsion dynamics can also be formulated in terms of a quasi-coordinate (we shall see later). In this case we have

$$\hat{\phi} = \phi - \frac{1}{2}\int_0^r v'' w' dr + O(\epsilon^2)$$

where $\hat{\phi}$ is an angle defining the rotation of a section, and ϕ is the elastic torsion of the section. The torsion moment is given by $GJ\phi'$. ϕ is a quasi-coordinate. Note how in the case of large deformations the rotation of a section depends not only on the elastic torsion but also on the flap and lag bending deflections.

2.2.6 Boundary Conditions

Consider the pure bending equation 2.20. It is fourth order in r, and requires four boundary conditions, two on either edge. The order-0 and order-1 boundary conditions are called the Dirichlet conditions. They are the geometric boundary conditions imposed on the displacement and slope at the boundary. The order-2 and order-3 boundary conditions are the Neumann conditions. They are the force boundary conditions imposed on the bending moment and shear force at the boundary. Figure 2.11 shows some of the important boundary conditions at an edge. They can be combined to provide more general conditions.

Figure 2.11(a) shows a cantilevered or fixed end condition where the deformation and slope are zero. Figure 2.11(b) shows a simple supported or pin ended condition where the deformation and bending moment are zero. Figure 2.11(c) shows a shear spring restrained condition. Here the bending moment is zero. The shear force is as restrained by the linear spring. For a positive deformation (upwards), the shear force is downwards on the left edge, hence the negative sign as per convention. Note that for a spring on the other end of the beam the boundary condition would be $(EIw'')' = +k_1 w$. Figure 2.11(d) shows a bending spring restrained boundary condition. Here the shear force is zero. The bending moment is as restrained by the rotary spring. Again, for a positive deformation (counter-clockwise rotation) the bending moment is clockwise, and on the left face, hence positive as per convention. Note that for a spring on the other end of the beam the boundary condition would be $EIw'' = -k_2 w$.

The beam equation is second order in time, hence requires two initial conditions to obtain response solution. At time = 0, the velocity $\dot{w}(r)$ and displacement w are prescribed.

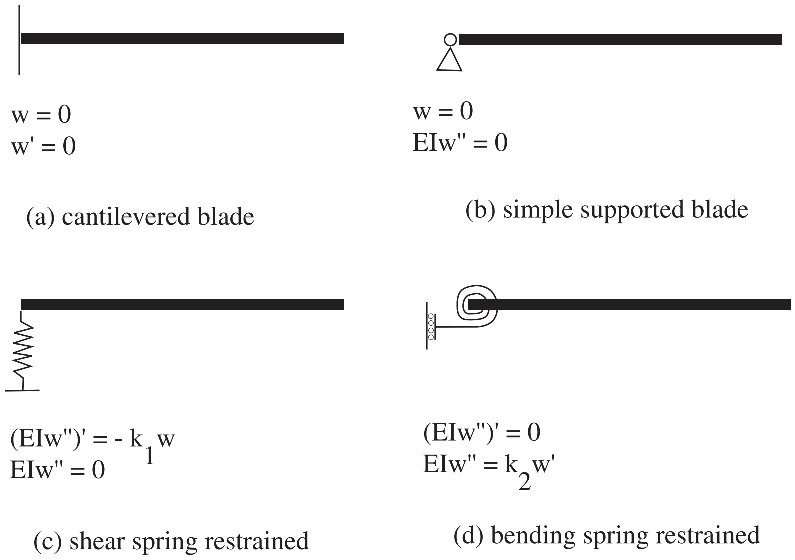


Figure 2.11: **Boundary conditions of a beam**

2.3 Non-rotating beam vibration

To understand beam vibration, we begin without any axial force and without rotation. See Fig. 2.12. Assuming the beam to be uniform and with no axial force, the partial differential equation **??**

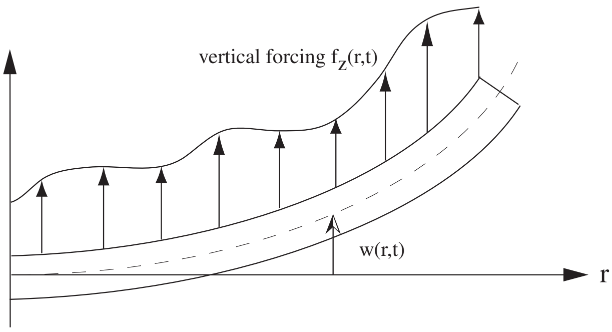


Figure 2.12: **Non-rotating beam in pure bending**

becomes

$$m\ddot{w} + (EIw'')'' = f_z(r,t) \tag{2.28}$$

If the beam is initially disturbed and then left to vibrate on its own, it follows what is called the natural vibration characteristics. During this time there is no forcing on the beam. Thus, to determine the natural vibration characteristics, one needs to calculate only the homogeneous solution by setting $f_z(r,t) = 0$.

$$m\ddot{w} + (EIw'')'' = 0 \tag{2.29}$$

The solution is of the following form.

$$w(r,t) = \phi(r)q(t)$$

One out of the two functions $\phi(r)$, or $q(t)$, can be assigned the dimension of $w(r,t)$, the other remains nondimensional. Here, we assume $q(t)$ to have the same dimension as $w(r,t)$, with units of displacement (m or in) in this case. $\phi(r)$ is considered nondimensional. Assuming that there is no damping in the structure, we seek $q(t)$ of the form

$$q(t) = q_0 e^{i\omega t}$$

In presence of damping, we seek $q(t)$ of the form

$$q(t) = q_0 e^{\alpha + i\omega t} = q_0 e^{st}$$

To obtain natural vibration characteristics we assume there is no damping. We seek such solutions because a linear combination of such solutions can be used to construct any continuous function of time. Substituting the solution type in the homogenous equation yields

$$\left[\phi^{IV} - \frac{m\omega^2}{EI}\phi\right] q_0 e^{i\omega t} = 0$$

For a non-trivial $q(t)$, i.e. a non-zero q_0, we must have

$$\phi^{IV} - \omega^2 \frac{m}{EI}\phi = 0 \tag{2.30}$$

which is a fourth order ordinary differential equation (ODE). This equation has an analytical solution. Assume

$$\phi(r) = Ce^{pr}$$

where C is a nondimensional constant and p has dimension of 1/length. Solve for p to obtain

$$p^4 = \frac{m\omega^2}{EI}$$

This gives four roots $p = \pm\lambda, \pm i\lambda$ where

$$\lambda = \left(\frac{m\omega^2}{EI}\right)^{1/4} \qquad \text{units: 1/m or 1/in} \tag{2.31}$$

Therefore $\phi(r)$ becomes

$$\phi(r) = C_1 \sinh \lambda r + C_2 \cosh \lambda r + C_3 \sin \lambda r + C_4 \cos \lambda r \tag{2.32}$$

The constants C_1, C_2, C_3 and C_4 are evaluated using the four boundary conditions specific to the beam. The constants have units of m or in.

2.3.1 Cantilevered Beam

For a cantivelered beam, the boundary conditions are as follows. At $r = 0$, displacement and slopes are zero at all times. Thus

$$\begin{aligned} w(0,t) = 0 &\implies \phi(0)q(t) = 0 \implies \phi(0) = 0 \\ w'(0,t) = 0 &\implies \phi'(0)q(t) = 0 \implies \phi'(0) = 0 \end{aligned} \tag{2.33}$$

At $r = R$, the bending moment and shear forces are zero at all times. Thus

$$\begin{aligned} EIw''(0,t) = 0 &\implies \phi''(0)q(t) = 0 \implies \phi(0) = 0 \\ EIw'''(0,t) = 0 &\implies \phi'''(0)q(t) = 0 \implies \phi'''(0) = 0 \end{aligned} \tag{2.34}$$

Substituting the four conditions in eqn. 2.32

$$\begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ \sinh\lambda R & \cosh\lambda R & -\sin\lambda R & -\cos\lambda R \\ \cosh\lambda R & \sinh\lambda R & -\cos\lambda R & -\sin\lambda R \end{bmatrix} \begin{bmatrix} C_1 \\ C_2 \\ C_3 \\ C_4 \end{bmatrix} = 0 \tag{2.35}$$

For non-trivial solution, the determinant of square matrix must be zero. This results in

$$\cos\lambda R \cosh\lambda R = -1$$

or

$$\cos\lambda R = -1/\cosh\lambda R$$

which is a transcendental equation. The solutions are obtained by plotting the left hand side and right hand side components individually and mark the points where these two graphs cross each other. See Fig. 2.13. The solution is given by

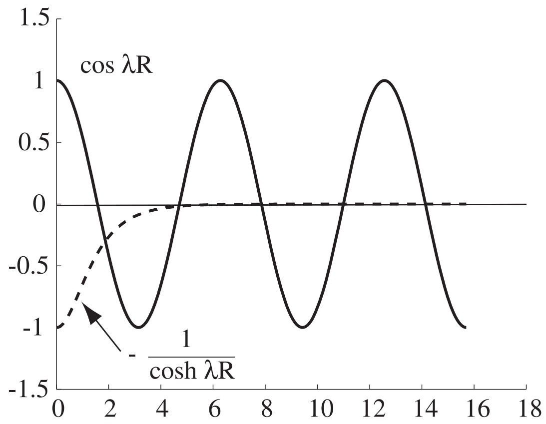


Figure 2.13: **Solution of transcendental equation for cantilevered beam natural frequencies**

$$\begin{aligned} (\lambda R)_1 &= 1.87 \\ (\lambda R)_2 &= 4.69 \\ (\lambda R)_3 &= 7.85 \\ (\lambda R)_j &\cong (2j-1)\frac{\pi}{2} \qquad \text{for j} > 3 \end{aligned} \tag{2.36}$$

The natural frequencies can now be easily calculated from equation 2.31, which can be reorganized as follows

$$\omega_j = (\lambda R)_j^2 \sqrt{\frac{EI}{mR^4}} = f_j \sqrt{\frac{EI}{mR^4}} \tag{2.37}$$

f_j in equation 2.37 represent an infinite set of eigenvalues which produce an infinite number of natural frequencies of the beam. The mode shape corresponding to any particular natural frequency is obtained by solving for C_1, C_2, C_3, and C_4 from equation 2.35. Note that, for an equation of this form, one cannot solve for C_1, C_2, C_4, and C_5. Any three can be solved for in terms of the fourth, for example solve for C_1, C_2, and C_3 in terms of C_4.

$$\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ \sinh \lambda R & \cosh \lambda R & -\cos \lambda R \end{bmatrix} \begin{bmatrix} C_1 \\ C_2 \\ C_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ -\cos \lambda R \end{bmatrix} C_4$$

Set $C_4 = 1$. The mode shape from equation 2.32 then becomes

$$\begin{aligned} \phi_j(r) &= \cosh \lambda_j r - \cos \lambda_j r - \frac{\cosh(\lambda R)_j + \cos(\lambda R)_j}{\sinh(\lambda R)_j + \sin(\lambda R)_j}(\sinh \lambda_j r - \sin \lambda_j r) \\ \phi_j(x) &= \cosh f_j x - \cos f_j x - \frac{\cosh f_j + \cos f_j}{\sinh f_j + \sin f_j}(\sinh f_j x - \sin f_j x) \\ f_j &= (\lambda R)_j \qquad x = r/R \end{aligned} \tag{2.38}$$

Corresponding to each λR, we have a different $\phi(r)$. These mode shapes are plotted in Fig. 2.14. These are the free vibration modes or natural modes. Note that the magnitude of the mode shapes are not unique. They depend on the value of C_4 chosen. Often it is chosen such that $\phi_j(R) = 1$. Each homogeneous solution is of the general form

$$w_j(r,t) = c_j \phi_j(r) e^{i(\omega_j t + d_j)}$$

where c_j and d_j are constants determined by the the initial displacement and velocity of each mode. If an initial velocity or displacement is applied on the beam, all modes get excited. The total response of the beam is the sum of all the modal responses

$$w(r,t) = \sum_{j=1}^{j=\infty} w_j(r,t)$$

However, most of the contribution comes from the first three or four modes.

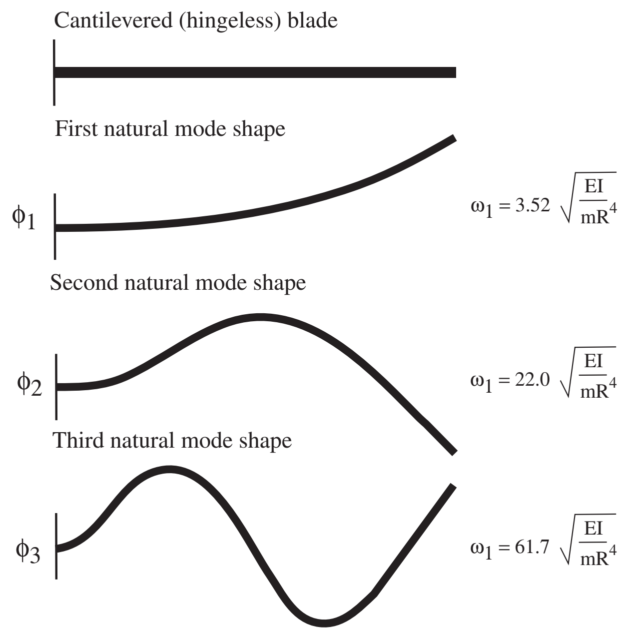


Figure 2.14: **Cantilevered beam natural frequencies and mode shapes**

2.3.2 Simple-Supported Beam

For a simple supported beam, the boundary conditions are as follows. Both at $r = 0$ and at $r = R$, displacement and bending moments are zero at all times. Thus

$$\begin{aligned} &w(0,t) = 0 \implies \phi(0) = 0 \\ &EIw''(0,t) = 0 \implies \phi''(0) = 0 \end{aligned} \tag{2.39}$$

$$\begin{aligned} &w(R,t) = 0 \implies \phi(R) = 0 \\ &EIw''(R,t) = 0 \implies \phi''(R) = 0 \end{aligned} \tag{2.40}$$

Substituting the four conditions in eqn. 2.32 and proceeding similarly as earlier, one obtains the eigenvalues of λR, natural frequencies, and mode shapes as follows

$$\begin{aligned} &(\lambda R)_j = j\pi \\ &\omega_j = (\lambda R)_j^2 \sqrt{\frac{EI}{mR^4}} \\ &\phi_j(r) = \sqrt{2}\ \sin \frac{r(\lambda R)_j}{R} \end{aligned} \tag{2.41}$$

2.3.3 Beam Functions

The nonrotating natural vibration characteristics are available for uniform beams with different boundary conditions. Felgar and Young (1952) tabulated the numerical values for the constants.

$$\begin{aligned} &\phi_j(x) = \cosh f_j x - \cos f_j x - \alpha_j(\sinh f_j x - \sin f_j x) \\ &x = r/R \end{aligned} \tag{2.42}$$

For cantilever beam

j	1	2	3	4	j > 4
f_j	1.8751	4.6941	7.8548	10.9955	$(2j-1)\frac{\pi}{2}$
α_j	.7341	1.0185	.9992	1.0000	1.0

The nonrotating natural frequency for uniform beam is obtained as

$$\omega_j = (f_j)^2 \sqrt{\frac{EI}{mR^4}} \tag{2.43}$$

An important property of these modes is that these are orthogonal

$$\begin{aligned} \int_0^R m\phi_i(r)\phi_j(r)\, dr &= 0 \quad i \neq j \\ &= \delta_{ij} M_i \end{aligned} \tag{2.44}$$

and

$$\begin{aligned} \int_0^R \phi_i \frac{d^2}{dr^2}\left(EI\frac{d^2\phi_j}{dr^2}\right) dr &= 0 \quad i \neq j \\ &= \delta_{ij}\omega_i^2 M_i \end{aligned} \tag{2.45}$$

where δ_{ij} is Kronecker's delta and M_i is generalized mass

$$M_i = \int_0^R m\phi_i^2\, dr$$

The beam functions can be of great value for calculating an approximate solution, by assuming the deflection in terms of a series of beam functions. Because of the orthogality properties, these functions generally result into good convergence characteristics.

2.4 Rotating Beam Vibration

A rotating beam, unlike a stationary beam, can generate aerodynamic forces which affects its response to an initial disturbance. By natural vibration characteristics of a rotating beam we mean characteristics obtained in a vaccuum. This is obtained from the homogeneous solution of the PDE given in eqn. **??**.

$$\begin{aligned} &m\ddot{w} + \frac{\partial^2}{\partial r^2}\left(EI_{\eta\eta}\frac{\partial^2 w}{\partial r^2}\right) - \frac{\partial}{\partial r}\left(T\frac{\partial w}{\partial r}\right) = 0 \\ &T(r) = \int_r^R mr\Omega^2 dr \end{aligned} \tag{2.46}$$

As in the case of a non-rotating beam we seek a solution of the form

$$w(r,t) = \phi(r)q(t) = \phi(r)q_0 e^{i\omega t}$$

where $\phi(r)$ is assumed to be nondimensional. $q(t)$ is assumed to have the same dimension as $w(r,t)$, here that of length, m or in. Substitution in the homogenous equation leads to the following ODE, for non-trivial $q(t)$,

$$\frac{\partial^2}{\partial r^2}\left(EI_{\eta\eta}\frac{\partial^2 \phi}{\partial r^2}\right) - \frac{\partial}{\partial r}\left(\int_r^R m\Omega^2 r\, dr\frac{\partial \phi}{dr}\right) - m\omega^2\phi = 0 \tag{2.47}$$

Unlike the non-rotating case, the above ODE cannot be solved analytically even for a beam with uniform properties. Approximate solution methods must be employed. Before we discuss approximate methods, it is instructive to re-write the governing eqn. **??** in a nondimensional form. To this end substitute

$$x = \frac{r}{R}, \qquad \overline{w} = \frac{w}{R}, \qquad \psi = \Omega t$$

The PDE then becomes

$$m\Omega^2 R \overset{**}{\overline{w}} + \frac{\partial^2}{\partial x^2}\left(\frac{EI_{\eta\eta}}{R^3}\frac{\partial^2 \overline{w}}{\partial x^2}\right) - \frac{\partial}{\partial x}\left(\frac{T}{R}\frac{\partial \overline{w}}{\partial x}\right) = f_z(x\,R,t) \tag{2.48}$$

where $\overset{**}{\overline{w}} = \partial^2\overline{w}/\partial\psi^2$. Divide by $m_0\Omega^2 R$ throughout, where m_0 is a reference mass per unit span, to obtain

$$\begin{aligned} &\frac{m}{m_0}\overset{**}{\overline{w}} + \left(\frac{EI_{\eta\eta}}{m_0\Omega^2 R^4}\overline{w}''\right)'' - \left(\frac{T}{m_0\Omega^2 R^2}\overline{w}'\right)' = \frac{f_z(x\,R,t)}{m_0\Omega^2 R} \\ &\frac{T}{m_0\Omega^2 R^2} = \int_x^1 \frac{m}{m_0} x dx \end{aligned} \tag{2.49}$$

where $(\ )' = \partial/\partial x$. To specify the reference mass per unit span, it is convenient to use an imaginary uniform beam which has the same flap inertia about the root as the real beam

$$I_b = \frac{m_0 R^3}{3} \qquad \text{or} \qquad m_0 = \frac{3I_b}{R}$$

Note than for an uniform beam $m/m_0 = 1$. The homogenous equation corresponding to eqn. 2.46 is then

$$\frac{m}{m_0}\overset{**}{\overline{w}} + \left(\frac{EI_{\eta\eta}}{m_0\Omega^2 R^4}\overline{w}''\right)'' - \left(\overline{w}'\int_x^1 \frac{m}{m_0} x dx\right)' = 0 \tag{2.50}$$

where $EI/m_0\Omega^2R^4$ is the nondimensional flexural stiffness. Proceeding similarly as before, assume a solution of the form

$$\overline{w}_a(x,\psi) = \phi_j(x)\overline{q}_j(\psi) = \phi_j(x)\overline{q}_{0j}e^{j\nu\psi} \tag{2.51}$$

where $\phi_j(x)$ are the same nondimensional shape functions as before, expressed now as a function of $x = r/R$, $\overline{q}(t)$ is now nondimensional $\overline{q}(t) = q(t)/R$, and ν is the nondimensional frequency in per rev $\nu = \omega/\Omega$. The nondimensional modal equation, corresponding to eqn. 2.47, then takes the following form

$$\frac{d^2}{dx^2}\left(\frac{EI_{\eta\eta}}{m_0\Omega^2R^4}\frac{d^2\phi}{dx^2}\right) - \frac{d}{dx}\left(\int_x^1 \frac{m}{m_0}x\,dx\,\frac{d\phi}{dx}\right) - \frac{m}{m_0}\nu^2\phi = 0 \tag{2.52}$$

The nondimensional mass and stiffness distributions determine the nondimensional frequency ν (in /rev). For the non-rotating case, $\Omega = 0$ the entire equation is multiplied by $m_0\Omega^2R^4$ to prevent division by zero. The equation then takes the following form

$$\frac{d^2}{dx^2}\left(EI_{\eta\eta}\frac{d^2\phi}{dx^2}\right) - m_0\Omega^2R^4\frac{d}{dx}\left(\int_x^1 \frac{m}{m_0}x\,dx\,\frac{d\phi}{dx}\right) - mR^4\omega^2\phi = 0 \tag{2.53}$$

which is the appropriate form for solving a non-rotating case. Note that $\Omega = 0$ gives back eqn. 2.30 with the substitution of $x = r/R$.

2.4.1 Approximate solution Methods

Two types of approximate methods are described: (1) a weighted residual method, and (2) a variational method. An example of each is discussed. An example of the first type is the Galerkin method. An example of the second type is the Rayleigh-Ritz method. The weighted residual methods are used to solve the governing PDEs. The governing PDEs, as we saw before are derived based on the principles of Newtonian mechanics (force and moment equilibrium). The weighted residual methods reduces the governing PDEs to a set of ODEs. The variational methods bypass the governing PDEs and formulates the ODEs directly. They are based on the principles of Analytical mechanics, a branch distinct and independant from Newtonian mechanics. The only variational principle in analytical mechanics is the principle of least action. Methods which formulate the governing ODEs using this principle are called variational methods. Variational methods are also called energy methods.

All modern structural dynamic analyses use a discretization technique called the Finite Element Method (FEM). It is a method of discretizing a system, while using any type of an approximate method of solution, that renders the formulation more suitable for numerical analysis. Thus one can have a FEM of Galerkin type, FEM of Rayleigh-Ritz type, etc. FEM is discussed as a seperate section later on. In this section, two approximate methods are described, one of each type.

2.4.2 Galerkin Method

This method is based on the equilibrium equation. The deflection is expressed as the sum of a series of assumed functions each of which satisfies all the boundary conditions. If we substitute this approximate deflection into the equilibrium equation it will result in an error, or a residual. There are many ways to reduce this residual. In the weighted residual method, it is reduced by projecting it orthogonally onto a subspace spanned by a chosen set of weighing functions. When the chosen set of weighing functions are same as the assumed functions the method is called the Galerkin method. The process is similar to the error orthogonalization of polynomials. Consider the equilibrium equation for the rotating beam blade, eqn. **??**

$$m\ddot{w} + (EI_{\eta\eta}w'')'' - (Tw')' = f_z(r,t) \tag{2.54}$$

Assume an approximate deflection $w_a(r,t)$ of the form

$$w_a(r,t) = \sum_{j=1}^{n} \phi_j(r) q_j(t) \tag{2.55}$$

where $\phi_i(r)$, assumed deflection shape, must satisfy both geometric and force boundary conditions. If it would have been an exact solution, then its substitution in the governing partial differential equation would be identically satisfied. Being an approximate solution, it will result into error

$$\epsilon(r,t) = \sum_{j=1}^{n} \{ m\phi_j \ddot{q}_j + (EI\phi_j'')'' q_j - (T\phi_j')' q_j - f_z \}$$

To determine q_i that will reduce this error, the error is projected orthogonally onto a subspace spanned by a set of weighing functions. In the Galerkin method the assumed deflection shapes are chosen as the weighing functions.

$$\text{i.e.} \quad \int_0^R \phi_i \epsilon(r,t)\, dr = 0 \qquad i = 1, 2, \ldots, n$$

or

$$\sum_{j=1}^{n} \left\{ \left(\int_0^R \phi_i m \phi_j \, dr \right) \ddot{q}_j + \left(\int_0^R \phi_i (EI\phi_j'')'' dr - \int_0^R \phi_i (T\phi_j')' dr \right) q_j - \left(\int_0^R \phi_i f_z dr \right) \right\} = 0$$

or,

$$\sum_{j=1}^{n} \{ m_{ij} \ddot{q}_i + k_{ij} q_i \} = Q_j \qquad i = 1, 2, \ldots, N$$

where

$$m_{ij} = \int_0^R m\phi_i \phi_j dr$$

$$k_{ij} = \int_0^R \phi_i (EI\phi_j'')'' dr - \int_0^R \phi_i (T\phi_j')' \, dr$$

$$Q_i = \int_0^R \phi_i f_z(r,t) dr$$

In matrix notation

$$M\underset{\sim}{\ddot{q}} + K\underset{\sim}{q} = \underset{\sim}{Q} \tag{2.56}$$

where M and K are mass and stiffness matrices of size $(n \times n)$. $\underset{\sim}{Q}$ is the forcing vector of size n, and $\underset{\sim}{q}$ are the degrees of freedom. The degrees of freedom $\underset{\sim}{q}$ have units of radians. M has units of kg. K has units of N/m. Q has units of N. For natural response, set $\underset{\sim}{Q} = \underset{\sim}{0}$ and seek solution of the form $\underset{\sim}{q} = \underset{\sim}{q_0} e^{j\omega t}$. This leads to

$$\begin{aligned} K\underset{\sim}{q_0} &= \omega^2 M \underset{\sim}{q_0} \\ \left(K - \omega^2 M\right) \underset{\sim}{q_0} &= 0 \end{aligned} \tag{2.57}$$

The above is an algebraic eigenvalue problem. It means that there are only n values of ω^2 for which a non-trivial solution of $\underset{\sim}{q_0}$ exists, for all other values of ω^2, $\underset{\sim}{q_0} = \underset{\sim}{0}$. For non-trivial solution,

$$|K - \omega^2 M| = 0$$

which leads to the solutions ω_i where $i = 1, 2, \ldots, n$. Corresponding to each ω_i there exists a solution $\underset{\sim}{q_{0i}}$ which satisfies the equation

$$K\underset{\sim}{q_{0i}} = \omega_i^2 M \underset{\sim}{q_{0i}}$$

ω_i and its corresponding $\underset{\sim}{q_{0i}}$ are called the eigenvalues and eigenvectors of the system.

As an example, consider a case where the assumed deflection functions are the non-rotating beam functions as given in eqn. 2.42.

$$w_a(r,t) = \sum_{j=1}^{n} \phi_j(r)\, q_j(t) \tag{2.58}$$

where ϕ_j is the mode shape for a non-rotating beam corresponding to its j^{th} natural frequency of vibration. Using the orthogonality property of these functions (eqns. 2.44 and 2.45), we have

$$\begin{aligned} m_{ij} &= \delta_{ij} M_i \\ k_{ij} &= \delta_{ij}\omega_{0_i}^2 M_i - \int_o^R \phi_j (T\phi_i')'\, dr \end{aligned} \tag{2.59}$$

where

$$M_i = \int_0^R m\phi_i^2\, dr \qquad T = \int_r^R m\Omega^2 r dr = (1/2)m\Omega^2(R^2 - r^2) \quad \text{for an uniform beam}$$

$$\omega_{0i} = i^{th} \text{ non-rotating natural frequency}$$

Recall, that the governing PDE eqn. 2.54 can also be expressed in the following non-dimensional form (from eqn. 2.49)

$$\begin{aligned} \frac{m}{m_0}\overset{**}{\overline{w}} + \left(\frac{EI_{\eta\eta}}{m_0\Omega^2 R^4}\overline{w}''\right)'' - \left(\frac{T}{m_0\Omega^2 R^2}\overline{w}'\right)' &= \frac{f_z(x\,R,t)}{m_0\Omega^2 R} \\ \frac{T}{m_0\Omega^2 R^2} &= \int_x^1 \frac{m}{m_0} x dx \end{aligned} \tag{2.60}$$

An assumed solution of the form

$$\overline{w}_a(x,\psi) = \sum_{i=1}^{n} \phi_j(x)\overline{q}_j(\psi) = \sum_{i=1}^{n} \phi_j(x)\overline{q}_{0j} e^{j\nu\psi}$$

where $\overline{w} = w/R$ and $\overline{q}(t) = q(t)/R$ leads to the eigenvalue problem

$$\begin{aligned} \overline{K}\underset{\sim}{\overline{q}_0} &= \nu^2 \overline{M}\underset{\sim}{\overline{q}_0} \\ \left(\overline{K} - \nu^2\overline{M}\right)\underset{\sim}{\overline{q}_0} &= 0 \end{aligned} \tag{2.61}$$

which is the nondimensional form of the eigenvalue eqn. 2.57, where $\overline{K}$ and $\overline{M}$ are the nondimensional stiffness and mass matrices given by

$$\begin{aligned} \overline{m}_{ij} &= \int_0^1 \frac{m}{m_0}\phi_i\phi_i \, dx \\ \overline{k}_{ij} &= \int_0^1 \phi_j \frac{d^2}{dx^2}\left(\frac{EI}{m_0\Omega^2R^4}\frac{d^2\phi_i}{dx^2}\right) dx - \int_0^1 \phi_j \frac{d}{dx}\left(\frac{T}{m_0\Omega^2R^2}\frac{d\phi_i}{dx}\right) dx \end{aligned}$$

The procedure is simple, but it is difficult to choose functions which satisfies all the boundary conditions. The Galerkin method overestimates the natural frequencies, the error progressively increasing for higher mode frequencies. For accurate prediction of higher mode frequencies a large number of modes must be assumed.

Example: 2.2

For a hingeless uniform rotating blade, calculate approximately the fundamental flap bending frequency using the Galerkin method. Assume a one term deflection series

$$w(r,t) = \left[3\left(\frac{r}{R}\right)^2 - 2\left(\frac{r}{R}\right)^3 + \frac{1}{2}\left(\frac{r}{R}\right)^4\right] q_1(t)$$

First note that

$$\phi_1(r) = 3\left(\frac{r}{R}\right)^2 - 2\left(\frac{r}{R}\right)^3 + \frac{1}{2}\left(\frac{r}{R}\right)^4 = 3x^2 - 2x^3 + \frac{1}{2}x^4$$

Cantilever boundary conditions

$$r = 0 \qquad w = \phi_1 = 0 \qquad \frac{dw}{dr} = \frac{1}{R}\frac{d\phi_1}{dx} = 0$$

$$r = R \qquad M = 0 \text{ or } \frac{1}{R^2}\frac{d^2\phi_1}{dx^2} = 0 \qquad S = 0 \text{ or } \frac{1}{R^3}\frac{d^3\phi_1}{dx^3} = 0$$

All boundary conditions are satisfied. Therefore, the assumed displacement function can be used to calculate Galerkin solution.

$$m_{11} = \int_0^R m\phi_1^2 dr = mR\int_0^1 \left(3x^2 - 2x^3 + \frac{1}{2}x^4\right)^2 dx = \frac{26}{45}mR$$

$$T = \frac{m}{2}\Omega^2(R^2 - r^2) = \frac{m}{2}\Omega^2R^2(1 - x^2)$$

$$\begin{aligned} k_{11} &= \int_0^R EI\phi_1\frac{d^4\phi_1}{dr^4}dr - \int_0^R T\phi_1\frac{d^2\phi_1}{dr^2}dr \\ &= \frac{EI}{R^3}\int_0^R 12\left(3x^2 - 2x^3 + \frac{1}{2}x^4\right) dx \\ &\quad -\frac{1}{2}m\Omega^2R\int_0^R (1-x^2)\left(x - 6x^2 + 2x^3\right)\left(3x^2 - 2x^3 + \frac{1}{2}x^4\right) dx \\ &= \frac{36}{5}\frac{EI}{R^3} + \frac{61}{90}m\Omega^2R \end{aligned}$$

$$\omega_1^2 = \frac{k_{11}}{m_{11}} = 12.46\frac{EI}{mR^4} + 1.173\Omega^2$$

$$\nu_1^2 = 12.46\frac{EI}{m\Omega^2R^4} + 1.173$$

$$\text{For } \Omega = 0 \quad \omega_1 = 3.53\sqrt{\frac{EI}{mR^4}} \quad \text{Exact soln: } 3.52\sqrt{\frac{EI}{mR^4}}$$

In this case, the assumed one term approximation was adequate for the fundamental mode.

2.4.3 Rayleigh-Ritz Method

Assume the deflection as a series of functions that need to satisfy only geometric boundary conditions. Using this deflection, calculate the kinetic energy, potential energy and virtual work. Substitute in the Euler-Lagrange equation of motion. This generates the governing ODEs directly.

The Euler-Lagrange equation of motion is a statement of the principle of Least Action in a differential form. The principle of Least Action, or more correctly the principle of stationary action, requires that the degrees of freedom $[q_1 q_2 q_3 \dots q_n]^T$ must evolve with time in such a manner so as to render a certain integral I stationary with respect to arbitrary variations in the manner of their evolution. The evolution of the degrees of freedom with time can be thought of as a path of motion of the system from a certain time t_1 to time t_2. The integral I is called the action integral. For mechanical systems, an appropriate action integral I is the integration of the kinetic energy U of the system, and work done W on the system, over the time t_1 to t_2, calculated along the path of motion. The action integral in this form was provided by Hamilton. Therefore, this formulation of the principle of Least Action for mechanical systems is known as Hamilton's Principle. Thus, Hamilton's Principle states that out of all possible paths of motion of a mechanical system between t_1 and t_2, the true motion is such that the integral $I = \int_{t_1}^{t_2} (U + W) dt$ attains an extremum, given that the end points of all possible paths at t_1 and t_2 are the same. Hamilton's Principle is a statement of the principle of Least Action, in an integral form. The differential form, as noted earlier, is the Euler-Lagrange equation. The connection between Hamilton's Principle and the Euler-Lagrange equation is given in the next section. Here, the Euler-Lagrange equation is used to provide an approximate solution of a rotating beam. Assume

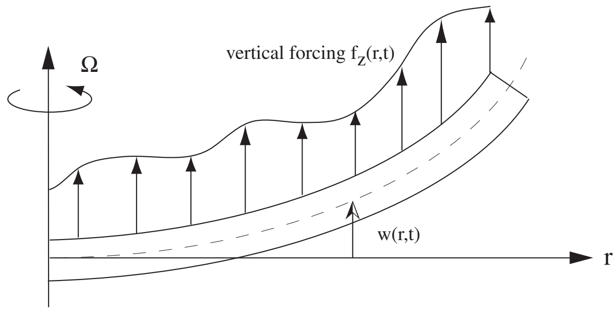


Figure 2.15: **Rotating cantilevered beam**

$$w_a(r,t) = \sum_{j=1}^{n} \phi_j(r)\, q_j(t) \tag{2.62}$$

where $\phi_j(r)$ need to satisfy only the geometric boundary conditions. For example for a cantilevered beam, it is enough that $\phi_j(0) = 0$ and $\phi_j'(0) = 0$. Now calculate the kinetic energy, potential energy and virtual work associated with the deflection of the beam. The kinetic energy U is given by

$$U = \frac{1}{2}\int_0^R m\dot{w}^2 dr = \frac{1}{2}\int_0^R m \left(\sum_{i=1}^{n} \phi_i(r)\dot{q}_i\right)\left(\sum_{j=1}^{n} \phi_j(r)\dot{q}_j\right) dr = \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} m_{ij}\dot{q}_i\dot{q}_j \tag{2.63}$$

where

$$m_{ij} = \int_0^R m\phi_i\phi_j dr \tag{2.64}$$

The potential energy or strain energy V is given by

$$
\begin{aligned}
V &= \text{strain energy due to flexure} + \text{strain energy due to centrifugal force} \\
&= \frac{1}{2}\int_0^R EI\left(\frac{d^2 w_a}{dr^2}\right)^2 dr + \frac{1}{2}\int_0^R T\left(\frac{dw_a}{dr}\right)^2 dr \\
&= \frac{1}{2}\int_0^R EI\left(\sum_{i=1}^{n}\phi_i'' q_i\right)\left(\sum_{j=1}^{N}\phi_j'' q_j\right) dr + \frac{1}{2}\int_0^R T\left(\sum_{i=1}^{n}\phi_i' q_i\right)\left(\sum_{j=1}^{n}\phi_j' q_j\right) dr \qquad (2.65)\\
&= \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} k_{ij} q_i q_j
\end{aligned}
$$

where

$$k_{ij} = \int_0^R EI\phi_i''\phi_j''\, dr + \int_0^R T\phi_i'\phi_j'\, dr \qquad (2.66)$$

and

$$T = \int_r^R m\Omega^2 r\, dr$$

Virtual work done by $f_z(r,t)$ through virtual displacements δw is given by

$$
\begin{aligned}
\delta W &= \int_0^R f_z(r,t)\delta w\, dr \\
&= \int_0^R f_z \sum_{i=1}^{n}\phi_i\, \delta q_i\, dr \qquad (2.67)\\
&= \sum_{i=1}^{n} Q_i\, \delta q_i
\end{aligned}
$$

where

$$Q_i = \int_0^R f_z\phi_i\, dr \qquad (2.68)$$

Substitute in the Euler-Lagrange equation

$$\frac{\partial}{\partial t}\left(\frac{\partial T}{\partial \dot{q}_i}\right) - \frac{\partial T}{\partial q_i} + \frac{\partial V}{\partial q_i} = Q_i \qquad i = 1, 2, \ldots n \qquad (2.69)$$

to obtain

$$\sum_{j=1}^{n}(m_{ij}\ddot{q}_j + k_{ij}q_j) = Q_i \qquad i = 1, 2, ...n$$

These are n coupled equations and can be put together in matrix form

$$M\underset{\sim}{\ddot{q}} + K\underset{\sim}{q} = \underset{\sim}{Q}$$

where M and K are mass and stiffness matrices of size $(n \times n)$. $\underset{\sim}{Q}$ is the forcing vector of size n, and $\underset{\sim}{q}$ are the degrees of freedom. The degrees of freedom $\underset{\sim}{q}$ have units of displacement w in m or in. M has units of kg^2. K has units of N/m. Q has units of N. For natural response, set $\underset{\sim}{Q} = \underset{\sim}{0}$ and seek solution of the form $\underset{\sim}{q} = \underset{\sim}{q_0} e^{j\omega t}$. This leads to the same algebraic eigenvalue problem as discussed earlier in the case of Galerkin method,

$$K\underset{\sim}{q_0} = \omega^2 M \underset{\sim}{q_0}$$
$$\left(K - \omega^2 M\right)\underset{\sim}{q_0} = 0 \tag{2.70}$$

It leads to the solutions ω_i where $i = 1, 2, \ldots, n$. Corresponding to each ω_i there exists a solution $\underset{\sim}{q_{0i}}$ which satisfies the equation

$$K\underset{\sim}{q_{0i}} = \omega_i^2 M \underset{\sim}{q_{0i}}$$

ω_i and its corresponding $\underset{\sim}{q_{0i}}$ are called the eigenvalues and eigenvectors of the system. For example, consider the simplest case of a one term solution. This is also called a Rayleigh solution.

$$w_a(r,t) = \phi_1(r)\, q_1(t)$$

This results in

$$M = m_{11} = \int_0^R m\phi_1^2\, dr$$
$$K = k_{11} = \int_0^R EI(\phi_1'')^2 dr + \int_0^R T(\phi_1')^2 dr$$

Eqn. 2.88 then leads to

$$\omega^2 = \frac{k_{11}}{m_{11}} = \frac{\int_0^R EI(\phi_1'')^2 dr + \int_0^R T(\phi_1')^2 dr}{\int_0^R m\phi_1^2\, dr}$$

The energy expressions can be nondimensionalized using $m_0\Omega^2 R^3$, with units of N-m. The kinetic energy, potential energy, and virtual work corresponding to eqns. 9.93, 2.65 and 2.67 become

$$\frac{U}{m_0\Omega^2 R^3} = \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} \overline{m}_{ij}\, \overset{*}{\overline{q}}_i \overset{*}{\overline{q}}_j$$
$$\frac{V}{m_0\Omega^2 R^3} = \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} \overline{k}_{ij}\overline{q}_i\overline{q}_j \tag{2.71}$$
$$\frac{\delta W}{m_0\Omega^2 R^3} = \sum_{i=1}^{n} \overline{Q}_i \delta\overline{q}_i$$

where $\overline{q} = q/R$. The nondimensional mass corresponding to eqn. 2.64 is then

$$\overline{m}_{ij} = \frac{\int_0^R m\phi_i\phi_j dr}{m_0 R}$$
$$= \frac{\int_0^1 m\,\phi_i(x)\,\phi_j(x) d(xR)}{m_0 R} \tag{2.72}$$
$$= \int_0^1 \frac{m}{m_0}\phi_i(x)\phi_j(x)dx$$

where $x = r/R$. Similarly, the the nondimensional stiffness corresponding to eqn. 2.66 is

$$\begin{aligned}
\overline{k}_{ij} &= \frac{\int_0^R EI\phi_i''\phi_j''\,dr + \int_0^R T\phi_i'\phi_j'\,dr}{m_0\Omega^2R^3} \\
&= \frac{\int_0^1 EI\frac{d^2\phi_i}{d(xR)^2}\frac{d^2\phi_j}{d(xR)^2}d(xR) + \int_0^1 T\frac{d\phi_i}{d(xR)}\frac{d\phi_j}{d(xR)}d(xR)}{m_0\Omega^2R} \\
&= \int_0^1 \frac{EI}{m_0\Omega^2R^4}\frac{d^2\phi_i}{dx^2}\frac{d^2\phi_j}{dx^2}dx + \int_0^1 \frac{T}{m_0\Omega^2R^2}\frac{d\phi_i}{dx}\frac{d\phi_j}{dx}dx
\end{aligned} \tag{2.73}$$

where

$$\frac{T}{m_0\Omega^2R^2} = \int_0^1 \frac{m}{m_0}xdx$$

The nondimensional force vector corresponding to eqn. 2.68 is

$$\overline{Q}_i = \int_0^1 \frac{f_z}{m_0\Omega^2R}\phi_i\,dx \tag{2.74}$$

The energy and work expressions as given in eqns. 2.71, when substituted in the Euler-Lagrange equation generates

$$\overline{M}\,\overset{**}{\underset{\sim}{\overline{q}}} + \overline{K}\underset{\sim}{\overline{q}} = \underset{\sim}{Q}$$

with the corresponding eigenvalue problem

$$\begin{aligned}
\overline{K}\underset{\sim}{\overline{q}_0} &= \nu^2\overline{M}\underset{\sim}{\overline{q}_0} \\
\left(\overline{K} - \nu^2\overline{M}\right)\underset{\sim}{\overline{q}_0} &= 0
\end{aligned} \tag{2.75}$$

where $\nu = \omega/\Omega$.

Example: 2.3

Calculate Rayleigh's solution for a uniform rotating blade. Assume displacement as

$$w(r,t) = \left(\frac{r}{R}\right)^2 q_i(t)$$

First note that

$$\phi_1(r) = \left(\frac{r}{R}\right)^2 = x^2$$

The geometric boundary conditions are satisfied

$$r = 0 \quad w = \frac{dw}{dr} = 0$$

thus the assumed form is a permissible deflection.

$$m_{11} = \int_0^R m\phi_1^2\,dr = \int_0^R m\frac{r^4}{R^4}dr = \frac{mR}{5}$$

$$T = \frac{m}{2}(R^2 - r^2)\Omega^2$$

$$
\begin{aligned}
k_{11} &= \int_0^R EI(\phi_1'')^2 dr + \int_0^R T(\phi_1')^2 dr \\
&= EI\int_0^R \frac{4}{R^4} dr + \int_0^R \frac{1}{2}m(R^2 - r^2)\Omega^2 \frac{4r^2}{R^4} dr \\
&= \frac{4EI}{R^3} + \frac{4}{15} m\Omega^2 R
\end{aligned}
$$

$$\omega_1^2 = \frac{k_{11}}{m_{11}} = \frac{\frac{4EI}{R^3} + \frac{4}{15}m\Omega^2 R}{\frac{mR}{5}} = 20\frac{EI}{mR^4} + \frac{4}{3}\Omega^2$$

$$\text{For} \quad \Omega = 0 \quad \omega_1 = 4.47\sqrt{\frac{EI}{mR^4}} \quad \text{Exact soln:} \; 3.52\sqrt{\frac{EI}{mR^4}}$$

In this case, the assumed one term approximation was not adequate. Only a very rough estimate of the solution is obtained.

Example: 2.4

A radially tapered solid blade is idealized into two uniform segments of equal lengths, with EI of 0.8 and 0.5 of root value EI_0 and mass distribution m of .9 and .7 of root value m_0. Calculate the fundamental flap bending frequency approximately using the Rayleigh-Ritz method. Assume a one term deflection series

$$w(r,t) = \left(\frac{r}{R}\right)^2 q_1(t)$$

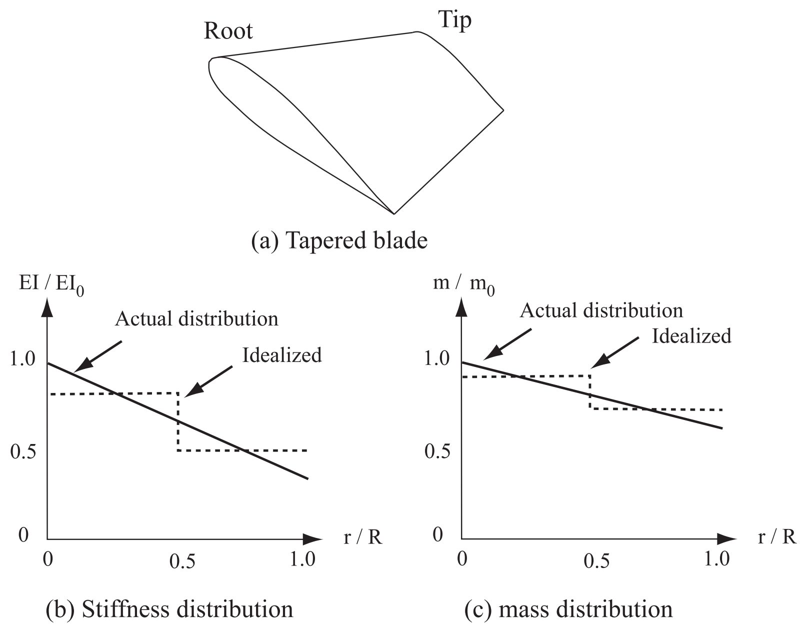


Figure 2.16: **Non-uniform tapered blade natural frequencies and mode shapes**

$$\phi_1 = x^2$$

The geometric boundary conditions are satisfied.

$$r = 0 \quad w = \frac{dw}{dr} = 0$$

The mass is given by

$$\begin{aligned} m_{11} &= \int_0^R m\phi_1^2\, dr \\ &= \int_0^{R/2} m\phi_1^2 dr + \int_{R/2}^R m\phi_1^2 dr \\ &= 0.9 m_0 R \int_0^{1/2} x^4 dx + 0.7 m_0 R \int_{1/2}^1 x^4 dx \\ &= 0.14 m_0 R \end{aligned}$$

The stiffness is given by

$$k_{11} = (k_{11})_{\text{bending}} + (k_{11})_{\text{centrifugal}}$$

$$\begin{aligned} (k_{11})_{\text{bending}} &= \int_0^{R/2} EI_1(\phi_1'')^2 dr + \int_{R/2}^R EI_2(\phi_1'')^2 dr \\ &= 0.8\frac{EI_0}{R^3}\int_0^{1/2} 4\, dx + 0.5\frac{EI_0}{R^3}\int_{1/2}^2 4\, dx \\ &= 2.6\frac{EI_0}{R^3} \end{aligned}$$

The centrifugal stiffness depends on the tensile force T

$$T = \int_r^R m\Omega^2 r\, dr$$

For $\frac{R}{2} < r < R$

$$T = \Omega^2 R^2 \int_x^1 0.7 m_0 x\, dx = 0.35 m_0 \Omega^2 R^2 (1 - x^2) = 0.35 m_0 \Omega^2 (R^2 - r^2)$$

For $r < R/2$

$$\begin{aligned} T &= \Omega^2 R^2 \int_x^{1/2} 0.9 m_0 x\, dx + \Omega^2 R^2 \int_{1/2}^1 0.7 m_0 x\, dx \\ &= m_0 \Omega^2 R^2 (0.375 - 0.45 x^2) = m_0 \Omega^2 (0.375 R^2 - 0.45 r^2) \end{aligned}$$

Therefore

$$\begin{aligned} (k_{11})_{\text{centrifugal}} &= \int_0^R T\,(\phi_1)^2\, dr \\ &= \int_0^{R/2} (m_0\Omega^2)(0.375R^2 - 0.45r^2)\frac{4r^2}{R^4} dr + \int_{R/2}^R 0.35 m_0 \Omega^2 (R^2 - r^2)\frac{4r^2}{R^4} dr \\ &= m_0\Omega^2 R \int_0^{1/2} (0.375 - 0.45x^2) 4x^2 dx + 0.35 m_0 \Omega^2 R \int_{1/2}^1 (1 - x^2) 4x^2 dx \\ &= 0.188 m_0 \Omega^2 R \end{aligned}$$

$$k_{11} = 2.6\frac{EI_0}{R^3} + 0.188 m_0 \Omega^2 R$$

$$\begin{aligned} \omega_1^2 &= \frac{k_{11}}{m_{11}} = 18.41\frac{EI_0}{m_0 R^4} + 1.33\Omega^2 \\ \nu^2 &= 18.41\frac{EI_0}{m_0\Omega^2 R^4} + 1.33 \end{aligned}$$

2.5 Finite Element Method (FEM)

The Finite Element Method (FEM) forms the basis of all modern structural analysis because of its easy adaptibility to different configurations and boundary conditions. The present section introduces a simple displacement based one dimensional FEM. FEM is a discretization technique. Any of the approximate methods, e.g. Galerkin or Rayleigh-Ritz, can be applied in combination with FEM. Thus, one can have a Galerkin type FEM, a Rayleigh-Ritz type FEM, etc. In general, one can have a FEM based on weighted residual methods or a FEM based on energy methods. In this section we will use a Rayleigh-Ritz type FEM to solve the Euler-Bernoulli rotating beam problem.

The first step is to discretize the rotating beam into a number of finite elements as shown in Fig. 2.17. Each element is free of constraints. This is followed by three major steps: (a) Development of elemental properties, (b) Assembly of elemental properties, and (3) Application of constraints and determination of solution. These steps are discussed below.

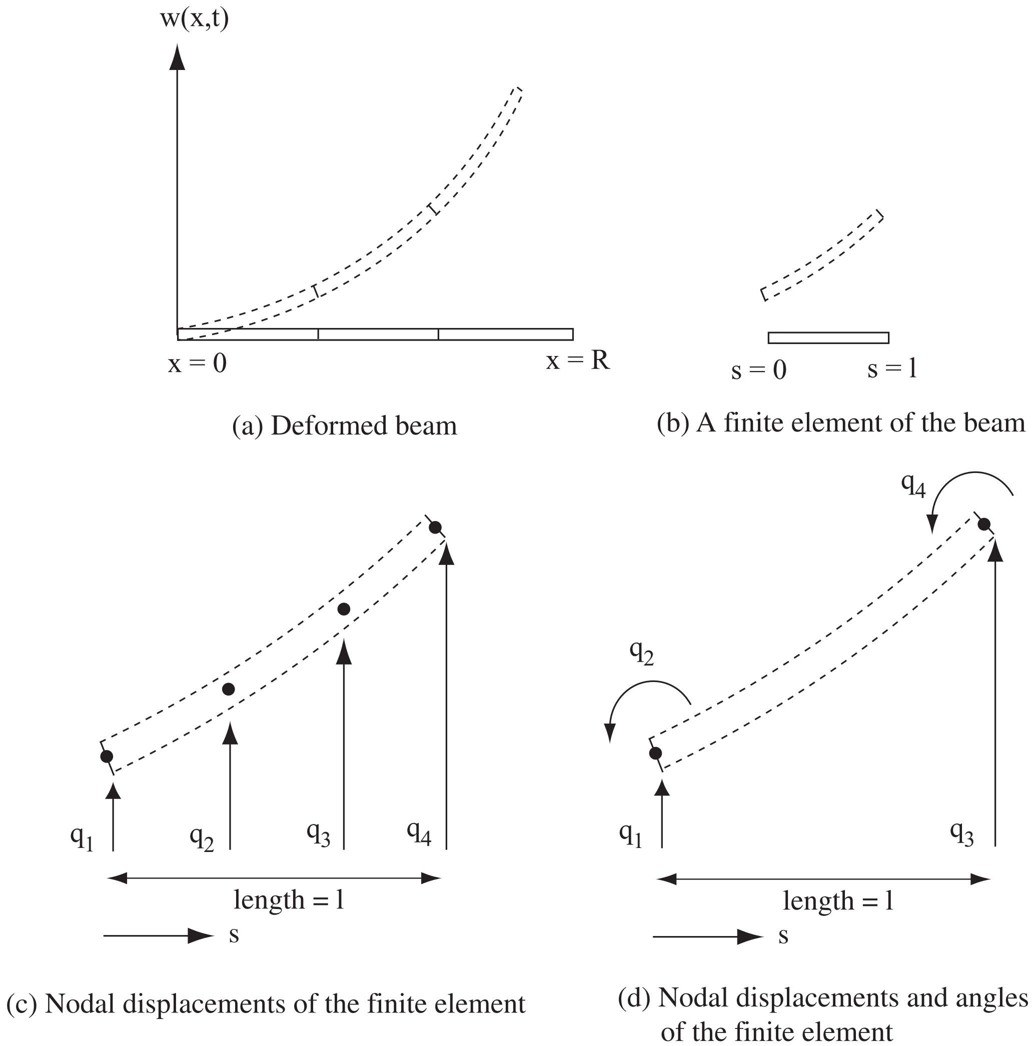


Figure 2.17: **Finite element discretization of a beam**

2.5.1 Element properties

Figure 2.17(a) shows a rotating beam in its undeformed and deformed positions. Figure 2.17(b) shows a finite element of the beam. The beam extends from $x = 0$ to $x = R$. Each element extends from, say, $s = 0$ to $s = l$ where s is a local variable within each element. The goal is to represent the deformation within each element w as a function of s and in the following form.

$$w(s) = \sum_{i=1}^{i=n} H_i(s) q_i(t) \tag{2.76}$$

Here $q_i(t), i = 1, 2, 3, \ldots, n$ are displacements at n 'chosen' points within the element. These points are also called nodes. $H_i(s), i = 1, 2, 3, \ldots, n$ are interpolation functions, also called shape functions, automatically extracted based on this choice. The order of each $H(s)$ depend on the chosen value of n. The type of each $H(s)$ depend on the chosen nature of $q_i(t)$. The above form is generated in the following manner.

First, assume a polynomial distribution for displacement w in the element

$$w(s,t) = \alpha_1 + \alpha_2 s + \alpha_3 s^2 + \alpha_4 s^3 \tag{2.77}$$

A third order polynomial is chosen because anything less will provide zero shear forces. The order must be at least three, i.e. the highest derivative for loads. The order can be greater than three, however, this implies that added number of unknowns need to be determined. For the third order polynomial, as chosen above, we have the unknowns α_{0-3} to be determined. It is here that the choice of $q_i(t)$ plays a role.

To determine the four constants α_{0-3}, four $q_i(t)$ need to be chosen, i.e. $i = 1, 2, 3, n = 4$. Consider first a choice of the type shown in Fig. 2.17(c). $q_i(t)$ are the displacements (same dimension as w, in m or in) at four equidistant nodes within the element. Thus

$$\begin{aligned}
w(0,t) &= q_1 = \alpha_0 \\
w(l/3,t) &= q_2 = \alpha_0 + \alpha_1(l/3) + \alpha_2(l/3)^2 + \alpha_3(l/3)^3 \\
w(2l/3,t) &= q_3 = \alpha_0 + \alpha_1(2l/3) + \alpha_2(2l/3)^2 + \alpha_3(2l/3)^3 \\
w(l,t) &= q_4 = \alpha_0 + \alpha_1(l) + \alpha_2(l)^2 + \alpha_3(l)^3
\end{aligned}$$

Solving for α_{0-3} in terms of q_{1-4}, and substitution into eqn. 2.77 leads to a form given by eqn. 2.76. The shape functions $H(s)$ are Lagrange polynomials.

A more suitable choice of nodal displacements $q_i(t)$ for beam problems is shown in Fig. 2.17(d). Here q_1 and q_2 are the displacement and angles at node 1 (in m or in, and in rads), q_2 and q_3 are the displacement and angles at node 1 (in m or in, and in rads). This is a more suitable choice because it ensures continuity of both displacement and slope between adjacent finite elements. Based on this choice we have

$$\begin{aligned}
w(0,t) &= q_1 = \alpha_0 \\
w'(0,t) &= \frac{dw}{ds}(\text{at s=0}) = q_2 = \alpha_2 \\
w(l,t) &= q_3 = \alpha_0 + \alpha_1(l) + \alpha_2(l)^2 + \alpha_3(l)^3 \\
w'(l,t) &= \frac{dw}{ds}(\text{at s=l}) = q_4 = \alpha_2 + 2\alpha_3(l) + 3\alpha_4(l)^2
\end{aligned}$$

Solving for α_{0-3} in terms of q_{1-4}, and substitution into eqn. 2.77 leads to a form given by eqn. 2.76. The shape functions $H(s)$ are in this case Hermite polynomials.

$$H_1 = 2\left(\frac{s}{l}\right)^3 - 3\left(\frac{s}{l}\right)^2 + 1$$

$$H_2 = \left[\left(\frac{s}{l}\right)^3 - 2\left(\frac{s}{l}\right)^2 + \frac{s}{l}\right] l$$
$$H_3 = -2\left(\frac{s}{l}\right)^3 + 3\left(\frac{s}{l}\right)^2$$
$$H_4 = \left[\left(\frac{s}{l}\right)^3 - \left(\frac{s}{l}\right)^2\right] l$$

Now calculate the elemental energies using the Rayleigh-Ritz method. Note that this step is same as that done earlier in the section on Rayleigh-Ritz method, except that here the integration is only over each element $s = 0$ to $s = l$, not the entire beam. The kinetic energy of the element U_e is given by

$$U_e = \frac{1}{2}\int_0^l m\dot{w}^2 ds = \frac{1}{2}\int_0^l \left(\sum_{i=1}^4 H_i \dot{q}_i\right)\left(\sum_{j=1}^4 H_j \dot{q}_j\right) ds = \frac{1}{2}\sum_{i=1}^4\sum_{j=1}^4 m_{ij}\,\dot{q}_i\,\dot{q}_j$$

where

$$m_{ij} = \int_0^l m H_i H_j \, ds \tag{2.78}$$

The potential energy of the element is given by the strain energy V_e

$$\begin{aligned} V_e &= \frac{1}{2}\int_0^l EI\left(\frac{d^2w}{ds^2}\right)^2 ds + \frac{1}{2}\int_0^R T\left(\frac{dw}{ds}\right)^2 ds \\ &= \frac{1}{2}\int_0^l EI\left(\sum_{i=1}^4 \frac{d^2H_i}{ds^2} q_i\right)\left(\sum_{j=1}^4 \frac{d^2H_j}{ds^2} q_j\right) ds + \frac{1}{2}\int_0^l T\left(\sum_{i=1}^4 \frac{dH_i}{ds} q_i\right)\left(\sum_{j=1}^4 \frac{dH_j}{ds} q_j\right) ds \\ &= \frac{1}{2}\sum_{i=1}^4\sum_{j=1}^4 k_{ij}\, q_i\, q_j \end{aligned}$$

where

$$k_{ij} = \int_0^l EI \frac{d^2H_i}{ds^2}\frac{d^2H_j}{ds^2} ds + \int_0^l T \frac{dH_i}{ds}\frac{dH_j}{ds} ds \tag{2.79}$$

The virtual work done by the external forces are given by

$$\delta W_e = \int_0^l f_z\, \delta w(s,t) = \int_0^l f_z \sum_{i=1}^4 H_i\, \delta q_i\, ds = \sum_{i=1}^4 Q_i\, \delta q_i$$

where

$$Q_i = \int_0^l f_z H_i ds \tag{2.80}$$

The above energy expressions can be put together in matrix form

$$T_e = \frac{1}{2}\dot{\underset{\sim}{q}}^T M_e\, \underset{\sim}{q} \qquad V_e = \frac{1}{2}\underset{\sim}{q}^T K_e\, \underset{\sim}{q} \qquad \delta W_e = \underset{\sim}{Q_e}^T\, \delta\underset{\sim}{q}$$

where $\underset{\sim}{q} = [q_1\, q_2\, q_3\, q_4]^T$. Consider a beam element with uniform properties within it, i.e., EI and m constant within the element. The tensile force T depends on the distance of a point $s = s$ from the rotation axis. For this purpose consider that the left hand edge of a general element i is at a distance x_i from the rotation axis. The length of the element is l. The mass matrix and the EI dependant part of the stiffness matrix calculation are straight forward. The centrifugal stiffness part can be easily treated by noting

$$T(s) = \int_{x_i+s}^{R} m\Omega^2 \rho d\rho = \int_{x_i}^{R} m\Omega^2 \rho d\rho - \int_{x_i}^{x_i+s} m\Omega^2 \rho d\rho$$

The first term is a successive integration over all elements from i to N and leads to $\sum_{j=i}^{N} m_j \Omega^2 (x_{j+1}^2 - x_j^2)/2$. The second term, with a change in integration variable, leads to

$$\int_{x_i}^{x_i+s} m\Omega^2 \rho d\rho = \int_0^s m\Omega^2 (x_i + \eta) d\eta = m_i \Omega^2 (x_i s + s^2/2)$$

Thus

$$\int_0^l T(s) \frac{dH_i}{ds}\frac{dH_j}{ds} ds = \sum_{j=i}^{N} \frac{m_j \Omega^2}{2} \left(x_{j+1}^2 - x_j^2\right) \int_0^l \frac{dH_i}{ds}\frac{dH_j}{ds} ds - m_i \Omega^2 \int_0^l \left(x_i s + \frac{1}{2} s^2\right) \frac{dH_i}{ds}\frac{dH_j}{ds} ds$$

Finally we have the following elemental matrices for the element i

$$M_e = m \begin{bmatrix} \frac{13}{35} l & \frac{11}{210} l^2 & \frac{9}{70} l & -\frac{13}{420} l^2 \\ \frac{11}{210} l^2 & \frac{1}{105} l^3 & \frac{13}{420} l^2 & -\frac{1}{140} l^3 \\ \frac{9}{70} l & \frac{13}{420} l^2 & \frac{13}{35} l & -\frac{11}{210} l^2 \\ -\frac{13}{420} l^2 & -\frac{1}{140} l^3 & -\frac{11}{210} l^2 & \frac{1}{105} l^3 \end{bmatrix} \tag{2.81}$$

$$K_e = EI \begin{bmatrix} \frac{12}{l^3} & \frac{6}{l^2} & -\frac{12}{l^3} & \frac{6}{l^2} \\ \frac{6}{l^2} & \frac{4}{l} & -\frac{6}{l^2} & \frac{2}{l} \\ -\frac{12}{l^3} & -\frac{6}{l^2} & \frac{12}{l^3} & -\frac{6}{l^2} \\ \frac{6}{l^2} & \frac{2}{l} & -\frac{6}{l^2} & \frac{4}{l} \end{bmatrix} + \frac{\Omega^2 A_i}{2} \begin{bmatrix} \frac{6}{5l} & \frac{1}{10} & -\frac{6}{5l} & \frac{1}{10} \\ \frac{1}{10} & \frac{2l}{15} & -\frac{1}{10} & -\frac{l}{30} \\ -\frac{6}{5l} & -\frac{1}{10} & \frac{6}{5l} & -\frac{1}{10} \\ \frac{1}{10} & -\frac{l}{30} & -\frac{1}{10} & \frac{2l}{15} \end{bmatrix} \tag{2.82}$$

$$- m_i \Omega^2 \begin{bmatrix} \frac{3}{5} x_i + \frac{6l}{35} & \frac{l x_i}{10} + \frac{l^2}{28} & -\frac{3}{5} x_i - \frac{6l}{35} & -\frac{l^2}{70} \\ \frac{l x_i}{10} + \frac{l^2}{28} & \frac{l x_i}{30} + \frac{l^3}{105} & -\frac{l x_i}{10} - \frac{l^2}{28} & -\frac{l^2 x_i}{60} + \frac{l^3}{70} \\ -\frac{3}{5} x_i - \frac{6l}{35} & -\frac{l x_i}{10} - \frac{l^2}{28} & \frac{3}{5} x_i + \frac{6l}{35} & \frac{l^2}{70} \\ -\frac{l^2}{70} & -\frac{l^2 x_i}{60} + \frac{l^3}{70} & +\frac{l^2}{70} & \frac{l^2 x_i}{10} + \frac{3l^3}{70} \end{bmatrix}$$

and

$$A_i = \sum_{j=i}^{N} m_j (x_{j+1}^2 - x_j^2)$$

2.5.2 Assembly of elements

We have the energies and virtual work for each element. The next step is to assemble them to obtain the global or total energies and virtual work T, V, and δW. For illustration of the assembly procedure, consider a case where the beam is discretized into three finite elements, with a total of eight degrees of freedom. The total potential energy is the sum of the elemental energies.

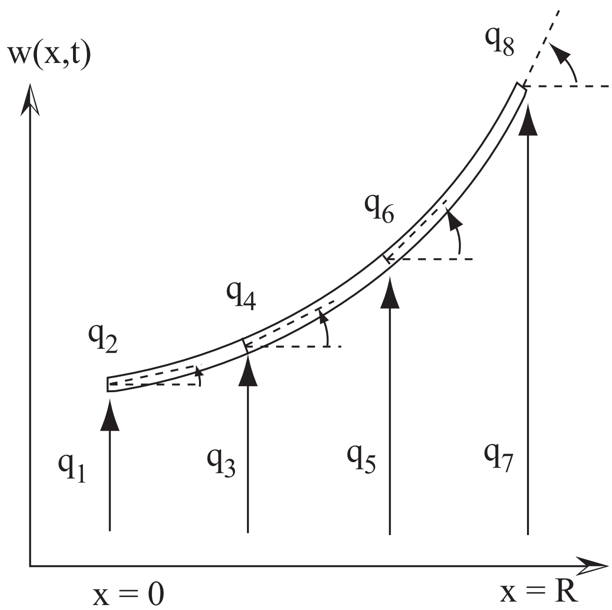


Figure 2.18: **Finite element discretization of a beam into three elements using Hermite polynomial interpolation within each element**

$$V = (V_e)_1 + (V_e)_2 + (V_e)_3$$

The elemental energies involve only four degrees of freedom. Thus they can be written as follows

$$(V)_1 = \frac{1}{2}\begin{Bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \\ q_5 \\ q_6 \\ q_7 \\ q_8 \end{Bmatrix}^T \begin{array}{c} \begin{array}{cccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \end{array} \\ \begin{bmatrix} \times & \times & \times & \times & & & & \\ \times & \times & \times & \times & & & & \\ \times & \times & \times & \times & & & & \\ \times & \times & \times & \times & & & & \\ & & & & & & & \\ & & & & & & & \\ & & & & & & & \\ & & & & & & & \end{bmatrix} \end{array} \begin{Bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \\ q_5 \\ q_6 \\ q_7 \\ q_8 \end{Bmatrix} = \frac{1}{2}\underset{\sim}{q}^T (K_e)_1 \underset{\sim}{q}$$

The matrix $(K_e)_1$ contains nonzero values only at marked places. The column numbers have been marked over the matrix. Similarly

$$(V)_2 = \frac{1}{2}\begin{Bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \\ q_5 \\ q_6 \\ q_7 \\ q_8 \end{Bmatrix}^T \begin{array}{c} \begin{array}{cccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \end{array} \\ \begin{bmatrix} & & & & & & & \\ & & & & & & & \\ & & + & + & + & + & & \\ & & + & + & + & + & & \\ & & + & + & + & + & & \\ & & + & + & + & + & & \\ & & & & & & & \\ & & & & & & & \end{bmatrix} \end{array} \begin{Bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \\ q_5 \\ q_6 \\ q_7 \\ q_8 \end{Bmatrix} = \frac{1}{2}\underset{\sim}{q}^T (K_e)_2 \underset{\sim}{q}$$

$$
(V)_3 = \frac{1}{2} \begin{Bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \\ q_5 \\ q_6 \\ q_7 \\ q_8 \end{Bmatrix}^T
\begin{array}{c} \begin{array}{cccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \end{array} \\
\begin{bmatrix}
 & & & & & & & \\
 & & & & & & & \\
 & & & & & & & \\
 & & & & & & & \\
 & & & & \bullet & \bullet & \bullet & \bullet \\
 & & & & \bullet & \bullet & \bullet & \bullet \\
 & & & & \bullet & \bullet & \bullet & \bullet \\
 & & & & \bullet & \bullet & \bullet & \bullet
\end{bmatrix} \end{array}
\begin{Bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \\ q_5 \\ q_6 \\ q_7 \\ q_8 \end{Bmatrix} = \frac{1}{2} \underset{\sim}{q}^T (K_e)_3 \, \underset{\sim}{q}
$$

The total potential energy then becomes

$$
V = (V_e)_1 + (V_e)_2 + (V_e)_3 = \frac{1}{2}\underset{\sim}{q}^T (K_e)_1 \underset{\sim}{q} + \frac{1}{2}\underset{\sim}{q}^T (K_e)_2 \underset{\sim}{q} + \frac{1}{2}\underset{\sim}{q}^T (K_e)_3 \underset{\sim}{q} = \frac{1}{2}\underset{\sim}{q}^T K \underset{\sim}{q}
$$

where the global stiffness matrix K has the following form

$$
K = (K_e)_1 + (K_e)_2 + (K_e)_3 =
\begin{array}{c|cccccccc|}
 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\
1 & \times & \times & \times & \times & & & & \\
2 & \times & \times & \times & \times & & & & \\
3 & \times & \times & \times\!\!\!\!+ & \times\!\!\!\!+ & + & + & & \\
4 & \times & \times & \times\!\!\!\!+ & \times\!\!\!\!+ & + & + & & \\
5 & & & + & + & +\!\!\!\!\bullet & +\!\!\!\!\bullet & \bullet & \bullet \\
6 & & & + & + & +\!\!\!\!\bullet & +\!\!\!\!\bullet & \bullet & \bullet \\
7 & & & & & \bullet & \bullet & \bullet & \bullet \\
8 & & & & & \bullet & \bullet & \bullet & \bullet
\end{array}
\qquad
\begin{array}{l} \times \text{ from element 1} \\ + \text{ from element 2} \\ \bullet \text{ from element 3} \end{array}
$$

In the same manner the total kinetic energy U and virtual work δW can be assembled as follows

$$
U = (U_e)_1 + (U_e)_2 + (U_e)_3 = \frac{1}{2}\underset{\sim}{q}^T (M_e)_1 \underset{\sim}{q} + \frac{1}{2}\underset{\sim}{q}^T (M_e)_2 \underset{\sim}{q} + \frac{1}{2}\underset{\sim}{q}^T (M_e)_3 \underset{\sim}{q} = \frac{1}{2}\underset{\sim}{q}^T M \underset{\sim}{q}
$$

$$
\delta W = (\delta W_e)_1 + (\delta W_e)_2 + (\delta W_e)_3 = (\underset{\sim}{Q_e})_1^T \, \delta\underset{\sim}{q} + (\underset{\sim}{Q_e})_2^T \, \delta\underset{\sim}{q} + (\underset{\sim}{Q_e})_3^T \, \delta\underset{\sim}{q} = \underset{\sim}{Q}^T \delta\underset{\sim}{q}
$$

where the total mass and load vectors are

$$
M = (M_e)_1 + (M_e)_2 + (M_e)_3 =
\begin{array}{c|cccccccc|}
 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\
1 & \times & \times & \times & \times & & & & \\
2 & \times & \times & \times & \times & & & & \\
3 & \times & \times & \times\!\!\!\!+ & \times\!\!\!\!+ & + & + & & \\
4 & \times & \times & \times\!\!\!\!+ & \times\!\!\!\!+ & + & + & & \\
5 & & & + & + & +\!\!\!\!\bullet & +\!\!\!\!\bullet & \bullet & \bullet \\
6 & & & + & + & +\!\!\!\!\bullet & +\!\!\!\!\bullet & \bullet & \bullet \\
7 & & & & & \bullet & \bullet & \bullet & \bullet \\
8 & & & & & \bullet & \bullet & \bullet & \bullet
\end{array}
\qquad
\begin{array}{l} \times \text{ from element 1} \\ + \text{ from element 2} \\ \bullet \text{ from element 3} \end{array}
$$

$$
Q = (Q_e)_1 + (Q_e)_2 + (Q_e)_3 =
\begin{array}{c|c|}
1 & \times \\
2 & \times \\
3 & \times\!\!\!\!+ \\
4 & \times\!\!\!\!+ \\
5 & +\!\!\!\!\bullet \\
6 & +\!\!\!\!\bullet \\
7 & \bullet \\
8 & \bullet
\end{array}
\qquad
\begin{array}{l} \times \text{ from element 1} \\ + \text{ from element 2} \\ \bullet \text{ from element 3} \end{array}
$$

Substitute the total energies and the total load vector into the Euler-Lagrange equation to determine the ODE's governing the degrees of freedom $q_1, q_2, q_3, \ldots, q_N$. Note that the virtual work expression includes only the external loading on each element, not the work done by the constraint forces acting on each face. The work done by the constraint forces however cancell when the elements are assembled. This is the reason why the elemental properties must be assembled before substitution into the Euler-Lagrange equation.

$$\frac{\partial}{\partial t}\left(\frac{\partial U}{\partial \dot{q}_i}\right) - \frac{\partial U}{\partial q_i} + \frac{\partial V}{\partial q_i} = Q_i \quad i = 1,\ 2,\ ...,\ N. \tag{2.83}$$

For the case of three elements we have the following ODE's.

$$\begin{bmatrix} m_{11} & m_{12} & \ldots & m_{18} \\ m_{21} & m_{22} & \ldots & m_{28} \\ \vdots & \vdots & \vdots & \vdots \\ m_{81} & m_{82} & \ldots & m_{88} \end{bmatrix} \begin{bmatrix} \ddot{q}_1 \\ \ddot{q}_2 \\ \vdots \\ \ddot{q}_8 \end{bmatrix} + \begin{bmatrix} k_{11} & k_{12} & \ldots & k_{18} \\ k_{21} & k_{22} & \ldots & k_{28} \\ \vdots & \vdots & \vdots & \vdots \\ k_{81} & k_{82} & \ldots & k_{88} \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \\ \vdots \\ q_8 \end{bmatrix} = \begin{bmatrix} Q_1 \\ Q_2 \\ \vdots \\ Q_8 \end{bmatrix} \tag{2.84}$$

Note that at this point the stiffness matrix above K is singular. This is because the elements were free-free in nature and no constraints have yet been implemented on either end of the beam. Thus the entire beam is still free-free and as such, the above ODE's include the rigid body modes of the beam. The next step is to apply the constraints posed by the geometric boundary conditions.

2.5.3 Constraint conditions

Constraints conditions can be easily incorporated by modifying eqn. 2.84, by simple removal of certain degrees of freedom. For example, for a cantilevered boundary condition at the root end we have

$$\begin{aligned} w(0,t) = 0 &\implies q_1(t) = 0 \\ w'(0,t) = 0 &\implies q_2(t) = 0 \end{aligned} \tag{2.85}$$

which can be incorporated by removing the first two rows and columns of eqn. 2.84. Thus in this case the governing ODE's become

$$\begin{bmatrix} m_{33} & m_{34} & \ldots & m_{38} \\ m_{43} & m_{44} & \ldots & m_{48} \\ \vdots & \vdots & \vdots & \vdots \\ m_{83} & m_{84} & \ldots & m_{88} \end{bmatrix} \begin{bmatrix} \ddot{q}_3 \\ \ddot{q}_4 \\ \vdots \\ \ddot{q}_8 \end{bmatrix} + \begin{bmatrix} k_{33} & k_{34} & \ldots & k_{38} \\ k_{43} & k_{44} & \ldots & k_{48} \\ \vdots & \vdots & \vdots & \vdots \\ k_{83} & k_{84} & \ldots & k_{88} \end{bmatrix} \begin{bmatrix} q_3 \\ q_4 \\ \vdots \\ q_8 \end{bmatrix} = \begin{bmatrix} Q_3 \\ Q_4 \\ \vdots \\ Q_8 \end{bmatrix}$$

The new K matrix is no longer singular. For a simply-supported beam we have

$$\begin{aligned} w(0,t) = 0 &\implies q_1(t) = 0 \\ w(R,t) = 0 &\implies q_7(t) = 0 \end{aligned} \tag{2.86}$$

which can be incorporated by removing the first and seventh rows and columns of eqn. 2.84. Similarly a statically indeterminate problem where one end is cantilevered and the other end simply-supported

$$\begin{aligned} w(0,t) = 0 &\implies q_1(t) = 0 \\ w'(0,t) = 0 &\implies q_2(t) = 0 \\ w(R,t) = 0 &\implies q_7(t) = 0 \end{aligned} \tag{2.87}$$

can be easily realized by removing the first, second and seventh rows and columns. The mass and stiffness matrices are in general banded, a fact that can be used to reduce computations and memory