Static Electric Fields

Introduction to Electromagnetism with Practice

Theory & Applications

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Remind: Electrostatics – Maxwell's Equations

Remind: Postulates: Differential Form

$\varepsilon_{\mathrm{r}} = 1$ in the Vacuum

$$\nabla \times \mathbf{E} = \mathbf{O}$$

$$\nabla \cdot \mathbf{D} = \rho$$

$$\mathbf{D} = \varepsilon_0 \varepsilon_r \mathbf{E}$$

$$\Rightarrow$$

$$\boxed{\begin{aligned} \nabla \times \mathbf{E} &= \mathbf{O} \\ \nabla \cdot \mathbf{E} &= \frac{\rho}{\varepsilon_0} \end{aligned}}$$

Remind: Postulates: Integral Form

$\varepsilon_{\mathrm{r}} = 1$ in the Vacuum

$$\nabla \times \mathbf{E} = \mathbf{O}$$

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\varepsilon_0}$$

Gauss & Stokes

$$\boxed{\begin{aligned} \oint_C \mathbf{E} \cdot d\mathbf{l} &= 0 \\ \oint_S \mathbf{E} \cdot d\mathbf{S} &= \frac{Q}{\varepsilon_0} \end{aligned}}$$

Gauss's Law

Gauss's Law

$$\varepsilon_\mathrm{r} = 1 \text{ in the Vacuum}$$

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\varepsilon_0} \Rightarrow \oint_S \mathbf{E} \cdot d\mathbf{S} = \frac{Q}{\varepsilon_0}$$

Total outward flux over any closed surface
= ***Total charge*** enclosed in the surface (divided by ε_0)

Applying Gauss's Law

$\varepsilon_{\mathrm{r}} = 1$ in the Vacuum

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\varepsilon_0} \Rightarrow \oint_S \mathbf{E} \cdot d\mathbf{S} = \frac{Q}{\varepsilon_0}$$

The key is on estimating the surface integral…
Symmetry is necessary for analytical solutions!

It is relatively easy to estimate the total Q
Usually simple volume integral…

$$Q = \int_V \rho dv$$

Special Topic: What is "symmetry"?

The Principle of Least Action I

We'd like to express the equations of motion
with ***a single real-valued function***

Lagrangian Function $L(\mathbf{q}, \dot{\mathbf{q}}, t)$ **Kinetic Energy – Potential Energy**

'Action'

$$\boxed{S = \int_{t_i}^{t_f} L(\mathbf{q}, \dot{\mathbf{q}}, t)dt}$$

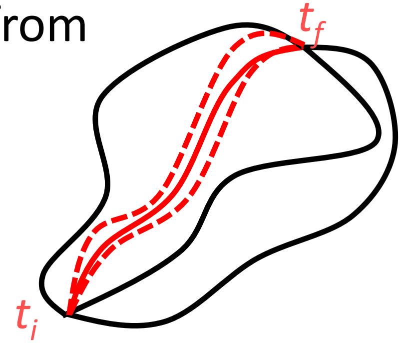
The Principle of Least Action II

The Principle of Least (or stationary) Action

: The equation of motion of the system is obtained from

$$\boxed{\delta S = 0}$$

Small change of 'Action' to First Order = 0



$$\delta S = \int_{t_i}^{t_f} L(\mathbf{q} + \delta\mathbf{q}, \dot{\mathbf{q}} + \delta\dot{\mathbf{q}}, t)dt - \int_{t_i}^{t_f} L(\mathbf{q}, \dot{\mathbf{q}}, t)dt$$

First Order

$$\approx \int_{t_i}^{t_f} \sum_k \left(\frac{\partial L}{\partial q_k} \delta q_k + \frac{\partial L}{\partial \dot{q}_k} \delta \dot{q}_k \right) dt$$

The Principle of Least Action III

$$\delta S = \int_{t_i}^{t_f} \sum_k \left(\frac{\partial L}{\partial q_k} \delta q_k + \frac{\partial L}{\partial \dot{q}_k} \delta \dot{q}_k \right) dt$$

Integrating by parts

$$= \int_{t_i}^{t_f} \sum_k \frac{\partial L}{\partial q_k} \delta q_k dt + \underline{\sum_k \left[\frac{\partial L}{\partial \dot{q}_k} \delta q_k \right]_{t_i}^{t_f} - \int_{t_i}^{t_f} \sum_k \left(\frac{d}{dt} \frac{\partial L}{\partial \dot{q}_k} \right) \delta q_k dt}$$

$$= \cancelto{0}{\sum_k \left[\frac{\partial L}{\partial \dot{q}_k} \delta q_k \right]_{t_i}^{t_f}} + \int_{t_i}^{t_f} \sum_k \underbrace{\left[\frac{\partial L}{\partial q_k} - \left(\frac{d}{dt} \frac{\partial L}{\partial \dot{q}_k} \right) \right]}_{=0} \delta q_k dt = 0$$

$$\delta \mathbf{q}(t_i) = 0$$

$$\delta \mathbf{q}(t_f) = 0$$

Lagrangian Mechanics

Euler–Lagrange equation

$$\boxed{\frac{\partial L}{\partial q_k} - \left(\frac{d}{dt} \frac{\partial L}{\partial \dot{q}_k} \right) = 0}$$

Lagrangian Function

$$L = \frac{1}{2} m\dot{x}^2 - mgx$$

$$q = x$$

$$-mg - \left(\frac{d}{dt} m\dot{x} \right) = 0$$

$$m\frac{d^2x}{dt^2} = -mg$$

Newtonian Mechanics

Noether's Theorem

$$\frac{\partial L}{\partial q_k} - \left(\frac{d}{dt} \frac{\partial L}{\partial \dot{q}_k} \right) = 0$$

$$\frac{\partial L}{\partial q_k} = 0 \quad \rightleftarrows \quad \frac{d}{dt} \frac{\partial L}{\partial \dot{q}_k} = 0$$

Symmetry = Conserved Quantity

A certain "*symmetry*" leads to
the *conservation* of *the corresponding physical quantity*

Symmetry for Gauss's Law

Examples of Gaussian Surfaces from Symmetries

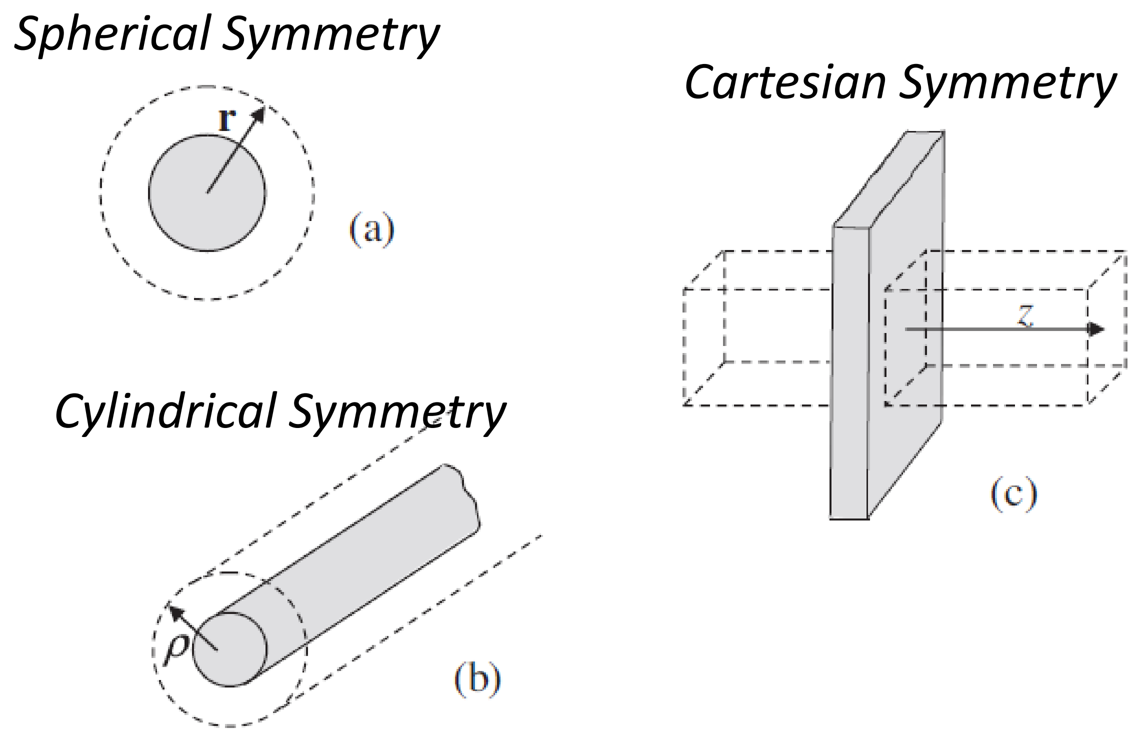
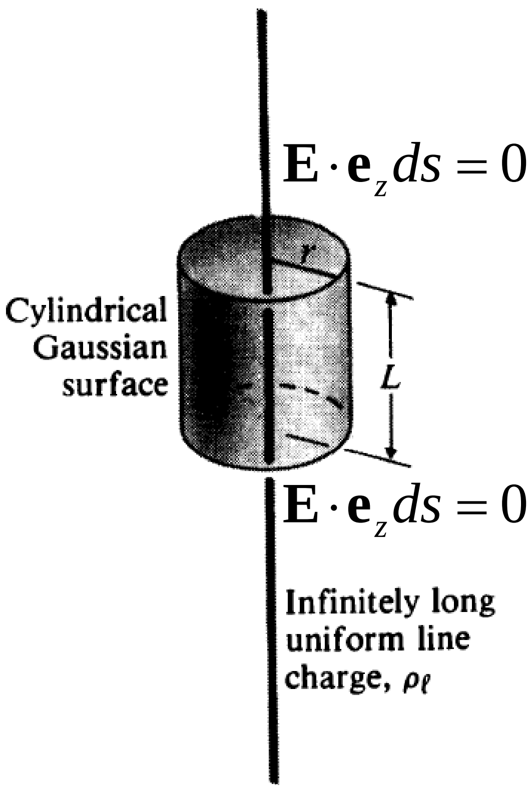


Figure 3.8: Shaded areas are highly symmetrical distributions of charge: (a) $\rho_0(r, \theta, \phi) = \rho_0(r)$; (b) $\rho_0(\rho, \phi, z) = \rho_0(\rho)$; (c) $\rho_0(x, y, z) = \rho_0(z)$. Dashed lines outline possible choices for Gaussian surfaces.

Example 003

EXAMPLE 3–5 Use Gauss's law to determine the electric field intensity of an infinitely long, straight, line charge of a uniform density ρ_ℓ in air.



Due to the symmetry

$$\mathbf{E} = \mathbf{e}_\rho E_\rho$$

We use r instead of ρ here…

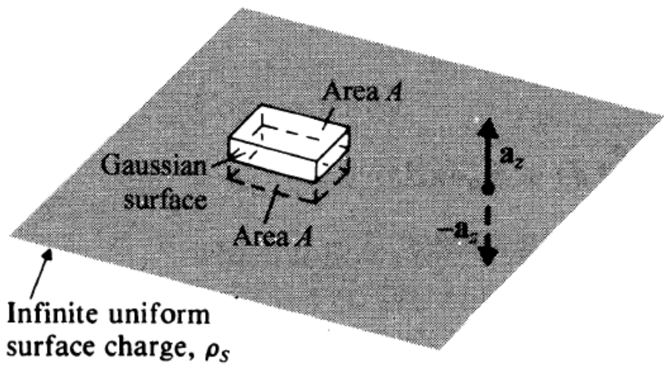
$$\oint_S \mathbf{E}\cdot d\mathbf{S} = 2\pi r L E_\rho = \frac{1}{\varepsilon_0}\int_V \rho_l dv = \frac{1}{\varepsilon_0}\rho_l L$$

$$E_\rho = \frac{\rho_l}{2\pi\varepsilon_0 r}$$

$$\boxed{\mathbf{E} = \mathbf{e}_\rho \frac{\rho_l}{2\pi\varepsilon_0 r}}$$

Example 004

EXAMPLE 3–6 Determine the electric field intensity of an infinite planar charge with a uniform surface charge density ρ_s.



Due to the symmetry

$$\mathbf{E} = \mathbf{e}_z E_z \qquad E_z(z>0) = -E_z(z<0)$$

$$\oint_S \mathbf{E}\cdot d\mathbf{S} = AE_z(z>0) - AE_z(z<0)$$

$$= \frac{1}{\varepsilon_0}\int_V \rho_s dv = \frac{1}{\varepsilon_0}\rho_s A$$

$$E_z(z>0) = \frac{1}{2\varepsilon_0}\rho_s$$

$$\boxed{\mathbf{E} = +\mathbf{e}_z \frac{\rho_s}{2\varepsilon_0} \quad (z>0), \quad \mathbf{E} = -\mathbf{e}_z \frac{\rho_s}{2\varepsilon_0} \quad (z<0)}$$

Remind: Example 002

Example 3.1 (a) Find **E** on the symmetry axis of a ring with radius R and uniform charge per unit length λ. (b) Use the results of part (a) to find **E** on the symmetry axis of a disk with radius R and uniform charge per unit area σ. (c) Use the results of part (b) to find **E** for an infinite sheet with uniform charge density σ. Discuss the matching condition at $z = 0$.

(c)

Consider some interesting cases… ($z > 0$)

If applicable, Gauss's law is significantly useful!

Ring

$$\boxed{\mathbf{E} = \frac{\lambda R}{2\varepsilon_0}\frac{z}{\left(z^2+R^2\right)^{\frac{3}{2}}}\mathbf{e}_z}$$

Disk

$$\boxed{\mathbf{E} = \frac{\sigma}{2\varepsilon_0}\mathbf{e}_z\left(1-\frac{z}{\sqrt{z^2+R^2}}\right)}$$

Discontinuity! (Discussed later)

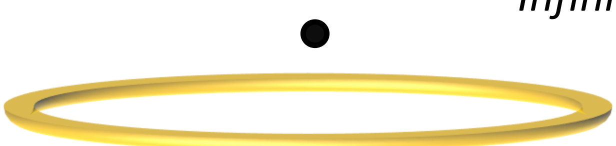
$R \to \infty$

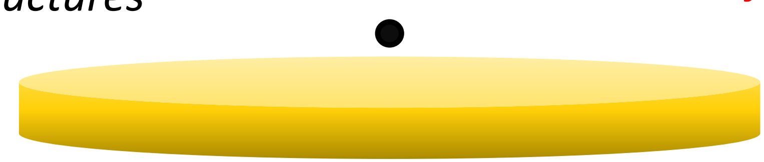
$$\mathbf{E} \sim 0$$

$$\mathbf{E} \sim +\frac{\sigma}{2\varepsilon_0}\mathbf{e}_z \quad (z>0), \qquad \mathbf{E} \sim -\frac{\sigma}{2\varepsilon_0}\mathbf{e}_z \quad (z<0)$$

Uniform

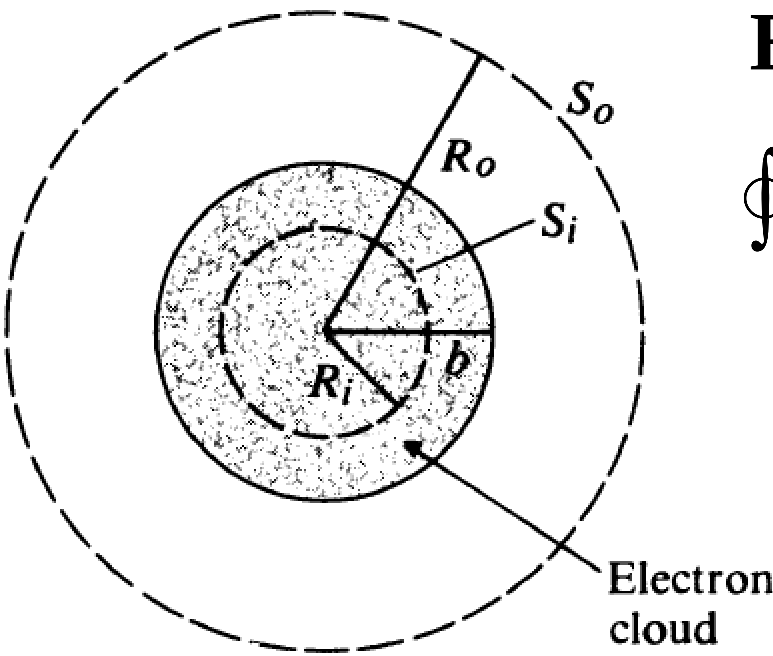
Infinite structures





Example 005

EXAMPLE 3–7 Determine the **E** field caused by a spherical cloud of electrons with a volume charge density $\rho = -\rho_o$ for $0 \le R \le b$ (both ρ_o and b are positive) and $\rho = 0$ for $R > b$.



Due to the symmetry

$$\mathbf{E} = \mathbf{e}_r E_r$$

$$\oint_S \mathbf{E} \cdot d\mathbf{S} = 4\pi R^2 E_r = \frac{1}{\varepsilon_0} \int_V \rho(R) dv$$

$0 \le R \le b$ $\quad \int_V \rho(R) dv = -\frac{4}{3}\pi R^3 \rho_0$

$R > b$ $\quad \int_V \rho(R) dv = -\frac{4}{3}\pi b^3 \rho_0$

Example 005

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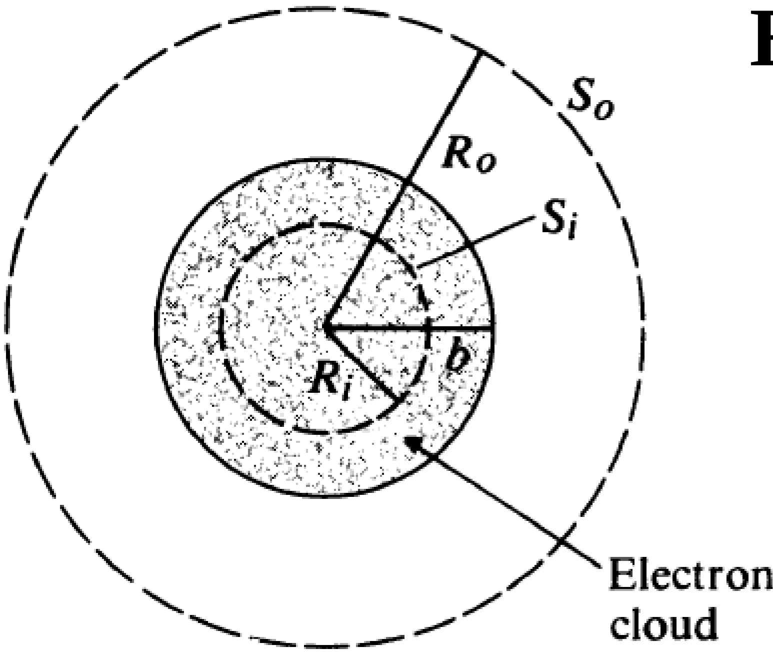
Due to the symmetry

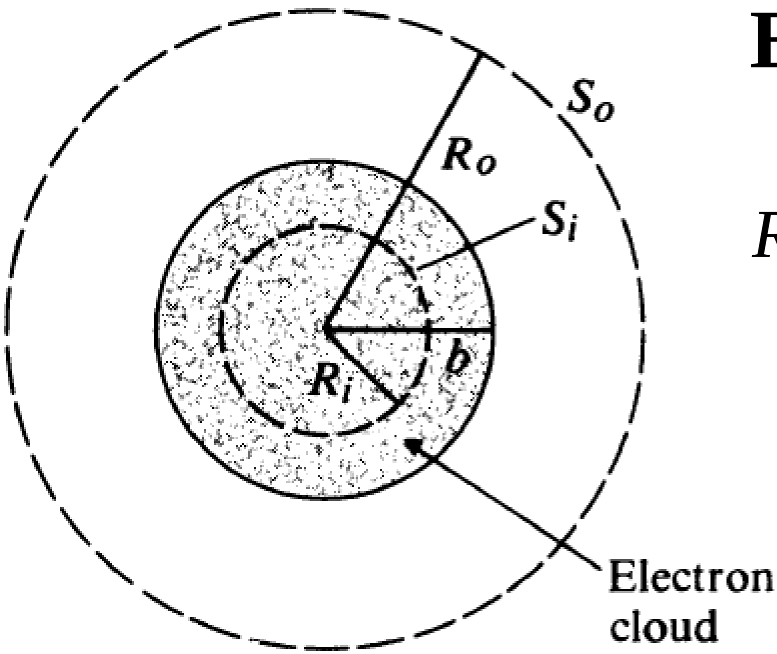


$$\mathbf{E} = \mathbf{e}_r E_r$$

$0 \le R \le b$	$\mathbf{E} = -\dfrac{\rho_0 R}{3\varepsilon_0}\mathbf{e}_r$
$R > b$	$\mathbf{E} = -\dfrac{\rho_0 b^3}{3\varepsilon_0 R^2}\mathbf{e}_r$

Example 005

EXAMPLE 3–7 Determine the **E** field caused by a spherical cloud of electrons with a volume charge density $\rho = -\rho_o$ for $0 \le R \le b$ (both ρ_o and b are positive) and $\rho = 0$ for $R > b$.



Due to the symmetry

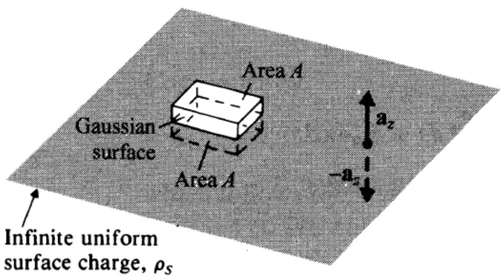
$$\mathbf{E} = \mathbf{e}_r E_r$$

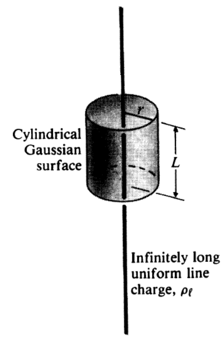
$$R > b: \ \mathbf{E} = -\frac{\rho_0 b^3}{3\varepsilon_0 R^2}\mathbf{e}_r$$

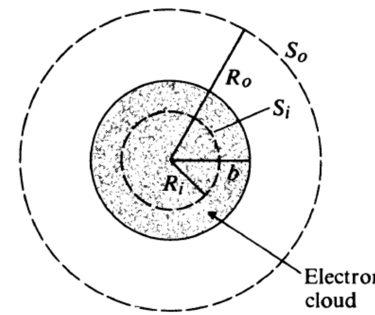
$$= \left(-\frac{4\pi b^3}{3}\rho_0\right)\frac{1}{4\pi\varepsilon_0 R^2}\mathbf{e}_r$$

$$= \frac{Q}{4\pi\varepsilon_0 R^2}\mathbf{e}_r$$ ~ Point Charge

Geometry Effect on Electric Fields







$z > 0$

$R > b$

$$\mathbf{E} = +\mathbf{e}_z \frac{\rho_s}{2\varepsilon_0} = +\mathbf{e}_z \frac{2\pi\rho_s z^2}{4\pi\varepsilon_0 z^2}$$

$$\mathbf{E} = \mathbf{e}_\rho \frac{\rho_l}{2\pi\varepsilon_0 r} = \mathbf{e}_\rho \frac{2\rho_l r}{4\pi\varepsilon_0 r^2}$$

$$\mathbf{E} = -\frac{\rho_0 b^3}{3\varepsilon_0 R^2}\mathbf{e}_r = \mathbf{e}_r \frac{\left(-\frac{4\pi b^3}{3}\rho_0\right)}{4\pi\varepsilon_0 R^2}$$

Concept of Effective Media & Metamaterials

Meta-materials: "Meta" ~ "Beyond"

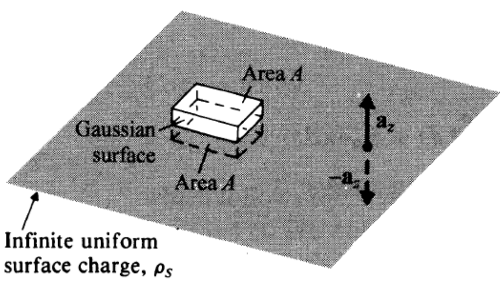
1. *Focusing on* ***the property "A"*** *among the entire material properties (A & B)*
2. We can find "several" different candidates of materials,
   which provide ***the identical A property***,
   while allowing ***the distinctions in the B property***
3. This degrees of freedom can impose uniqueness on the "B" property,
   even allowing for "unnatural" behaviors: ***Beyond natural materials***

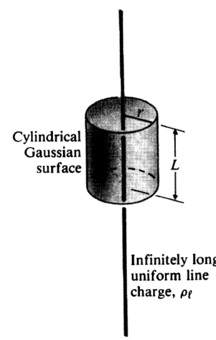
Example of a Concept of "Meta"

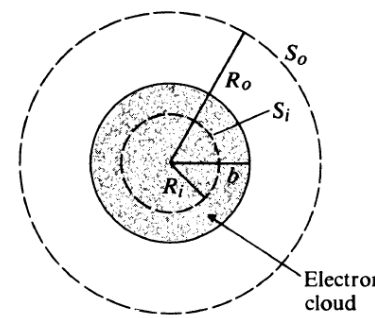
$$\mathbf{E} = \frac{q}{4\pi\varepsilon_0}\frac{\hat{\mathbf{r}}}{|\mathbf{r}|^2}$$

$$\mathbf{E} = \frac{q_{\text{eff}}}{4\pi\varepsilon_0}\frac{\text{Directional Vector}}{\text{Distance}^2}$$

*Interpreting Coulomb's law in **a narrow (pointwise) sense***







$$\boxed{q_{\text{eff}} \sim 2\pi\rho_s z_0^{\,2}}$$

near $z = z_0$

$$\boxed{q_{\text{eff}} \sim 2\rho_l r_0}$$

near $r = r_0$

$$\boxed{q_{\text{eff}} = -\frac{4\pi b^3}{3}\rho_0}$$

for $R > b$

By controlling geometric parameters, we can manipulate q_{eff} & the other properties

Electric Potential

Remind: Helmholtz Theorem

Helmholtz Theorem (or Helmholtz Decomposition)

An arbitrary vector field can always be decomposed into the sum of two vector fields:
one with zero divergence **and** ***one with zero curl***

$$\mathbf{E} = \mathbf{E}_{\mathrm{D}} + \mathbf{E}_{\mathrm{C}}$$

Solenoidal (divergence-free)

$$\nabla \cdot \mathbf{E}_{\mathrm{D}} = 0$$

Irrotational (curl-free)

$$\nabla \times \mathbf{E}_{\mathrm{C}} = 0$$

Remembering Null Identities

$$\nabla \cdot \left(\nabla \times \mathbf{A} \right) = 0 \qquad \nabla \times \left(\nabla f \right) = \mathbf{0}$$

We can write **E** as follow:

$$\mathbf{E} = \nabla \times \mathbf{A} + \nabla f$$

The proper boundary condition (B.C.) allows the unique **E**

Helmholtz Theorem for Electrostatics

$$\mathbf{E} = \mathbf{E}_{\mathrm{D}} + \mathbf{E}_{\mathrm{C}}$$

$$\nabla \cdot \mathbf{E}_{\mathrm{D}} = 0$$

$$\nabla \times \mathbf{E}_{\mathrm{C}} = 0$$

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\varepsilon_0}$$

$$\nabla \times \mathbf{E} = \mathbf{O}$$

$$\nabla \cdot \mathbf{E} = \nabla \cdot \left(\mathbf{E}_{\mathrm{D}} + \mathbf{E}_{\mathrm{C}}\right)$$

$$= \nabla \cdot \mathbf{E}_{\mathrm{C}} = \frac{\rho}{\varepsilon_0}$$

$$\nabla \times \mathbf{E} = \nabla \times \left(\mathbf{E}_{\mathrm{D}} + \mathbf{E}_{\mathrm{C}}\right)$$

$$= \nabla \times \mathbf{E}_{\mathrm{D}} = \mathbf{O}$$

$$\mathbf{E}_{\mathrm{D}} = \mathbf{O}$$

Helmholtz Theorem for Electrostatics: Electric Potential

$$\mathbf{E} = \mathbf{E}_\mathrm{C} \qquad \nabla \times \mathbf{E}_\mathrm{C} = 0$$

Let's assign $\mathbf{E}_\mathrm{C}$ for the conventional notation

$$\mathbf{E}_\mathrm{C} = -\nabla V$$

Electric Potential (or Scalar Potential) V

$$\boxed{\mathbf{E} = -\nabla V}$$

Electric Field to Electric Potential – Path Invariance

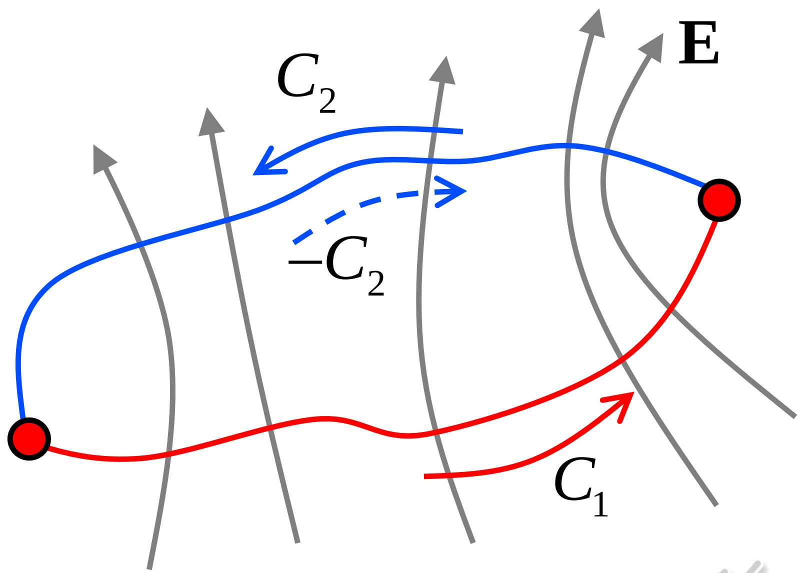
$$\mathbf{E} = -\nabla V \qquad \nabla \times \mathbf{E} = -\nabla \times \nabla V = \mathbf{O}$$

$$\int_S (\nabla \times \mathbf{E}) \cdot d\mathbf{s} = \oint_C \mathbf{E} \cdot d\mathbf{l} = 0$$

$$\oint_C \mathbf{E} \cdot d\mathbf{l} = \int_{C_1 + C_2} \mathbf{E} \cdot d\mathbf{l} = \int_{C_1 - (-C_2)} \mathbf{E} \cdot d\mathbf{l} = \int_{C_1} \mathbf{E} \cdot d\mathbf{l} - \int_{-C_2} \mathbf{E} \cdot d\mathbf{l} = 0$$

Path-invariant integral!

$$\int_{C_1} \mathbf{E} \cdot d\mathbf{l} = \int_{-C_2} \mathbf{E} \cdot d\mathbf{l}$$



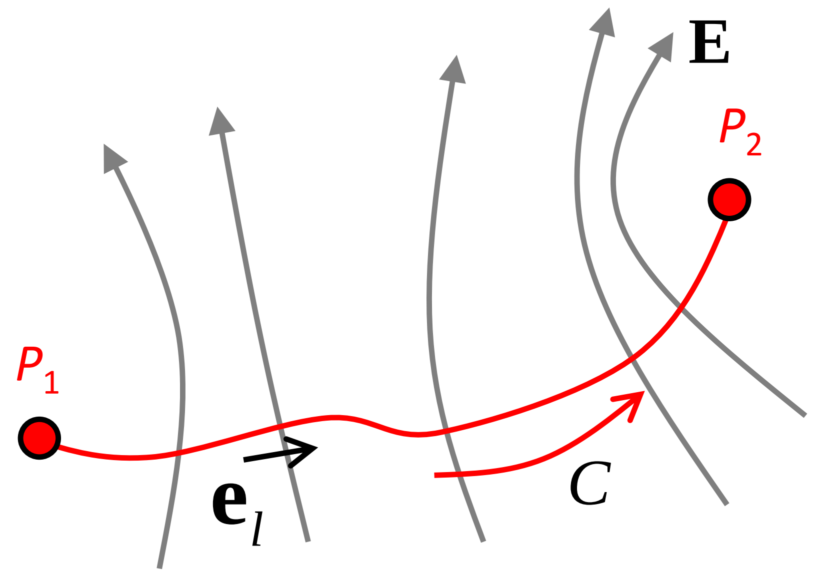
Electric Field to Electric Potential – Path Invariance

$$\int_C \mathbf{E} \cdot d\mathbf{l} = -\int_C \nabla V \cdot d\mathbf{l}$$

$$\nabla V \cdot d\mathbf{l} = \nabla V \cdot \mathbf{e}_l dl = dV$$

$$-\int_C \mathbf{E} \cdot d\mathbf{l} = \int_C dV = V_2 - V_1$$

$$\boxed{V_2 - V_1 = -\int_{P_1}^{P_2} \mathbf{E} \cdot d\mathbf{l}}$$



Static Electric Fields

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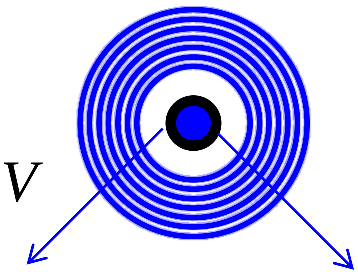
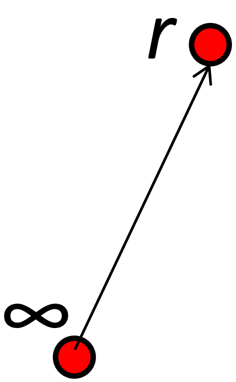
Electric Potential (cont.)

Electric Potential – Point Charge

$$\mathbf{E} = -\nabla V$$



Decrease of *V*



$$\mathbf{E} = \frac{1}{4\pi\varepsilon_0} q \frac{\mathbf{e}_r}{r^2}$$

$$V_2 - V_1 = -\int_{P_1}^{P_2} \mathbf{E} \cdot d\mathbf{l}$$

$$V(r) - V(\infty) = -\int_{\infty}^{r} \mathbf{E} \cdot d\mathbf{l}$$

$$V(r) - V(\infty)$$

$$= -\int_{P_\infty}^{P_r} \frac{1}{4\pi\varepsilon_0} q \frac{\mathbf{e}_r}{r^2} \cdot d\mathbf{l}$$

$$= -\frac{q}{4\pi\varepsilon_0} \int_0^{\infty - r} \frac{\mathbf{e}_r}{r'^2} \cdot (-\mathbf{e}_r dl)$$

$$= \frac{q}{4\pi\varepsilon_0} \int_{\infty}^{r} \frac{1}{r'^2} (-dr')$$

$l = \infty - r'$
$dl = -dr'$

$$= \frac{q}{4\pi\varepsilon_0} \frac{1}{r}$$

If we set $V(\infty) = 0$

$$\boxed{V(r) = \frac{q}{4\pi\varepsilon_0} \frac{1}{r}}$$

Superposition for Electric Potentials

$$\mathbf{E} = -\nabla V \qquad \sum_k \mathbf{E}_k = -\nabla \sum_k V_k$$

The superposition principle is also valid for the potential *V*

$$\mathbf{E} = \frac{1}{4\pi\varepsilon_0} q \frac{\mathbf{r}}{r^3} = \frac{1}{4\pi\varepsilon_0} q \frac{\mathbf{e}_r}{r^2}$$

$$\boxed{V = \frac{q}{4\pi\varepsilon_0} \frac{1}{r}}$$

$$\mathbf{E} = \frac{1}{4\pi\varepsilon_0} \sum_{k=1}^{N} q_k \frac{\mathbf{x} - \mathbf{x}_k}{|\mathbf{x} - \mathbf{x}_k|^3}$$

$$\boxed{\nabla \frac{1}{|\mathbf{x} - \mathbf{x}'|} = -\frac{\mathbf{x} - \mathbf{x}'}{|\mathbf{x} - \mathbf{x}'|^3}}$$

$$\boxed{V = \frac{1}{4\pi\varepsilon_0} \sum_{k=1}^{N} q_k \frac{1}{|\mathbf{x} - \mathbf{x}_k|}}$$

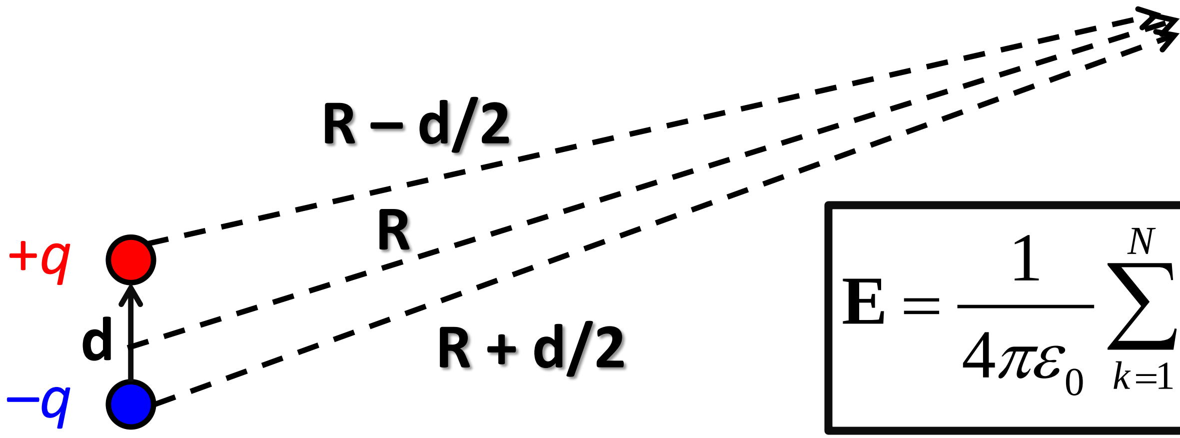
$$\mathbf{E} = \frac{1}{4\pi\varepsilon_0} \int_V \frac{\mathbf{x} - \mathbf{x}'}{|\mathbf{x} - \mathbf{x}'|^3} \rho(\mathbf{x}') d^3x'$$

$$\boxed{V = \frac{1}{4\pi\varepsilon_0} \int_V \frac{1}{|\mathbf{x} - \mathbf{x}'|} \rho(\mathbf{x}') d^3x'}$$

Electric Potential: Exercises

Remind: Example 001

***Electric Dipole*: Estimating an electric field far from the dipole?**



$$\mathbf{E} = \frac{1}{4\pi\varepsilon_0}\sum_{k=1}^{N} q_k \frac{\mathbf{x}-\mathbf{x}_k}{\left|\mathbf{x}-\mathbf{x}_k\right|^3}$$

$$\mathbf{E} = \frac{1}{4\pi\varepsilon_0}\left(q\frac{\mathbf{R}-\dfrac{\mathbf{d}}{2}}{\left|\mathbf{R}-\dfrac{\mathbf{d}}{2}\right|^3} - q\frac{\mathbf{R}+\dfrac{\mathbf{d}}{2}}{\left|\mathbf{R}+\dfrac{\mathbf{d}}{2}\right|^3} \right)$$

We cannot learn a lot from accurate but too complex equations!

Remind: Example 001

Electric Dipole: Estimating an electric field far from the dipole?

$$\left|\mathbf{R}-\frac{\mathbf{d}}{2}\right|^{-3}=\left|\left(\mathbf{R}-\frac{\mathbf{d}}{2}\right)\cdot\left(\mathbf{R}-\frac{\mathbf{d}}{2}\right)\right|^{-\frac{3}{2}}=\left(R^2+\frac{d^2}{4}-\mathbf{R}\cdot\mathbf{d}\right)^{-\frac{3}{2}}=\left[R^2\left(1-\frac{\mathbf{R}\cdot\mathbf{d}}{R^2}+\frac{d^2}{4R^2}\right)\right]^{-\frac{3}{2}}$$

$$=R^{-3}\left(1-\frac{\mathbf{R}\cdot\mathbf{d}}{R^2}+\frac{d^2}{4R^2}\right)^{-\frac{3}{2}}\sim R^{-3}\left(1+\frac{3}{2}\frac{\mathbf{R}\cdot\mathbf{d}}{R^2}\right)$$

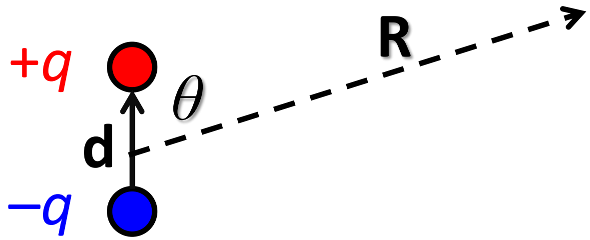
$$\mathbf{E}\sim\frac{q}{4\pi\varepsilon_0}\left[\left(\mathbf{R}-\frac{\mathbf{d}}{2}\right)R^{-3}\left(1+\frac{3}{2}\frac{\mathbf{R}\cdot\mathbf{d}}{R^2}\right)-\left(\mathbf{R}+\frac{\mathbf{d}}{2}\right)R^{-3}\left(1-\frac{3}{2}\frac{\mathbf{R}\cdot\mathbf{d}}{R^2}\right)\right]$$

$$=\frac{q}{4\pi\varepsilon_0 R^3}\left[\left(\mathbf{R}-\frac{\mathbf{d}}{2}\right)\left(1+\frac{3}{2}\frac{\mathbf{R}\cdot\mathbf{d}}{R^2}\right)-\left(\mathbf{R}+\frac{\mathbf{d}}{2}\right)\left(1-\frac{3}{2}\frac{\mathbf{R}\cdot\mathbf{d}}{R^2}\right)\right]$$

$$=\frac{q}{4\pi\varepsilon_0 R^3}\left[3\frac{\mathbf{R}\cdot\mathbf{d}}{R^2}\mathbf{R}-\mathbf{d}\right]$$

Remind: Example 001

Electric Dipole: Estimating an electric field far from the dipole?



Electric Dipole Moment

$$\mathbf{p} = q\mathbf{d}$$

$$\mathbf{E} \sim \frac{q}{4\pi\varepsilon_0 R^3}\left[3\frac{\mathbf{R}\cdot\mathbf{d}}{R^2}\mathbf{R} - \mathbf{d}\right]$$

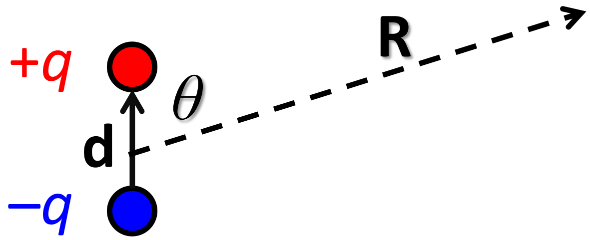
$$= \frac{1}{4\pi\varepsilon_0 R^3}\left[3\frac{\mathbf{R}\cdot\mathbf{p}}{R^2}\mathbf{R} - \mathbf{p}\right] = \frac{1}{4\pi\varepsilon_0 R^3}\left[3\frac{Rp\cos\theta}{R^2}R\mathbf{e}_r - \mathbf{e}_z p\right]$$

$$= \frac{1}{4\pi\varepsilon_0 R^3}\left[3\frac{Rp\cos\theta}{R^2}R\mathbf{e}_r - \left(\mathbf{e}_r\cos\theta - \mathbf{e}_\theta\sin\theta\right)p\right]$$

$$= \frac{p}{4\pi\varepsilon_0 R^3}\left(\mathbf{e}_r 2\cos\theta + \mathbf{e}_\theta\sin\theta\right)$$

Remind: Example 001

Electric Dipole: Estimating an electric field far from the dipole?



Electric Dipole Moment

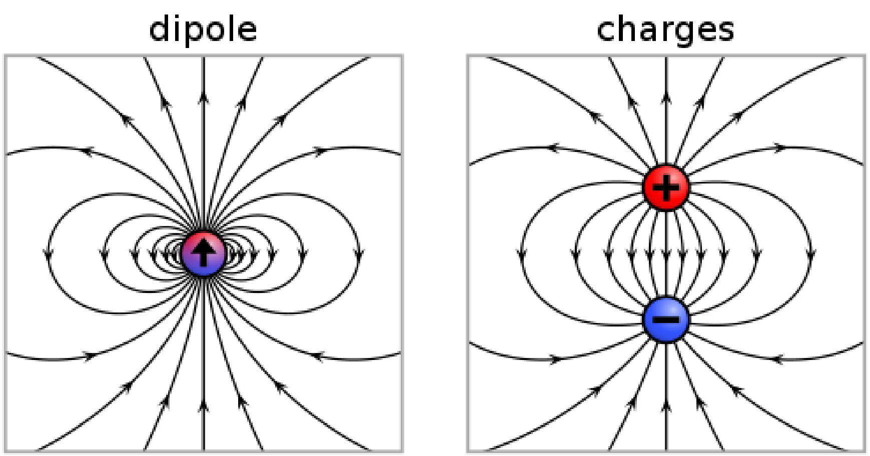
$$\mathbf{p} = q\mathbf{d}$$

$$\boxed{\mathbf{E} \sim \frac{1}{4\pi\varepsilon_0 R^3}\left[3\frac{\mathbf{R}\cdot\mathbf{p}}{R^2}\mathbf{R} - \mathbf{p}\right] = \frac{p}{4\pi\varepsilon_0 R^3}\left(\mathbf{e}_r 2\cos\theta + \mathbf{e}_\theta \sin\theta\right)}$$

$$\theta = 0 \qquad \mathbf{E} \sim \frac{2p}{4\pi\varepsilon_0 R^3}\mathbf{e}_r$$

$$\theta = \frac{\pi}{2} \qquad \mathbf{E} \sim \frac{p}{4\pi\varepsilon_0 R^3}\mathbf{e}_\theta$$

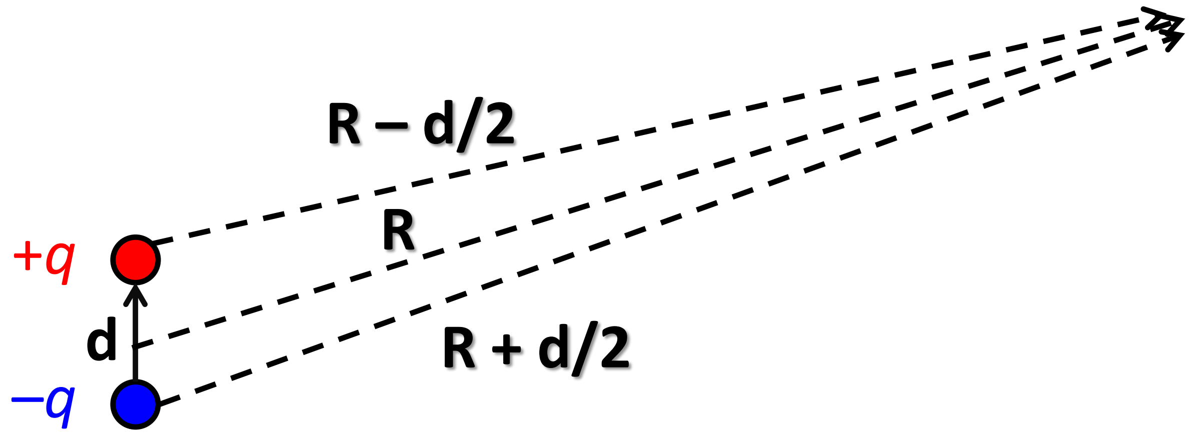
$$\theta = \pi \qquad \mathbf{E} \sim -\frac{2p}{4\pi\varepsilon_0 R^3}\mathbf{e}_r$$



WIKI

Example 006: Revisiting a Dipole

***Electric Dipole*: Estimating an electric field far from the dipole?**



$$\boxed{V = \frac{1}{4\pi\varepsilon_0}\sum_{k=1}^{N} q_k \frac{1}{|\mathbf{x}-\mathbf{x}_k|}}$$

$$V = \frac{1}{4\pi\varepsilon_0}\sum_{k=1}^{N} q_k \frac{1}{|\mathbf{x}-\mathbf{x}_k|} = \frac{q}{4\pi\varepsilon_0}\left(\frac{1}{|\mathbf{R}-\mathbf{d}/2|} - \frac{1}{|\mathbf{R}+\mathbf{d}/2|}\right)$$

Example 006: Revisiting a Dipole

$$\left|\mathbf{R}-\mathbf{d}/2\right|^{-1}=\left[\left(\mathbf{R}-\mathbf{d}/2\right)\cdot\left(\mathbf{R}-\mathbf{d}/2\right)\right]^{-1/2}$$

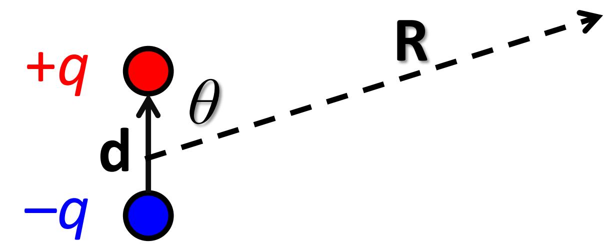
$$=\left[\left|\mathbf{R}\right|^2-\mathbf{R}\cdot\mathbf{d}+\left|\mathbf{d}\right|^2/4\right]^{-1/2}$$

$$=\frac{1}{\left|\mathbf{R}\right|}\left[1-\frac{\mathbf{R}\cdot\mathbf{d}}{\left|\mathbf{R}\right|^2}+\frac{\left|\mathbf{d}\right|^2}{4\left|\mathbf{R}\right|^2}\right]^{-1/2}\sim\frac{1}{\left|\mathbf{R}\right|}\left(1+\frac{\mathbf{R}\cdot\mathbf{d}}{2\left|\mathbf{R}\right|^2}\right)$$

$$V=\frac{q}{4\pi\varepsilon_0}\left(\frac{1}{\left|\mathbf{R}-\mathbf{d}/2\right|}-\frac{1}{\left|\mathbf{R}+\mathbf{d}/2\right|}\right)=\frac{q}{4\pi\varepsilon_0}\frac{\mathbf{R}\cdot\mathbf{d}}{\left|\mathbf{R}\right|^3}$$

Example 006: Revisiting a Dipole

$$V = \frac{q}{4\pi\varepsilon_0}\frac{\mathbf{R}\cdot\mathbf{d}}{|\mathbf{R}|^3} = \frac{q}{4\pi\varepsilon_0}\frac{d\cos\theta}{R^2}$$



Electric Dipole Moment

$\mathbf{p} = q\mathbf{d}$

$$V = \frac{p}{4\pi\varepsilon_0}\frac{\cos\theta}{R^2}$$

$$\mathbf{E} = -\nabla V = -\mathbf{e}_r\frac{\partial}{\partial R}\left(\frac{p\cos\theta}{4\pi\varepsilon_0 R^2}\right) - \mathbf{e}_\theta\frac{\partial}{R\partial\theta}\left(\frac{p\cos\theta}{4\pi\varepsilon_0 R^2}\right)$$

$$= \mathbf{e}_r 2\frac{p\cos\theta}{4\pi\varepsilon_0 R^3} + \mathbf{e}_\theta\frac{p\sin\theta}{4\pi\varepsilon_0 R^3} = \frac{p}{4\pi\varepsilon_0 R^3}\left(\mathbf{e}_r 2\cos\theta + \mathbf{e}_\theta\sin\theta\right)$$

Same result through a simpler process!

Remind: Example 002

Example 3.1 (a) Find **E** on the symmetry axis of a ring with radius R and uniform charge per unit length λ. (b) Use the results of part (a) to find **E** on the symmetry axis of a disk with radius R and uniform charge per unit area σ. (c) Use the results of part (b) to find **E** for an infinite sheet with uniform charge density σ. Discuss the matching condition at $z = 0$.

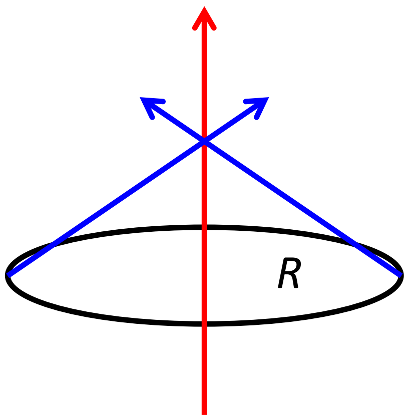
(a) $$\mathbf{E} = \frac{1}{4\pi\varepsilon_0}\int_V \frac{\mathbf{x}-\mathbf{x}'}{|\mathbf{x}-\mathbf{x}'|^3}\rho(\mathbf{x}')d^3x'$$

In 1D: $$\boxed{\mathbf{E} = \frac{1}{4\pi\varepsilon_0}\int_C \frac{\mathbf{x}-\mathbf{x}'}{|\mathbf{x}-\mathbf{x}'|^3}\rho(\mathbf{x}')dx'}$$

z-component

$$\mathbf{E} = \frac{1}{4\pi\varepsilon_0}\int_0^{2\pi}\mathbf{e}_z \frac{\sqrt{z^2+R^2}\dfrac{z}{\sqrt{z^2+R^2}}}{\left(\sqrt{z^2+R^2}\right)^3}\lambda R d\varphi$$

$$= \frac{2\pi R\lambda}{4\pi\varepsilon_0}\mathbf{e}_z \frac{z}{\left(\sqrt{z^2+R^2}\right)^3} = \frac{\lambda R}{2\varepsilon_0}\frac{z}{\left(z^2+R^2\right)^{\frac{3}{2}}}\mathbf{e}_z$$



$$\boxed{\mathbf{E} = \frac{\lambda R}{2\varepsilon_0}\frac{z}{\left(z^2+R^2\right)^{\frac{3}{2}}}\mathbf{e}_z}$$

Remind: Example 002

Example 3.1 (a) Find $\mathbf{E}$ on the symmetry axis of a ring with radius R and uniform charge per unit length λ. (b) Use the results of part (a) to find $\mathbf{E}$ on the symmetry axis of a disk with radius R and uniform charge per unit area σ. (c) Use the results of part (b) to find $\mathbf{E}$ for an infinite sheet with uniform charge density σ. Discuss the matching condition at $z = 0$.

(b)

Electric Field induced by $q = 2\pi R\lambda$

Electric Field induced by the unit charge

Ring: $$\mathbf{E} = \frac{\lambda R}{2\varepsilon_0}\frac{z}{\left(z^2+R^2\right)^{\frac{3}{2}}}\mathbf{e}_z \Rightarrow \mathbf{E} = \frac{1}{4\pi\varepsilon_0}\frac{z}{\left(z^2+R^2\right)^{\frac{3}{2}}}\mathbf{e}_z$$

Disk: Integration of rings ➔ *Superposition Principle*

$$\mathbf{E} = \frac{1}{4\pi\varepsilon_0}\int_0^R \frac{2\pi r\sigma z}{\left(z^2+r^2\right)^{\frac{3}{2}}}\mathbf{e}_z dr = \frac{\sigma z}{2\varepsilon_0}\mathbf{e}_z\int_0^R \frac{r}{\left(z^2+r^2\right)^{\frac{3}{2}}}dr$$

$$= \frac{\sigma z}{2\varepsilon_0}\mathbf{e}_z\int_0^R \frac{r}{\left(z^2+r^2\right)^{\frac{3}{2}}}dr = \frac{\sigma}{2\varepsilon_0}\mathbf{e}_z\left(1-\frac{z}{\sqrt{z^2+R^2}}\right),$$

$$\boxed{\mathbf{E} = \frac{\sigma}{2\varepsilon_0}\mathbf{e}_z\left(1-\frac{z}{\sqrt{z^2+R^2}}\right)}$$

Remind: Example 002

Example 3.1 (a) Find **E** on the symmetry axis of a ring with radius R and uniform charge per unit length λ. (b) Use the results of part (a) to find **E** on the symmetry axis of a disk with radius R and uniform charge per unit area σ. (c) Use the results of part (b) to find **E** for an infinite sheet with uniform charge density σ. Discuss the matching condition at $z = 0$.

(c)

Consider some interesting cases… ($z > 0$)

Ring

$$\boxed{\mathbf{E} = \frac{\lambda R}{2\varepsilon_0}\frac{z}{\left(z^2+R^2\right)^{\frac{3}{2}}}\mathbf{e}_z}$$

Disk

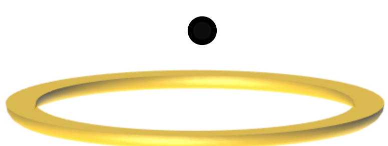
$$\boxed{\mathbf{E} = \frac{\sigma}{2\varepsilon_0}\mathbf{e}_z\left(1-\frac{z}{\sqrt{z^2+R^2}}\right)}$$

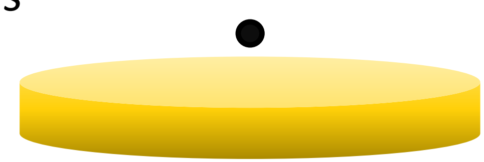
$R >> z$

$$\mathbf{E} \sim \frac{\lambda}{2\varepsilon_0}\frac{z}{R^2}\mathbf{e}_z = \frac{Q}{4\pi\varepsilon_0}\frac{z}{R^3}\mathbf{e}_z$$

$$\mathbf{E} \sim \frac{\sigma}{2\varepsilon_0}\mathbf{e}_z\left(1-\frac{z}{R}\right)$$

Near the structures





Remind: Example 002

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(c)

Consider some interesting cases… (z > 0)

Ring

$$\boxed{\mathbf{E} = \frac{\lambda R}{2\varepsilon_0} \frac{z}{\left(z^2 + R^2\right)^{\frac{3}{2}}} \mathbf{e}_z}$$

Disk

$$\boxed{\mathbf{E} = \frac{\sigma}{2\varepsilon_0} \mathbf{e}_z \left(1 - \frac{z}{\sqrt{z^2 + R^2}}\right)}$$

Discontinuity! (Discussed later)

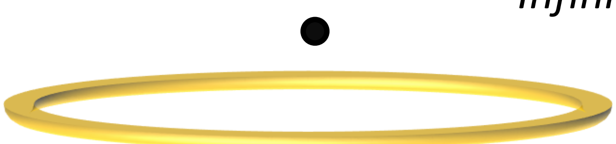
$R \to \infty$

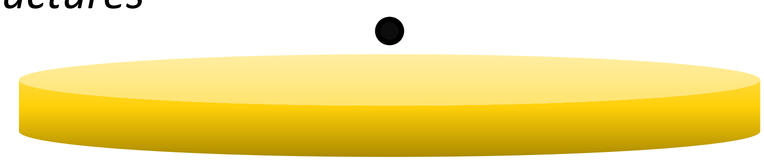
$$\mathbf{E} \sim 0$$

$$\mathbf{E} \sim +\frac{\sigma}{2\varepsilon_0}\mathbf{e}_z \quad (z > 0), \qquad \mathbf{E} \sim -\frac{\sigma}{2\varepsilon_0}\mathbf{e}_z \quad (z < 0)$$

Infinite structures

Uniform





Remind: Example 002

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(c)

Consider some interesting cases... ($z > 0$)

Ring

$$\boxed{\mathbf{E} = \frac{\lambda R}{2\varepsilon_0} \frac{z}{\left(z^2 + R^2\right)^{\frac{3}{2}}} \mathbf{e}_z}$$

Disk

$$\boxed{\mathbf{E} = \frac{\sigma}{2\varepsilon_0} \mathbf{e}_z \left(1 - \frac{z}{\sqrt{z^2 + R^2}}\right)}$$

$$\sim 1 - \frac{1}{2}\left(\frac{R}{z}\right)^2$$

Taylor Expansion $\frac{R}{z}$

$R << z$

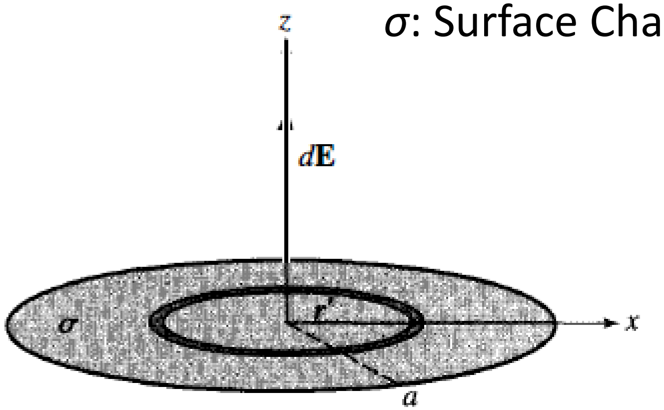
$$\mathbf{E} \sim \frac{\lambda}{2\varepsilon_0} \frac{R}{z^2} \mathbf{e}_z = \frac{Q}{4\pi\varepsilon_0} \frac{1}{z^2} \mathbf{e}_z$$

$$\mathbf{E} \sim \frac{Q}{4\pi\varepsilon_0} \frac{1}{z^2} \mathbf{e}_z$$

Far from the structures
~ Point-like behaviors!

Example 007: Revisiting a Disk

EXAMPLE 9 What is the electrostatic potential, and electric field, above a uniformly charged circular plate of radius a, on the axis of symmetry?



σ: Surface Charge Density

$$\boxed{V = \frac{1}{4\pi\varepsilon_0}\int_V \frac{1}{|\mathbf{x}-\mathbf{x}'|}\rho(\mathbf{x}')d^3x'}$$

$$V = \frac{1}{4\pi\varepsilon_0}\int_0^{2\pi}\int_0^a \frac{1}{\sqrt{r'^2+z^2}}\sigma r'dr'd\varphi$$

$$= \frac{\sigma}{2\varepsilon_0}\int_0^a \frac{r'}{\sqrt{r'^2+z^2}}dr' = \frac{\sigma}{2\varepsilon_0}\int_{z^2}^{a^2+z^2}\frac{1}{2\sqrt{t}}dt$$

$$= \frac{\sigma}{2\varepsilon_0}\left(\sqrt{a^2+z^2}-|z|\right)$$

Example 007: Revisiting a Disk

$$V = \frac{\sigma}{2\varepsilon_0}\left(\sqrt{a^2+z^2} - |z|\right)$$

$$z>0: \quad V = \frac{\sigma}{2\varepsilon_0}\left(\sqrt{a^2+z^2} - z\right)$$

$$\mathbf{E} = -\nabla V = -\mathbf{e}_z \frac{\sigma}{2\varepsilon_0}\frac{\partial}{\partial z}\left(\sqrt{a^2+z^2} - z\right) = \mathbf{e}_z \frac{\sigma}{2\varepsilon_0}\left(1 - \frac{z}{\sqrt{a^2+z^2}}\right)$$

Same result through a simpler process!

$$z<0: \quad V = \frac{\sigma}{2\varepsilon_0}\left(\sqrt{a^2+z^2} + z\right)$$

$$\mathbf{E} = -\nabla V = -\mathbf{e}_z \frac{\sigma}{2\varepsilon_0}\frac{\partial}{\partial z}\left(\sqrt{a^2+z^2} + z\right) = \mathbf{e}_z \frac{\sigma}{2\varepsilon_0}\left(-1 - \frac{z}{\sqrt{a^2+z^2}}\right)$$

Electrostatics – Poisson's Equation & More

Remind: Postulates: Differential Form

$\varepsilon_r = 1$ in the Vacuum

$$\nabla \times \mathbf{E} = \mathbf{O}$$

$$\nabla \cdot \mathbf{D} = \rho$$

$$\mathbf{D} = \varepsilon_0 \mathbf{E}$$

$$\Rightarrow$$

$$\boxed{\begin{aligned} \nabla \times \mathbf{E} &= \mathbf{O} \\ \nabla \cdot \mathbf{E} &= \frac{\rho}{\varepsilon_0} \end{aligned}}$$

Poisson's Equation in a Vacuum

$\varepsilon_r = 1$ in the Vacuum

$$\nabla \times \mathbf{E} = \mathbf{O} \qquad \nabla \cdot \mathbf{E} = \frac{\rho}{\varepsilon_0}$$

Already related...

Need to be related...

$$\mathbf{E} = -\nabla V$$

$$\nabla \cdot (-\nabla V) = \frac{\rho}{\varepsilon_0} \Rightarrow \boxed{-\nabla^2 V = \frac{\rho}{\varepsilon_0}}$$

Poisson's Equation

Governing Eq. for Electrostatics in a vacuum (with B.C.)

Discussed later...But, B.C.?

Poisson's Equation in a Homogeneous Material

$$\nabla \times \mathbf{E} = \mathbf{O} \qquad \nabla \cdot \mathbf{D} = \rho$$

Already related...

$$\mathbf{E} = -\nabla V$$

Need to be related...

$$\mathbf{D} = \varepsilon_0 \varepsilon_r \mathbf{E}$$

Poisson's Equation

$$\nabla \cdot \left(-\nabla V \right) = \frac{\rho}{\varepsilon_0 \varepsilon_r} \Rightarrow \boxed{-\nabla^2 V = \frac{\rho}{\varepsilon_0 \varepsilon_r}}$$

Governing Eq. for Electrostatics in a Homogeneous Material (with B.C.)

Discussed later…But, B.C.?

Governing Equation in an Inhomogeneous Material

$$\nabla \times \mathbf{E} = \mathbf{O} \qquad \nabla \cdot \mathbf{D} = \rho$$

Already related...

$$\mathbf{E} = -\nabla V$$

Need to be related...

$$\mathbf{D} = \varepsilon_0 \varepsilon_r(\mathbf{x})\mathbf{E}$$

$$\nabla \cdot \left(\varepsilon_r(\mathbf{x})\mathbf{E}\right) = \varepsilon_r(\mathbf{x})\nabla \cdot \mathbf{E} + \left(\nabla \varepsilon_r(\mathbf{x})\right) \cdot \mathbf{E}$$

$$\boxed{-\varepsilon_r(\mathbf{x})\nabla^2 V - \left(\nabla \varepsilon_r(\mathbf{x})\right) \cdot \left(\nabla V\right) = \frac{\rho}{\varepsilon_0}}$$

Governing Eq. for Electrostatics in an Inhomogeneous Material (with B.C.)

Mathematical View

Electrostatics – Boundary Conditions

Maxwell's Equations: Electrostatics

$$\nabla \times \mathbf{E} = \mathbf{O}$$

$$\nabla \cdot \mathbf{D} = \rho$$

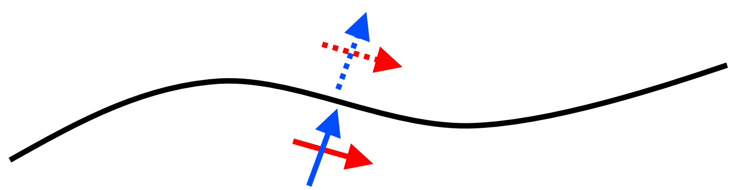
D will be discussed later…

Gauss & Stokes

$$\oint_C \mathbf{E} \cdot d\mathbf{l} = 0$$

$$\oint_S \mathbf{D} \cdot d\mathbf{S} = Q$$

Boundary Conditions: Connecting "Fields" across the boundary



Tangential & *Normal* Fields!

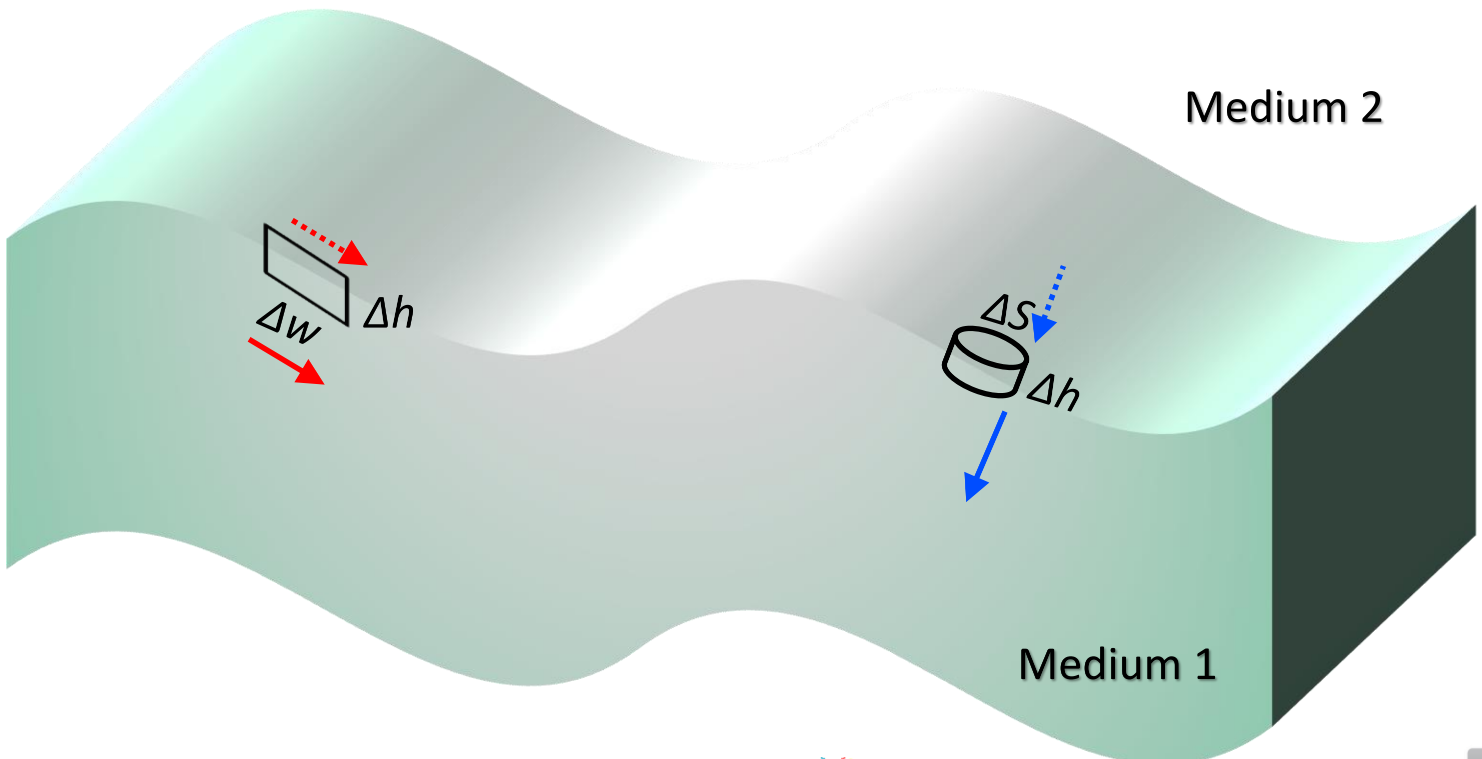
Strategy for Boundary Conditions

I. Boundary includes "different" materials ➔ Integral forms are proper

II. Stokes ➔ Closed "Loop" across materials
Gauss ➔ "Closed Surface" across materials

$$\oint_C \mathbf{E} \cdot d\mathbf{l} = 0 \qquad \oint_S \mathbf{D} \cdot d\mathbf{S} = Q$$

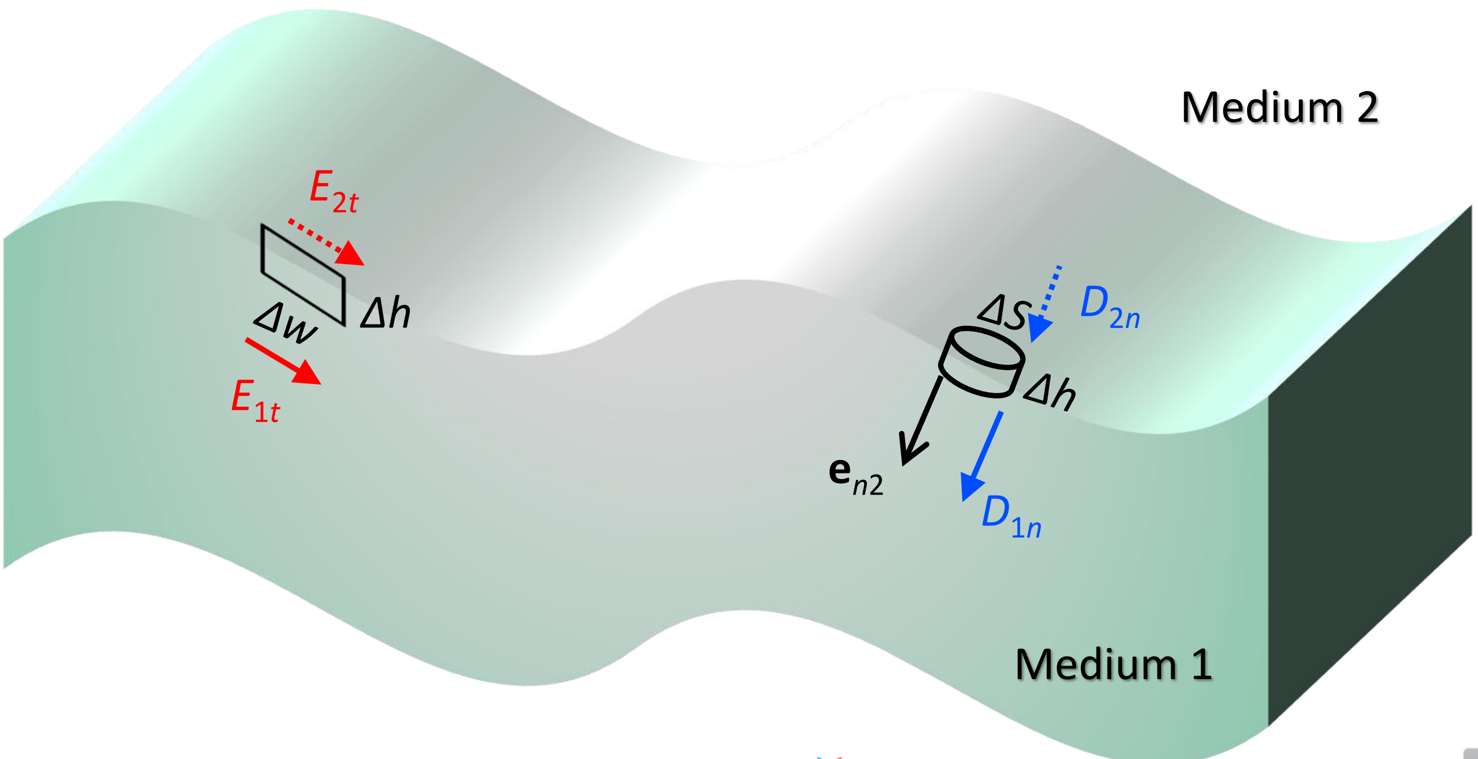
III. Loop measures tangential fields & Surface measures normal fields



Analyzing Boundary Conditions

$\Delta h \rightarrow 0$ to characterize the "boundary"

$$\oint_C \mathbf{E}\cdot d\mathbf{l}\Big|_{\Delta h=0} = E_{1t}\Delta w - E_{2t}\Delta w = 0 \qquad \oint_S \mathbf{D}\cdot d\mathbf{S} = \mathbf{e}_{n2}\cdot\left(\mathbf{D}_1 - \mathbf{D}_2\right)\Delta S = \rho_s \Delta S$$



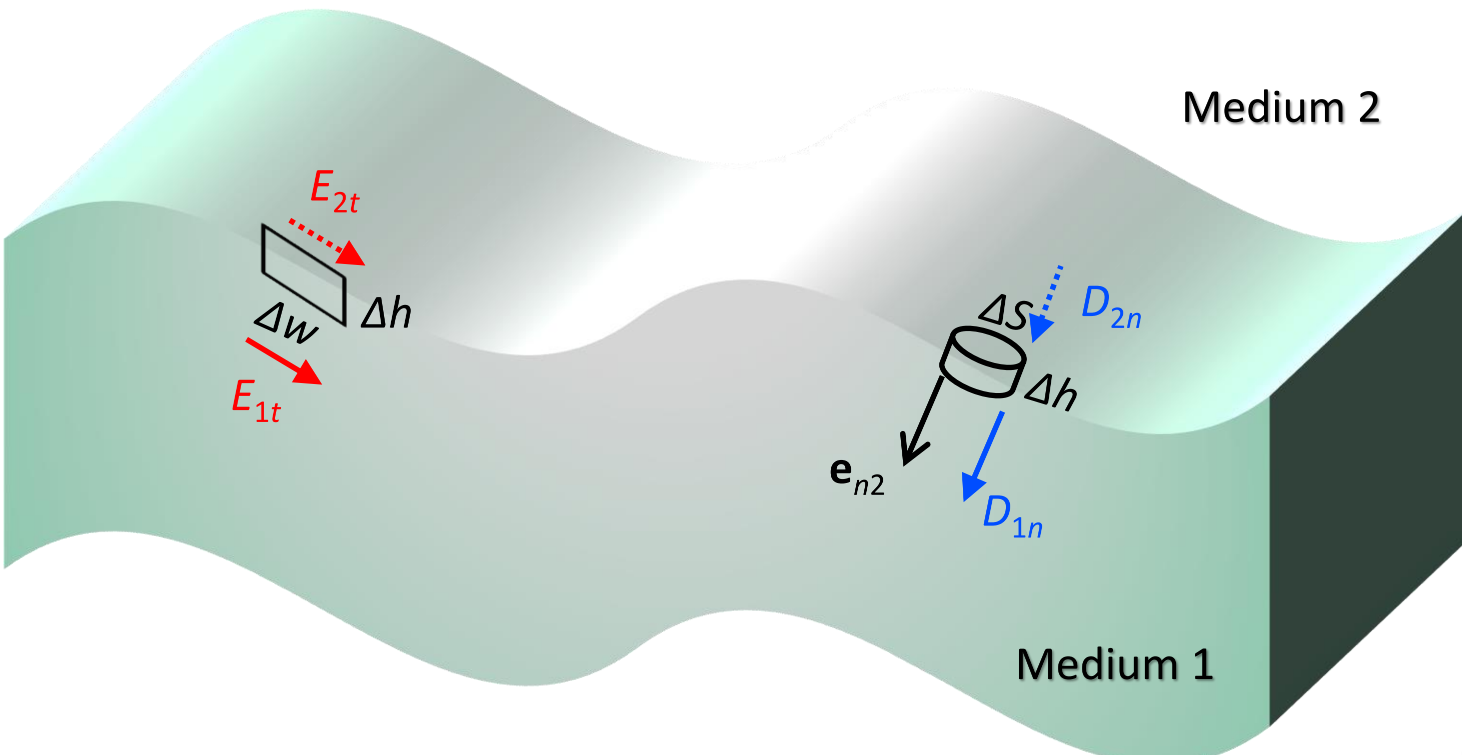
Boundary Conditions: Electrostatics

Tangential Fields

$$E_{1t} = E_{2t}$$

Normal Fields

$$\mathbf{e}_{n2} \cdot \left(\mathbf{D}_1 - \mathbf{D}_2 \right) = \rho_s$$



Special Topic: What if $E_{1t} \neq E_{2t}$?

$$E_{1t} \neq E_{2t} \quad \Rightarrow \quad \oint_C \mathbf{E} \cdot d\mathbf{l}\Big|_{\Delta h=0} = E_{1t}\Delta w - E_{2t}\Delta w \neq 0$$

$$\int_S \left(\nabla \times \mathbf{E}\right) \cdot d\mathbf{s}\Big|_{\Delta h=0} \neq 0$$

Singular Existence of $\nabla \times \mathbf{E}$

Special Topic: What if $E_{1t} \neq E_{2t}$?

Now, look at more general Maxwell's equations

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \qquad \nabla \times \mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t} \qquad \nabla \cdot \mathbf{D} = \rho \qquad \nabla \cdot \mathbf{B} = 0$$

Singular Existence of $\nabla \times \mathbf{E}$ $\Rightarrow$ $\nabla \cdot \mathbf{B} = \rho_M^{\neq 0}$ & $\nabla \times \mathbf{E} = -\mathbf{J}_M^{\neq 0} - \dfrac{\partial \mathbf{B}}{\partial t}$

$\therefore$ ***Magnetic Charges*** or ***Magnetic Currents*** should exist for the discontinuity in the tangential component of an electric field!

I. ***Until know, a magnetic monopole has not been discovered!***

II. ***Magnetic multipoles (dipoles, quadrupoles, …) require spatial "distances"***, *prohibiting the singular existence at the boundary*

$\therefore$ The tangential component of an electric field should be continuous (*until now*!)