storage requirements during the solution of the ODE's. After the application of constraints, the n coupled equations and can be put together in matrix form.

$$M\ddot{\underset{\sim}{q}} + K\underset{\sim}{q} = \underset{\sim}{Q}$$

where M and K are mass and stiffness matrices of size $(n \times n)$. $\underset{\sim}{Q}$ is the forcing vector of size n, and $\underset{\sim}{q}$ are the degrees of freedom. The degrees of freedom $\underset{\sim}{q}$ are generalized displacements (displacement or angles, with units of m or in, and radians). M has units of kg. K has units of N/m. Q has units of N. For natural response, set $\underset{\sim}{Q} = \underset{\sim}{0}$ and seek solution of the form $\underset{\sim}{q} = \underset{\sim}{q_0} e^{j\omega t}$. This leads to the same algebraic eigenvalue problem as discussed earlier in the case of Galerkin and Rayleigh-Ritz methods,

$$\begin{aligned} K\underset{\sim}{q_0} &= \omega^2 M \underset{\sim}{q_0} \\ \left(K - \omega^2 M\right)\underset{\sim}{q_0} &= 0 \end{aligned} \tag{2.88}$$

It leads to the solutions ω_i where $i = 1, 2, \ldots, n$. Corresponding to each ω_i there exists a solution $\underset{\sim}{q_{0i}}$ which satisfies the equation

$$K\underset{\sim}{q_{0i}} = \omega_i^2 M \underset{\sim}{q_{0i}}$$

ω_i and its corresponding $\underset{\sim}{q_{0i}}$ are called the eigenvalues and eigenvectors of the system. The mode shapes of the beam can be extracted from the eigenvectors. Consider the example of the beam discretized into three elements as before. Consider a simply-supported case at the root end. Let the i-th eigenvector be $\underset{\sim}{q_{0i}} = [q_{02}\, q_{03}\, q_{04}\, q_{05}\, q_{06}\, q_{07}\, q_{08}]^T$. The mode shape ϕ_i corresponding to this eigenvectors can then be constructed using the shape functions as follows.

$$\phi_j(r) = \begin{cases} w_1(r) = \qquad\qquad H_2(s)q_{02} + H_3(s)q_{03} + H_4(s)q_{04} & r_1 < r < r_2 \quad s = r - r_1 \\ w_2(r) = H_1(s)q_{03} + H_2(s)q_{04} + H_3(s)q_{05} + H_4(s)q_{06} & r_2 < r < r_3 \quad s = r - r_2 \\ w_3(r) = H_1(s)q_{05} + H_2(s)q_{06} + H_3(s)q_{07} + H_4(s)q_{08} & r_3 < r < R \quad s = r - r_3 \end{cases}$$

Note that the shape functions obtained here correspond to the rotating beam. Thus an important property is that they are orthogonal with respect to mass and rotational stiffness.

$$\int_0^R m\phi_i(r)\phi_j(r)\,dr \quad = \quad \delta_{ij} M_i \tag{2.89}$$

and

$$\int_0^R \left[EI \frac{d^2\phi_i(r)}{dr^2}\frac{d^2\phi_j(r)}{dr^2} + T\frac{d\phi_i(r)}{dr}\frac{d\phi_j(r)}{dr}\right] dr \quad = \quad \delta_{ij}\omega_i^2 M_i \tag{2.90}$$

where δ_{ij} is Kronecker's delta and M_i is generalized mass

$$M_i = \int_0^R m(r)\phi_i^2\,dr$$

2.6 Fan plot and frequency plots for rotating beams

The natural modes of a structure represent the unique ways it can vibrate in vacuum and without damping. The lowest frequency is called the fundamental frequency and the corresponding mode is called the fundamental mode. The natural frequencies of a rotating blade depend on its mass and stiffness properties, boundary conditions, and rotational speed. The rotational speed supplies centrifugal stiffness. At low rotational speeds, the beam stiffness is more important than the centrifugal stiffness. At higher rotational speeds, the centrifugal stiffness is more important than the beam stiffness. At still higher rotational speeds, the beam behaves like a string, the fundamental natural frequency assymptotes to the rotational frequency. The rotating frequencies are always greater than non-rotating frequencies. However, there is only a slight change in mode shapes from non-rotating to the rotating ones. In the following sub-sections the natural frequencies of a uniform, rotating beam are studied. The frequency and mode shape calculations are performed using a Rayleigh-Ritz type finite element analysis with ten equal length elements.

2.6.1 Rotating versus non-rotating frequencies

For a given rotational speed, the blade rotating frequencies are determined by the mass and stiffness of the blade, and the boundary conditions. Consider a cantilevered non-rotating beam with uniform properties EI and m. This is a simple model for a hingeless blade.

First, solve eqn.2.29 to get the non-rotating frequencies ω_{NR_1}, ω_{NR_2}, ω_{NR_3} etc. The first frequency, or the lowest, $\omega_{NR_1} = \omega_{NR}$ say, is called the fundamental frequency. Note that these frequencies are of the form $f_j\sqrt{EI/mR^4}$ as given in eqns.2.37 and 2.41. Now consider a rotational speed Ω. Corresponding to this Ω solve eqn.2.46 to obtain the rotating frequencies ω_{R_1}, ω_{R_2}, ω_{R_3} etc. Again, $\omega_{R_1} = \omega_R$ is the fundamental frequency, this time that of the rotating beam. Varying the stiffness EI, a set of ω_{NR} and ω_R can be obtained. Thus one can obtain a plot of ω_R versus ω_{NR}. This plot corresponds to the specific set of beam properties and a given Ω.

If the frequencies are non-dimensionalized with respect to rotational speed Ω, i.e. ω_R/Ω versus ω_{NR}/Ω, then the plot becomes representative of all uniform cantilevered beams at any given rotational speed. This is due to the following. We have

$$\frac{\omega_{NR}}{\Omega} = f_j\sqrt{\frac{EI}{m\Omega^2R^4}} \quad \text{where m = m}_0 \text{ for uniform beams}$$

Recall that, eqn.2.50 showed that the only parameter on which the non-dimensional rotational frequency, ω_R/Ω, of a uniform beam depend is $EI/m\Omega^2R^4$. This is the same parameter on the right hand side of the above expression. Varying ω_{NR}/Ω from zero onwards includes all variations of this parameter. Thus all beams, regardless of their properties EI, m, dimension R, and rotational speed Ω would correspond to a point on the plot of ω_R/Ω versus ω_{NR}/Ω. Such a plot, for the first mode, is shown in figure 2.19. Note that, different beams with different EI, m, R, and Ω can correspond to the same point on the plot as long as they have the same $EI/m_0\Omega^2R^4$. Therefore ω_{R_j}/Ω versus ω_{NR_j}/Ω plots are representative of all uniform beams of a specific boundary condition type. Figure 2.20 shows the variation of two higher modes in addition to the fundamental mode.

2.6.2 Rotating frequencies vs. rotational speed

For a given mass and stiffness, the rotating frequencies vary with the rotational speed (RPM). At zero RPM the frequency corresponds to a non-rotating beam. As RPM increases, the centrifugal force gradually stiffens the blade. Figure 2.21(a) shows the variation of rotating frequencies in Hz with RPM. The value at zero RPM is $3.52\sqrt{EI/mR^4}$ from eqns.2.36 and 2.37, where the following values have been assumed: $EI = 4.225\,e5$ Nm^2, $m = 13$ kg/m, and $R = 8.2$ m.

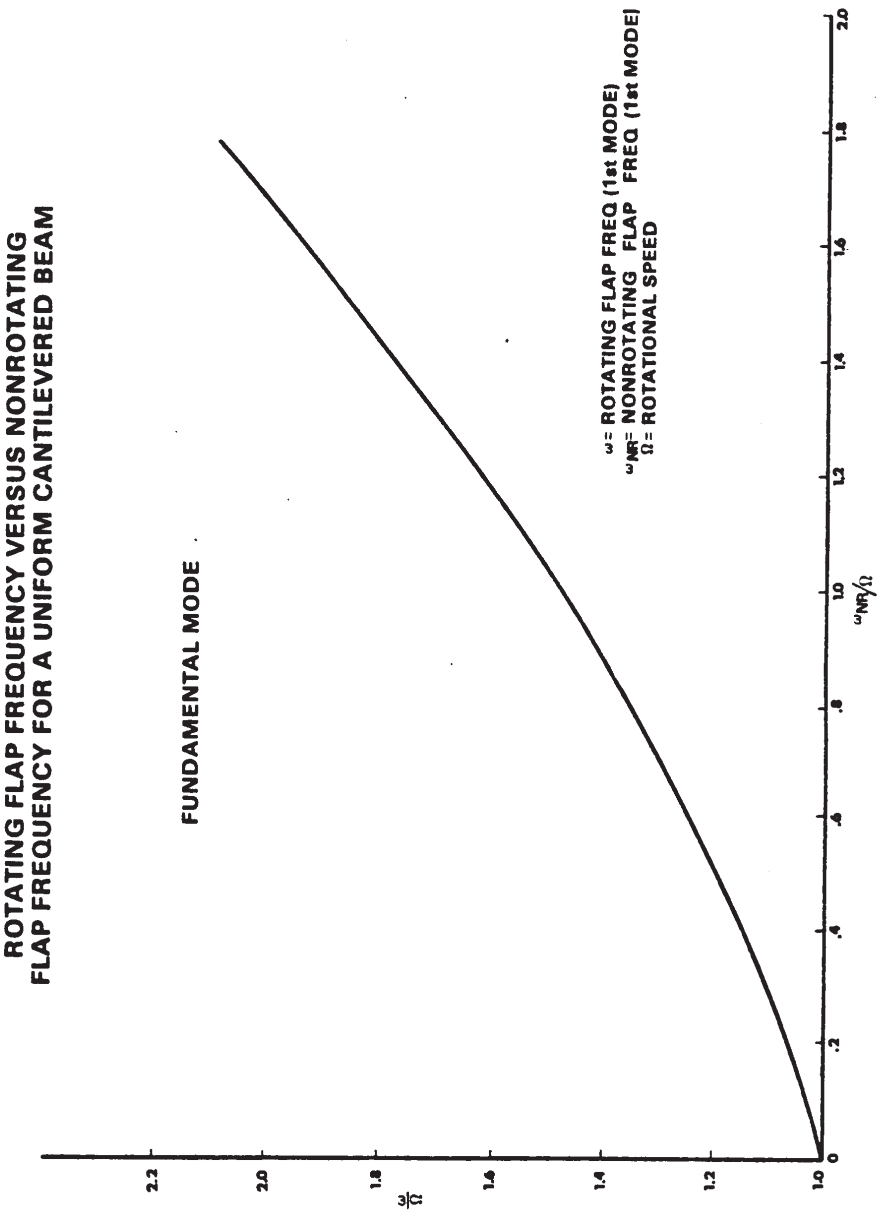


Figure 2.19: **Rotating natural frequencies as function of non-rotating natural frequencies for a uniform cantilevered beam: Fundamental mode**

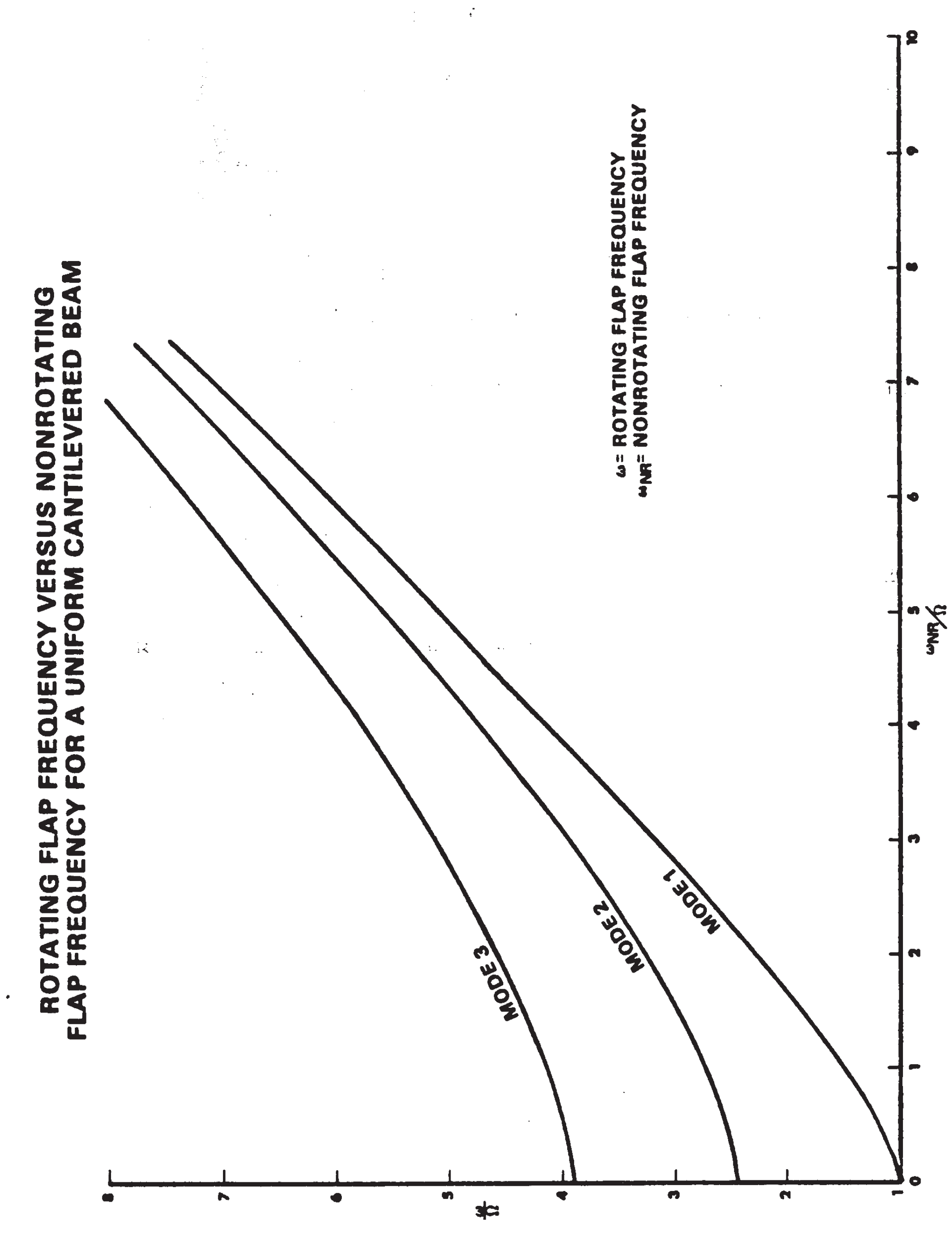


Figure 2.20: **Rotating natural frequencies as function of non-rotating natural frequencies for a uniform cantilevered beam: First three modes**

Let the operating RPM be 260. Then the x-axis is often conveniently representated in terms of the operating RPM, see Fig. 2.21(b)). The frequencies, instead of being in Hz can be non-dimensionalized at each rotor RPM. These frequencies, in per rev, are plotted in Figs. 2.21(c) and 2.21(d). These plots show the relative dominance of the centrifugal stiffness. A very high per rev value, as is the case for very lower RPM, signifies the dominance of bending stiffness. A lower per rev value, as is the case of higher RPM, signifies the dominance of centrifugal stiffness.

For design purposes it is often convenient to represent the frequencies in the following two formats. The first is called a fan plot. The second is called the non-dimensional frequency plot. The fan plot is same as the frequency plot of figure 2.21(b), except that it shows the 1/rev, 2/rev, 3/rev, etc lines in addition to the rotor frequencies. The rotor frequency can be read off in Hz. In addition, at any RPM an approximate per rev value can be estimated. For example, at the operating RPM the second mode lies between 3 and 4/rev, the third mode lies between 7 and 8/rev. It is desirable to design the blade structurally in such a way that the modal frequencies lie in between /rev lines. The aerodynamic forcing in steady flight occurs at 1/rev, 2/rev, 3/rev etc. Structural frequencies near these forcing harmonics expose the rotor to resonance. The non-dimensional frequency plot is same as the frequency plot of figure 2.21(b), except that the frequencies are non-dimensionalized with respect to the *operating RPM*. Note that this is different from figure 2.21(d) in that the frequencies are not divided by the rotor RPM, but the rotor RPM at the operating condition. Thus these are not /rev values. They equal the /rev values only at the operating RPM.

Frequency plots for a simple-supported beam (articulated rotor model) is shown in figures 2.23 and 2.24. The simple-supported beam has exactly the same properties as the cantilevered beam (hingeless rotor model). The only difference is in the boundary condition. The frequency trends are very similar for the higher modes. The key difference is in the fundamental mode. Figure 2.23(c) shows that the fundamental frequency is determined by the centrifugal stiffness regardless of the RPM. Thus it is always at 1/rev. Resonance is not a problem because of the high aerodynamic damping present in the flap mode (around 50%). On the contrary it is desirable to place the first frequency as close to 1/rev as possible to relieve the root bending moments. Under this condition the balance of the centrifugal and aerodynamic forces on the blade is used up completely by the blade flapping motion with zero moment transmitted to the root.

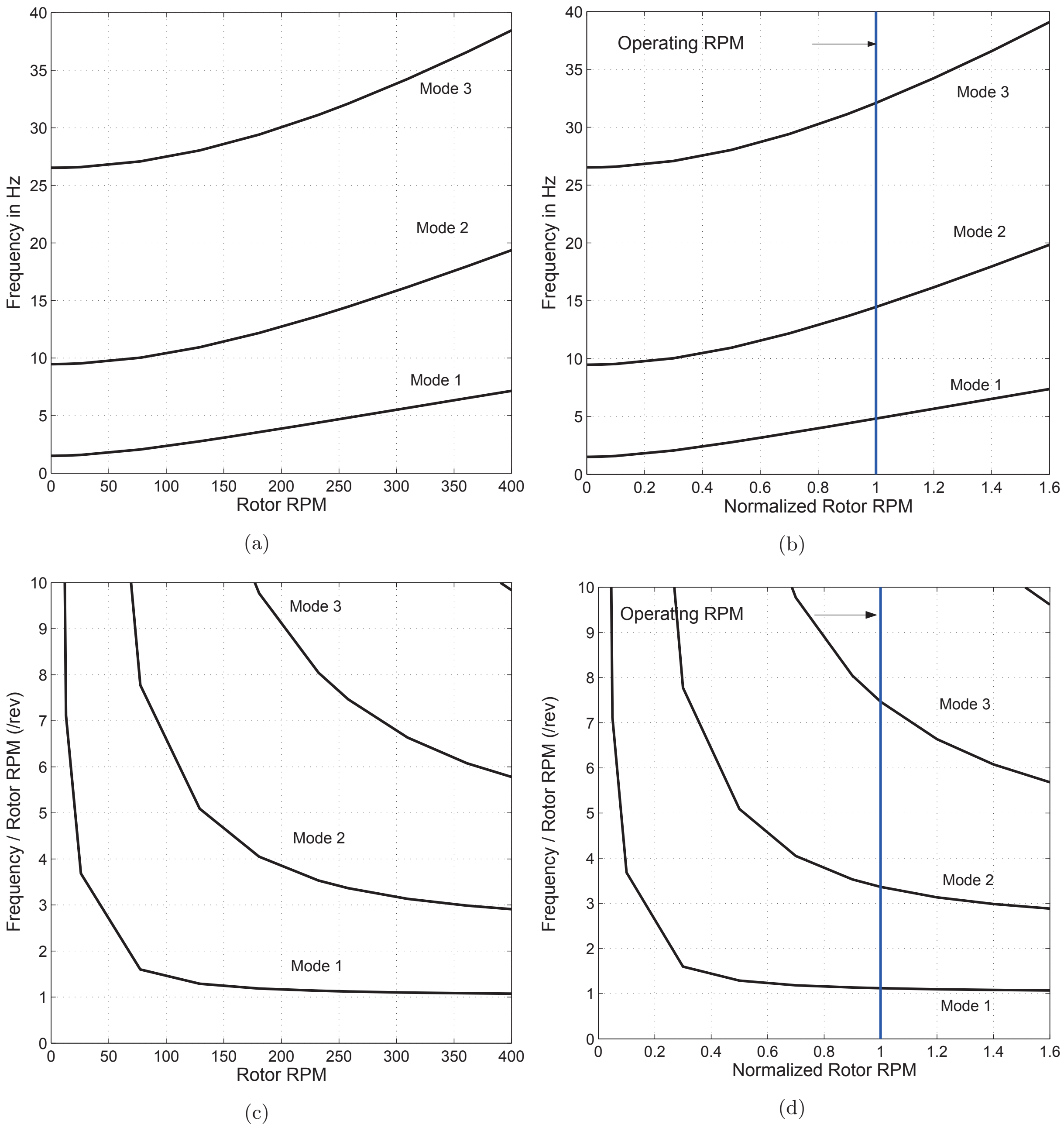
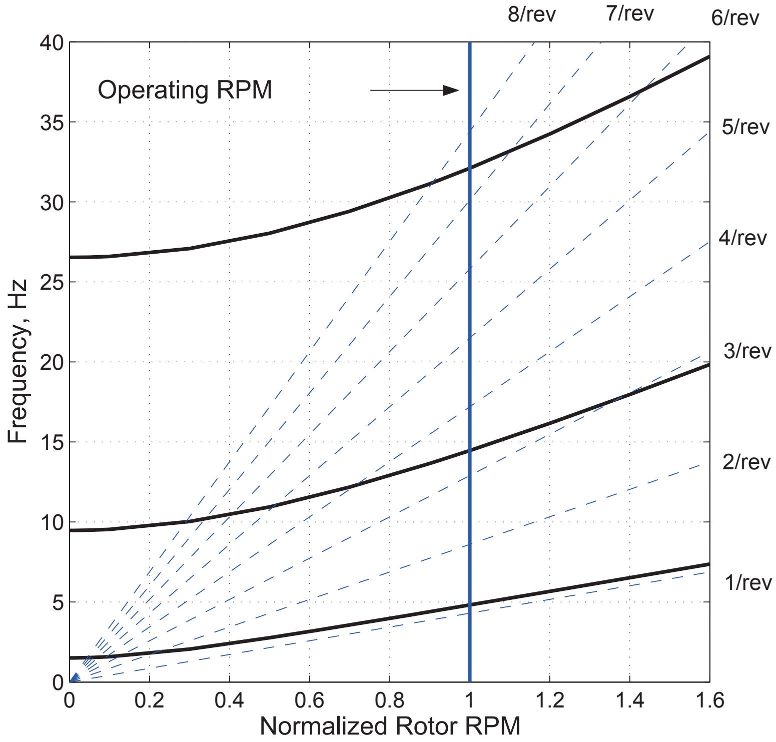
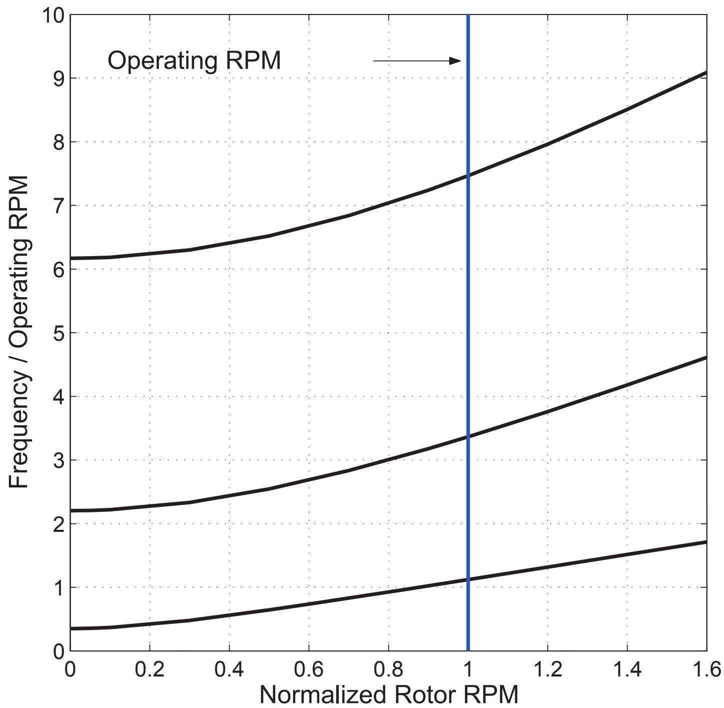


Figure 2.21: **Natural frequencies of a uniform cantilevered beam varying with RPM**



(a) Fan plot in Hz



(b) Frequency plot non-dimensionalized with respect to operating RPM

Figure 2.22: **Fan plot and Non-dimensional frequency plot for a hingeless rotor blade**

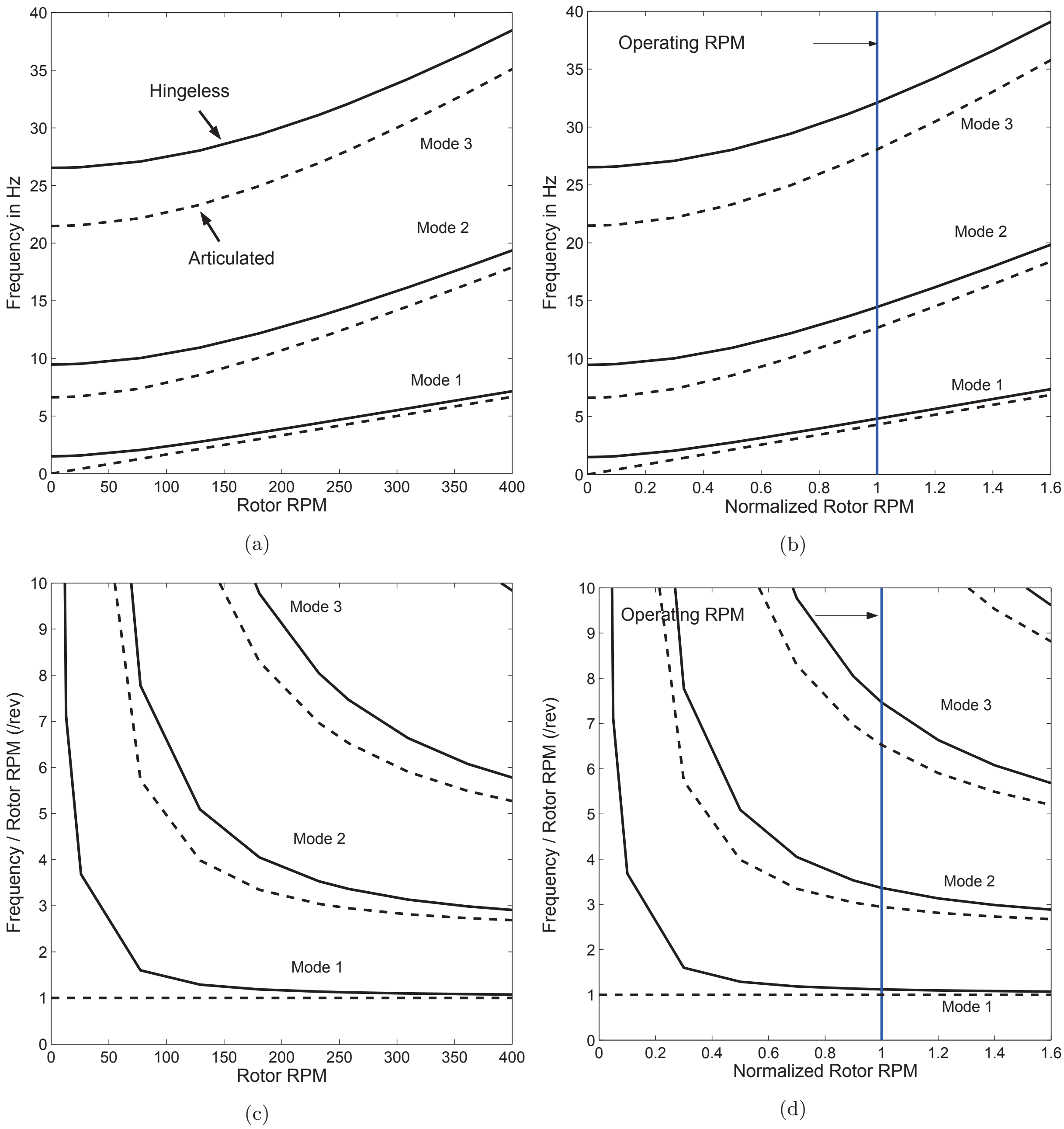
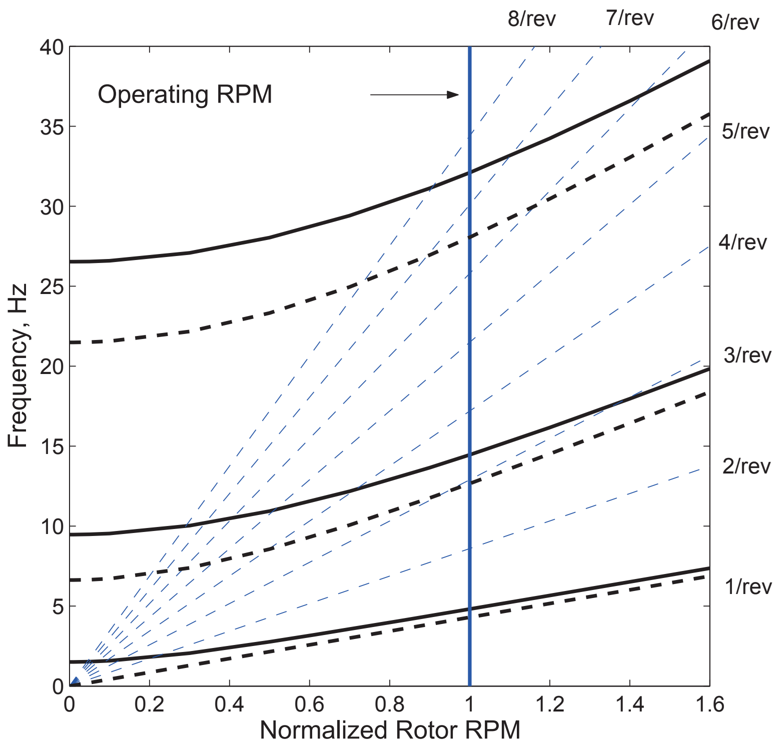
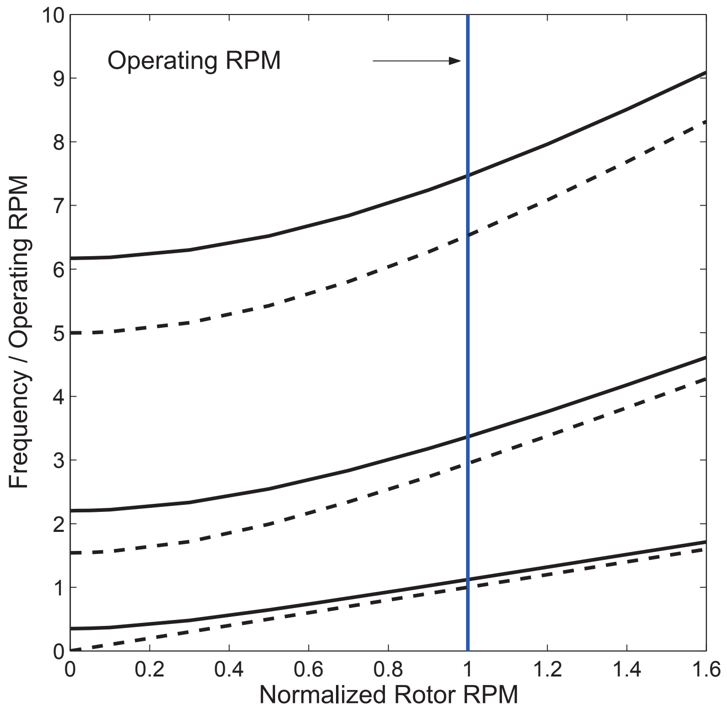


Figure 2.23: **Natural frequencies of an articulated rotor blade compared with a hingeless rotor blade**



(a) Fan plot in Hz



(b) Frequency plot non-dimensionalized with respect to operating RPM

Figure 2.24: **Fan plot and Non-dimensional frequency plot for a hingeless rotor blade**

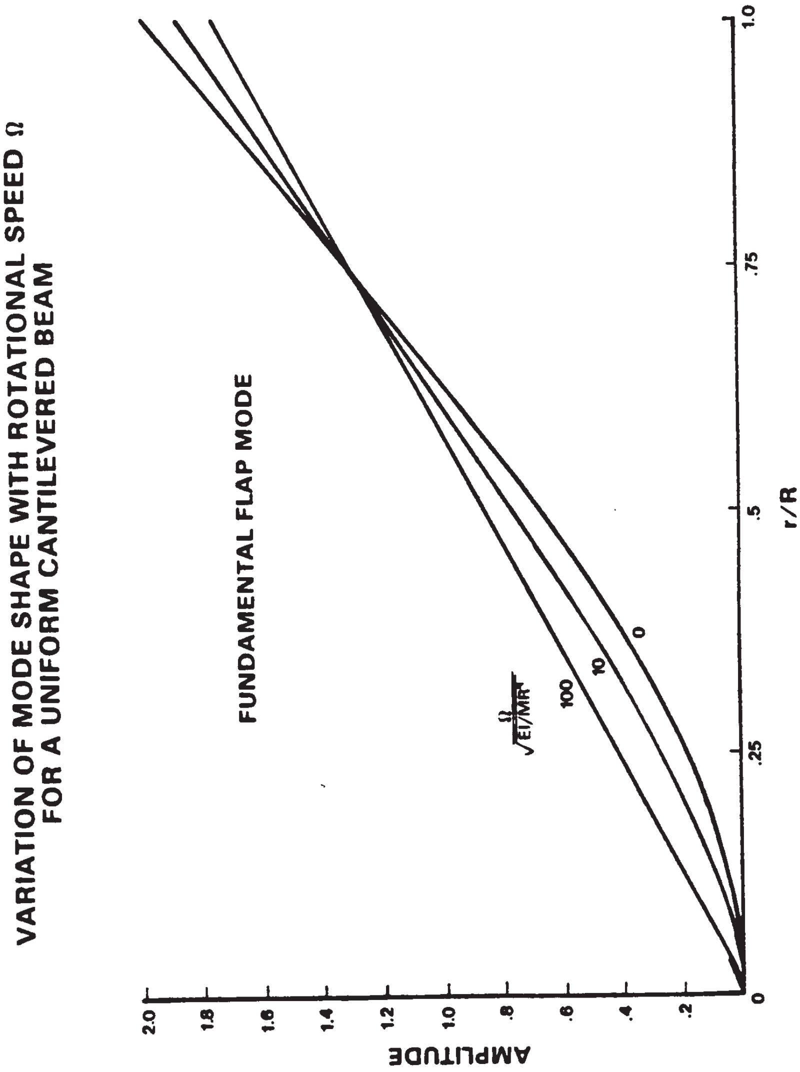


Figure 2.25: **Rotating mode shapes for a uniform cantilevered beam: Fundamental mode**

2.6.3 Rotating versus non-rotating mode shapes

It is clear that the parameter which makes the non-dimensional frequencies and mode shapes differ from one uniform beam to another is $EI/m\Omega^2 R^4$. The parameter can be re-arranged to read $\Omega/\sqrt{EI/mR^4}$. The frequencies and mode shapes of two beams operating at different values of Ω can still be same if EI, m, and R are such that the above parameter remains same. The effect of rotational speed on the mode shape can be seen only if this parameter is varied. Figure 3 and 4 shows such plots for the first and the second modes for cantilevered beams. Note that each line on a plot can represent the mode shape of different cantilevered beams with different rotational speeds, but all having the same $\Omega/\sqrt{EI/mR^4}$.

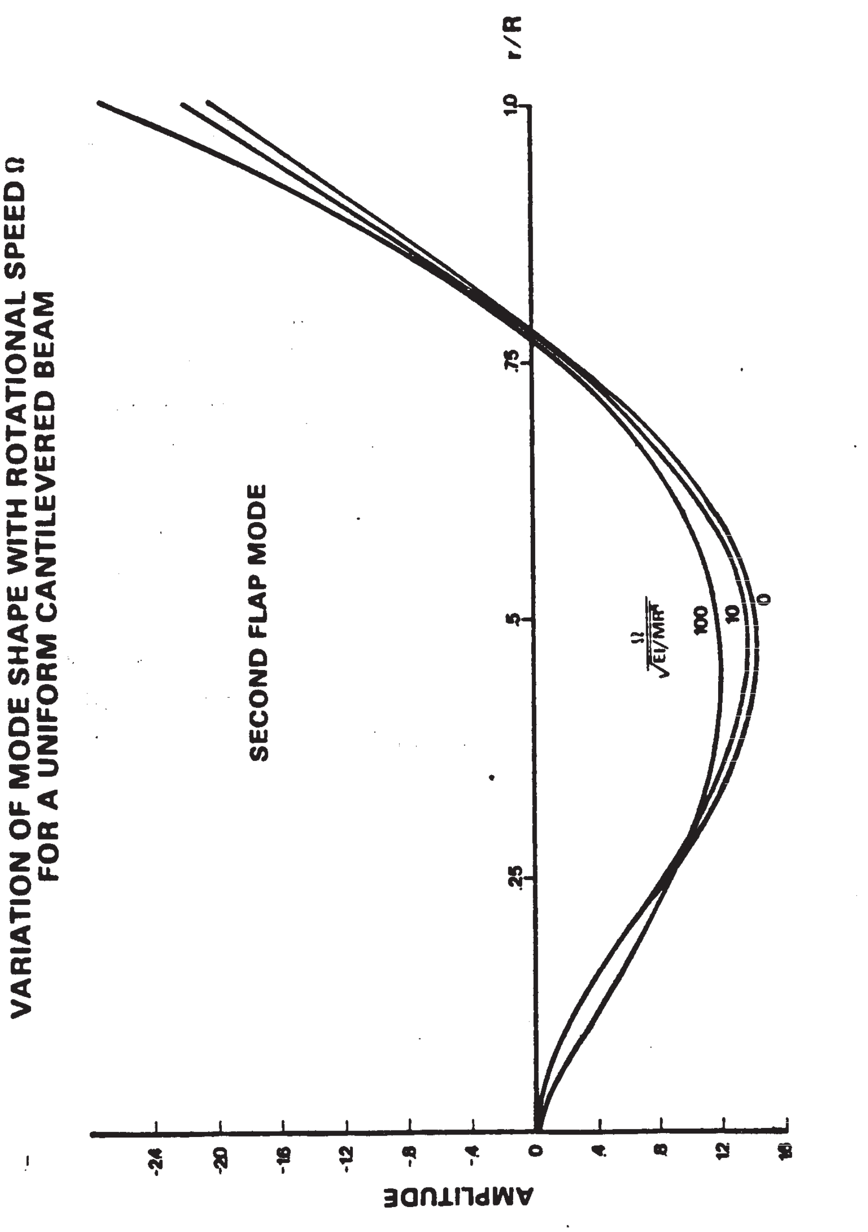


Figure 2.26: **Rotating mode shapes for a uniform cantilevered beam: First three modes**

2.7 Response Solution in time

After the natural vibration characteristics of the blade has been determined, the next step is to calculate dynamic response to a given forcing. Let us examine the equations of motion. For the rigid blade model, the flap equation was given by

$$\overset{**}{\beta} + \nu_\beta^2 \beta = \frac{\omega_0^2}{\Omega^2}\beta_p + \gamma \overline{M}_\beta \tag{2.91}$$

The aerodynamic moment term $\overline{M}_\beta$ may contain motion dependent terms like β and $\overset{*}{\beta}$. It may also contain periodic terms, particularly for forward flight condition. One of the simplest and most commonly used method is Fourier series. The method will be discussed later. For the flexible blade model, the flap bending equation was given by

$$\frac{d^2}{dr^2}\left(EI_{\eta\eta}\frac{d^2w}{dr^2}\right) + m\ddot{w} - \frac{d}{dr}\left(T\frac{dw}{dr}\right) = f_z(r,t) \tag{2.92}$$

where $f_z(r,t)$ was the aerodynamic force. Again, it may contain motion dependent terms as well as periodic terms. Recall, that the first step of the solution was to obtain the natural frequencies and mode shapes by solving the homogenous form of the equation, i.e. with $f_z(r,t) = 0$. The next step is to reduce the governing PDE to a set of ODE's using the mode shapes. The ODE's are then called normal mode equations. To this end, assume that the loading is a series of N natural modes

$$w(r,t) = \sum_{j=1}^{N} \phi_j(r)\, \xi_j(t) \tag{2.93}$$

where $\phi_j(r)$ is j^{th} natural mode. $\xi_j(t)$ is the j^{th} modal response. Substitute in the governing eqn.2.92, project the error onto a subspace spanned by the mode shapes themselves and set to zero.

$$\int_0^R \phi_i(r)\, \epsilon(r,t)\, dr = 0 \qquad i = 1, 2, \ldots, N \tag{2.94}$$

Use the orthogonality relations 2.89 and 2.90 to obtain

$$M_i\ddot{\xi}_i + \omega_i^2 M_i \xi_i = S_i \qquad i = 1, 2, \ldots, N \tag{2.95}$$

where

$$\begin{aligned} M_i &= \int_0^R m\phi_i^2(r)\, dr \\ S_i &= \int_0^R \phi_i f_z(r,t)\, dr \end{aligned} \tag{2.96}$$

These are N modally reduced equations.

The external forcing on the beam is f_z. For a pure structural dynamics problem, f_z is purely a function of r and t. In this case S_i is only a function of time. For a aeroelastic problem, as is the case for rotor blades, f_z is motion dependant, i.e. it depends on the response itself. In this case S_i contain deflection dependant terms. First, consider the case where S_i is only a function of time. The modally reduced equations simply represent a series of one degree of freedom spring-mass systems. Generally, $N = 2$ to 3 are adequate to describe the response of a system. The higher modes contribute comparatively little to the response. The normal mode coordinates $\xi_1(t)$, $\xi_2(t)$, ..., etc may be solved in time using various methods. The most commonly used methods for rotor problems are Fourier series and time integration technique and these will be discussed later. For non-rotor problems, Duhamel's integral is often used to calculate.

Now consider the case of motion dependant forcing. In this case $f_z(r,t)$ depends on displacement w as well as time t. In general, we have

$$f_z(r,t) = \overline{f}_z(r,t) + aw + b\dot{w} + cw' + d\dot{w}' + \ldots \text{etc} \tag{2.97}$$

In this case the modally reduced equations take the following form.

$$M_i\ddot{\xi} + M_i\omega_i^2\xi_i = S_i \quad i = 1,\, 2,\, \ldots,\, N$$

where

$$S_i = \sum_{j=1}^{N} \overline{S}_i(t) + (A_{ij} + C_{ij})\xi_i + (B_{ij} + D_{ij})\dot{\xi}_j + \ldots \text{etc} \tag{2.98}$$

and

$$\overline{S}_i = \int_0^R \phi_i\, \overline{f}_z(r,t)\, dr \quad A_{ij} = \int_0^R \phi_i\, a\, \phi_j\, dr \quad B_{ij} = \int_0^R \phi_i\, b\, \phi_j\, dr$$

$$C_{ij} = \int_0^R \phi_i\, c\, \phi_j'\, dr \quad D_{ij} = \int_0^R \phi_i\, d\, \phi_j'\, dr$$

The mode shapes ϕ are not orthogonal with respect to a, b, c, d. Thus the matrices A, B, C, D are not diagonal. Therefore the resultant ODE's are now coupled. $A + C$ represent aerodynamic stiffness. $B + D$ represent aerodynamic damping. Unlike the structural properties, the aerodynamic stiffness and damping matrices are no longer symmetric. Further, unlike the mechanical system without aerodynamics, the aerodynamic forcing adds a damping to the system. Thus the system is no longer a energy conserving system. The aerodynamic damping need not be necessarily positive. A negative damping can lead to instability, typically called aeroelastic instability. It is more involved to solve these equations. Three widely used variety of methods are: (1) Fourier series based methods, (2) Finite Element in Time method, and (3) Time integration methods. The first two methods provide the steady state forced response solution and are well suited for rotorcraft applications. The third, is a general time marching procedure with provide both the natural response as well as the forced response.

2.7.1 Fourier series methods

In the Fourier series method the response is assumed to be periodic and consisting of a sum of harmonics. For example, for the rigid blade model, the response $\beta(\psi)$ is assumed to be a linear combination of sine and cosine terms as

$$\begin{aligned} \beta(\psi) &= \beta_0 + \beta_{1c}\cos\psi + \beta_{1s}\sin\psi + \beta_{2c}\cos 2\psi + \beta_{2s}\sin 2\psi + \ldots\infty \\ &= \beta_0 + \sum_{n=1}^{\infty}(\beta_{nc}\cos n\psi + \beta_{ns}\sin n\psi) \end{aligned}$$

where the fundamental period is 2π. The fourier constants $\beta_0, \beta_{lc}, \beta_{ls}, \ldots$ are constant with time. They are given by

$$\begin{aligned} \beta_0 &= \frac{1}{2\pi}\int_0^{2\pi} \beta(\psi)\, d\psi \\ \beta_{nc} &= \frac{1}{\pi}\int_0^{2\pi} \beta(\psi)\cos n\psi\, d\psi \\ \beta_{ns} &= \frac{1}{\pi}\int_0^{2\pi} \beta(\psi)\sin n\psi\, d\psi \end{aligned} \tag{2.99}$$

The number of harmonics necessary for satisfactory solution depends on the intended results of the analysis. For preliminary performance and flight dynamic calculations the first harmonic is often adequate. For vibratory loads at least the first five harmonics must be retained. The flapping harmonics can be computed from measured data. If the sample of data points taken over one revolution is N_s, where N_s is the total number of azimuthal intervals such that $\beta(N_s + 1) = \beta(1)$, then

$$\begin{aligned}
\beta_0 &= \frac{1}{N_s}\sum_{i=1}^{N_s}\beta_i \\
\beta_{nc} &= \frac{2}{N_s}\sum_{i=1}^{N_s}\beta_i \cos n\psi_i \\
\beta_{ns} &= \frac{2}{N_s}\sum_{i=1}^{N_s}\beta_i \sin n\psi_i
\end{aligned} \tag{2.100}$$

where $\psi_i = 2\pi(i-1)/N_s$. Using the fourier series method, the governing ODE's can be solved using two approaches: (1) the Substitutional or Harmonic Balance method and (2) the Operational method.

The harmonic balance method is well suited for analytical solution. In this method the fourier series is truncated to a finite number of terms

$$\beta(\psi) = \beta_0 + \sum_{n=1}^{N}(\beta_{nc}\cos n\psi + \beta_{ns}\sin n\psi)$$

and substituted in the ODE. The coefficients of the equation are also written as fourier series by reducing the products of sines and cosines to sums of sines and cosines. The coefficients of the sine and cosine components are then collected

$$(\ldots) + (\ldots)\sin\psi + (\ldots)\cos\psi + (\ldots)\sin 2\psi + (\ldots)\cos 2\psi + \ldots = 0$$

These coefficients are then separately set to zero leading to $2N+1$ algebraic equations for the same number of unknown fourier coefficients.

In the operational method, the following operators are used directly on the ODEs.

$$\begin{aligned}
&\frac{1}{2\pi}\int_0^{2\pi}(de)\,d\psi = 0 \\
&\frac{1}{\pi}\int_0^{2\pi}(de)\cos n\psi\,d\psi = 0 \qquad n = 1,\, 2,\, \ldots,\, \mathrm{N} \\
&\frac{1}{\pi}\int_0^{2\pi}(de)\sin n\psi\,d\psi = 0 \qquad n = 1,\, 2,\, \ldots,\, \mathrm{N}
\end{aligned} \tag{2.101}$$

The coefficients of the equation are again written as fourier series but the degrees of freedom are not. The operators act on the product of the degrees of freedom and the sin or cosine harmonics reducing them to appropriate fourier coeffients. Both the harmonic balance and the operational method yield the same algebraic equations. In the later, the equations can be derived one at a time. The following standard formulas are helpful in reducing the products of sines and cosines to sums of sines and cosines.

$$\sin\psi\cos\psi = \frac{1}{2}\sin 2\psi \qquad \sin^2\psi = \frac{1}{2}(1-\cos 2\psi) \qquad \cos^2\psi = \frac{1}{2}(1+\cos 2\psi)$$

$$\sin^3\psi = \frac{3}{4}\sin\psi - \frac{1}{4}\sin 3\psi \qquad \cos^3\psi = \frac{3}{4}\cos\psi + \frac{1}{4}\cos 3\psi$$

$$\sin\psi\cos^2\psi = \frac{1}{4}(\sin\psi + \sin 3\psi) \qquad \sin^2\psi\cos\psi = \frac{1}{4}(\cos\psi - \cos 3\psi)$$

Example 2.5:

A rotor blade is idealized into a rigid blade with spring at the hinge ($\nu_\beta = 1.10$/rev) and is in hovering flight condition. The blade is excited by an oscillatory aerodynamic lift produced by oscillating the outermost 25% of the blade segment so that $\Delta\theta = 1^\circ \cos\psi$. Calculate the vibratory response assuming the following fourier series

$$\beta(\psi) = \beta_0 + \beta_{1c}\cos\psi + \beta_{1s}\sin\psi$$

Use $\gamma = 8.0$ and assume uniform remains constant.

Figure 2.27: **Excitation of outboard blade segment to generate oscillatory lift**

We have the flap equation as follows

$$\overset{**}{\beta} + \nu_\beta^2\beta = \gamma\overline{M}_\beta$$

where

$$\overline{M}_\beta = \frac{1}{2}\int_0^1 x\left\{\left(\frac{U_T}{\Omega R}\right)^2\theta - \frac{U_P}{\Omega R}\frac{U_T}{\Omega R}\right\}dx$$

$$\frac{U_T}{\Omega R} = x \qquad \frac{U_P}{\Omega R} = \lambda + x\overset{*}{\beta}$$

$$\overline{M}_\beta = \frac{1}{2}\int_0^{3/4}(x^3\theta - \lambda x^2 - \overset{*}{\beta}x^3)dx + \frac{1}{2}\int_{3/4}^1\{x^3(\theta+\Delta\theta) - \lambda x^2 - \overset{*}{\beta}x^3\}dx = \frac{\theta}{8} - \frac{\lambda}{6} - \frac{\overset{*}{\beta}}{8} + 0.0854\Delta\theta$$

The flapping equation is then

$$\overset{**}{\beta} + \nu_\beta^2\beta = \frac{\gamma\theta}{8} - \frac{\gamma\lambda}{6} + \gamma\ 0.0854\ \frac{1\times\pi}{180}\cos\psi$$

Substituting

$$\beta = \beta_0 + \beta_{lc}\cos\psi + \beta_{ls}\sin\psi$$

in the flapping equation, collect $\cos\psi$ and $\sin\psi$ terms and set to zero.

$$\text{constant term:} \quad \beta_0 = \frac{\gamma}{2}\left(\frac{\theta_0}{8} - \frac{\gamma}{6}\right)$$

$$\text{cosine term:} \quad (\nu_\beta^2 - 1)\beta_{lc} + \frac{\gamma}{8}\beta_{ls} = \gamma\ 0.0854\ \frac{1\times\pi}{180}$$

$$\text{sine term:} \quad (\nu_\beta^2 - 1)\beta_{ls} - \frac{\gamma}{8}\beta_{lc} = 0$$

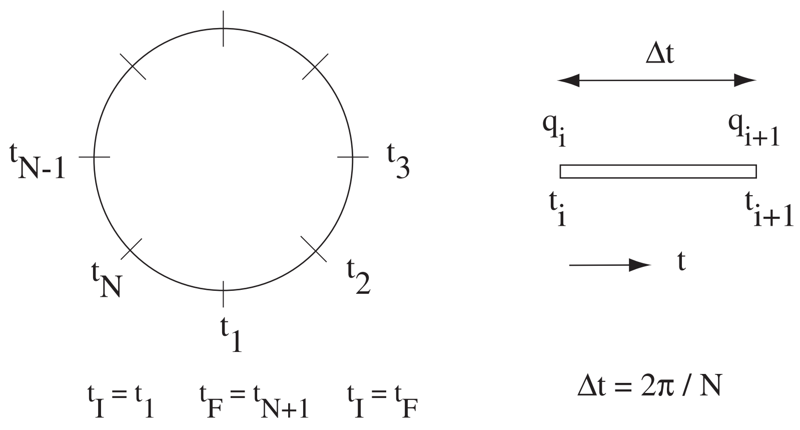
It follows

$$\begin{bmatrix} 0.21 & 1.00 \\ -1.00 & 0.21 \end{bmatrix}\begin{bmatrix} \beta_{1c} \\ \beta_{1s} \end{bmatrix} = \begin{bmatrix} 0.0119 \\ 0.0000 \end{bmatrix}$$

$$\beta_{1c} = 0.137^\circ, \qquad \beta_{1s} = 0.65^\circ$$

2.7.2 Finite Element in Time (FET) method

Finite element in time is a method to calculate the periodic response of a rotor blade. The method can be formulated in two ways:(1) Direct Energy approach, and (2) Indirect Governing Equations approach. We will discuss the Indirect Governing Equations approach. The discretization procedure is the same in both.



(a) FET discretization
$t_I = t_1$; $t_F = t_{N+1}$

(b) A time element

Figure 2.28: **Finite Element in Time (FET) discretization of one period of oscillatory motion**

Consider a single period of oscillatory motion as shown in Fig. 2.28(a). Let the period be T be discretized into N time elements of length $T/2\pi$. For rotors $T = 2\pi$. The initial and final times are the same.

$$t_I = t_1 \quad t_F = t_{N+1} \quad \text{where} \quad t_{N+1} = t_I$$

Similarly the response, say q, at the initial and final times are also the same. For purposes of illustration consider a single degree of freedom system. Now consider a single time element as shown in Fig. 2.28(b). Within the element the degree of freedom q is assumed to vary as a function of time. For example, for a linear variation we have

$$q(t) = \alpha_1 + \alpha_2 t$$

where the constants α are determined in terms of values of q at certain chosen points, called nodes, within the time element. The procedure is same as that described in FEM in space earlier. For purposes of illustration consider the first order element. To determine the two constants α, two nodal degrees of freedom are needed. Let these be the values at the two end points. Then for element-1, for example, we have

$$\eta_1 = q(t_1) = \alpha_1 + \alpha_2 t_1$$

$$\eta_2 = q(t_2) = \alpha_1 + \alpha_2 t_2$$

Solve for $\alpha_{1,2}$

$$\alpha_2 = \frac{\eta_2 - \eta_1}{t_2 - t_1} \qquad \alpha_1 = \frac{\eta_1 t_2 - \eta_2 t_1}{t_2 - t_1}$$

It follows

$$
\begin{aligned}
q(t) &= H_1(t)\eta_1 + H_2(t)\eta_2 \\
H_1(t) &= \left(1 - \frac{t - t_1}{\Delta t}\right) \\
H_2(t) &= \left(\frac{t - t_1}{\Delta t}\right)
\end{aligned}
$$

where H_1 and H_2 are the time shape functions. The nodal values of q, denoted by η, have no time dependance. The derivatives are

$$
\begin{aligned}
\dot{q}(t) &= \dot{H}_1(t)\eta_1 + \dot{H}_2(t)\eta_2 \\
\dot{H}_1(t) &= -\frac{1}{\Delta t} \\
\dot{H}_2(t) &= \frac{1}{\Delta t}
\end{aligned}
$$

The solution procedure begins by putting the governing ODEs in a variational form.

$$
\int_{t_I}^{t_F} \delta \underset{\sim}{q}^T \left(m\underset{\sim}{\ddot{q}} + c\underset{\sim}{\dot{q}} + k\underset{\sim}{\dot{q}} - \underset{\sim}{f} \right) dt = 0 \tag{2.102}
$$

For a constant m the acceleration term reduces to

$$
\int_{t_I}^{t_F} \delta \underset{\sim}{q}^T m\underset{\sim}{\ddot{q}} dt = \delta \underset{\sim}{q}^T m\underset{\sim}{\dot{q}} \,|_{t_I}^{t_F} - \int_{t_I}^{t_F} \delta \underset{\sim}{\dot{q}}^T m\underset{\sim}{\dot{q}} dt = -\int_{t_I}^{t_F} \delta \underset{\sim}{\dot{q}}^T m\underset{\sim}{\dot{q}} dt \tag{2.103}
$$

where the first term is cancelled due to periodicity of the response. Using the above, eqn.2.102 becomes

$$
I = \int_{t_I}^{t_F} \left(-\delta \underset{\sim}{\dot{q}}^T m\underset{\sim}{\dot{q}} + \delta \underset{\sim}{q}^T c\underset{\sim}{\dot{q}} + \delta \underset{\sim}{q}^T k\underset{\sim}{q} - \delta \underset{\sim}{q}^T \underset{\sim}{f} \right) dt = 0 \tag{2.104}
$$

where

$$
I = \int_{t_1}^{t_2} (\;)dt + \int_{t_1}^{t_2} (\;)dt + \int_{t_1}^{t_2} (\;)dt \ldots + \int_{t_{N-1}}^{t_N} (\;)dt = I_1 + I_2 + I_3 + \cdots + I_N \tag{2.105}
$$

Each integral is of the following form. Consider for example, I_1.

$$
\begin{aligned}
I_1 = &-\int_{t1}^{t2} \left\{ \begin{array}{c} \delta\eta_1 \\ \delta\eta_2 \end{array} \right\}^T \left\{ \begin{array}{c} \dot{H}_1 \\ \dot{H}_2 \end{array} \right\} m \left[\dot{H}_1 \dot{H}_2 \right] \left\{ \begin{array}{c} \eta_1 \\ \eta_2 \end{array} \right\} dt \\
&+ \int_{t1}^{t2} \left\{ \begin{array}{c} \delta\eta_1 \\ \delta\eta_2 \end{array} \right\}^T \left\{ \begin{array}{c} H_1 \\ H_2 \end{array} \right\} c \left[\dot{H}_1 \dot{H}_2 \right] \left\{ \begin{array}{c} \eta_1 \\ \eta_2 \end{array} \right\} dt \\
&+ \int_{t1}^{t2} \left\{ \begin{array}{c} \delta\eta_1 \\ \delta\eta_2 \end{array} \right\}^T \left\{ \begin{array}{c} H_1 \\ H_2 \end{array} \right\} c \left[H_1 H_2 \right] \left\{ \begin{array}{c} \eta_1 \\ \eta_2 \end{array} \right\} dt \\
&- \int_{t1}^{t2} \left\{ \begin{array}{c} \delta\eta_1 \\ \delta\eta_2 \end{array} \right\}^T \left\{ \begin{array}{c} H_1 \\ H_2 \end{array} \right\} f dt \\
= &\left\{ \begin{array}{c} \delta\eta_1 \\ \delta\eta_2 \end{array} \right\}^T \left[\begin{array}{cc} A_{11} & A_{12} \\ A_{21} & A_{22} \end{array} \right] \left\{ \begin{array}{c} \eta_1 \\ \eta_2 \end{array} \right\} - \left\{ \begin{array}{c} \delta\eta_1 \\ \delta\eta_2 \end{array} \right\}^T \left\{ \begin{array}{c} Q_1 \\ Q_2 \end{array} \right\}
\end{aligned} \tag{2.106}
$$

where

$$
A_{11} = -\frac{m}{\Delta t} - \int_{t_1}^{t_2} \frac{c}{\Delta t} \left(1 - \frac{t - t_1}{\Delta t}\right) dt + \int_{t_1}^{t_2} k \left(1 - \frac{t - t_1}{\Delta t}\right)^2 dt
$$

$$A_{12} = \frac{m}{\Delta t} + \int_{t_1}^{t_2} \frac{c}{\Delta t}\left(1 - \frac{t-t_1}{\Delta t}\right) dt + \int_{t_1}^{t_2} k\frac{t-t_1}{\Delta t}\left(1 - \frac{t-t_1}{\Delta t}\right) dt$$
$$A_{21} = \frac{m}{\Delta t} - \int_{t_1}^{t_2} \frac{c}{\Delta t}(t-t_1)\, dt + \int_{t_1}^{t_2} k\,\frac{t-t_1}{\Delta t}\left(1 - \frac{t-t_1}{\Delta t}\right) dt$$
$$A_{22} = -\frac{m}{\Delta t} + \int_{t_1}^{t_2} \frac{c}{\Delta t}(t-t_1)\, dt + \int_{t_1}^{t_2} k\left(\frac{t-t_1}{\Delta t}\right)^2 dt$$
$$Q_1 = \int_{t_1}^{t_2} f\left(1 - \frac{t-t_1}{\Delta t}\right) dt$$
$$Q_2 = \int_{t_1}^{t_2} f\,\frac{t-t_1}{\Delta t} dt$$

Similar expressions can be found for I_2, I_3, ... etc. The following step is to add the individual integrals as in eqn.2.105. This is an assembly procedure. For illustration consider a case where the time period is discretized into 4 time elements, see Fig.2.29. For the four elements we have the

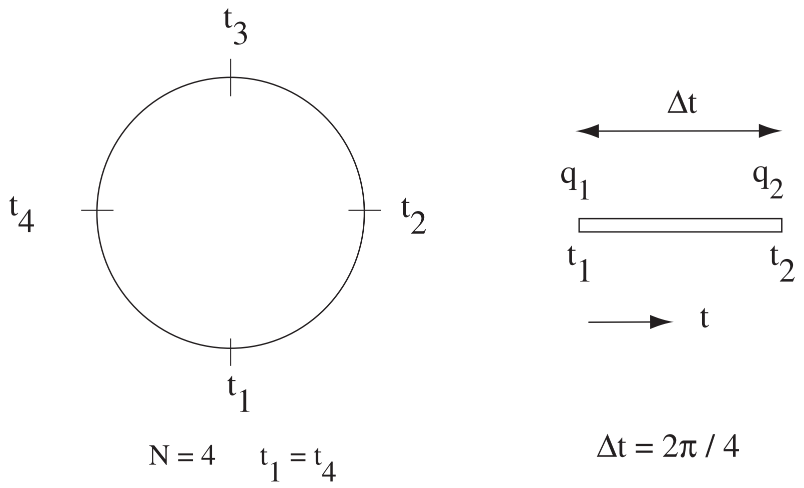


Figure 2.29: **Finite Element in Time (FET) discretization of one period of oscillatory motion**

following

$$\begin{aligned}
I_1 &= \left\{\begin{array}{c}\delta\eta_1\\ \delta\eta_2\end{array}\right\}^T \left[\begin{array}{cc}A_{11} & A_{12}\\ A_{21} & A_{22}\end{array}\right]_1 \left\{\begin{array}{c}\eta_1\\ \eta_2\end{array}\right\} - \left\{\begin{array}{c}\delta\eta_1\\ \delta\eta_2\end{array}\right\}^T \left\{\begin{array}{c}Q_1\\ Q_2\end{array}\right\}_1\\
I_2 &= \left\{\begin{array}{c}\delta\eta_2\\ \delta\eta_3\end{array}\right\}^T \left[\begin{array}{cc}A_{11} & A_{12}\\ A_{21} & A_{22}\end{array}\right]_2 \left\{\begin{array}{c}\eta_2\\ \eta_3\end{array}\right\} - \left\{\begin{array}{c}\delta\eta_2\\ \delta\eta_3\end{array}\right\}^T \left\{\begin{array}{c}Q_1\\ Q_2\end{array}\right\}_2\\
I_3 &= \left\{\begin{array}{c}\delta\eta_3\\ \delta\eta_4\end{array}\right\}^T \left[\begin{array}{cc}A_{11} & A_{12}\\ A_{21} & A_{22}\end{array}\right]_3 \left\{\begin{array}{c}\eta_3\\ \eta_4\end{array}\right\} - \left\{\begin{array}{c}\delta\eta_3\\ \delta\eta_4\end{array}\right\}^T \left\{\begin{array}{c}Q_1\\ Q_2\end{array}\right\}_3\\
I_4 &= \left\{\begin{array}{c}\delta\eta_4\\ \delta\eta_1\end{array}\right\}^T \left[\begin{array}{cc}A_{11} & A_{12}\\ A_{21} & A_{22}\end{array}\right]_4 \left\{\begin{array}{c}\eta_4\\ \eta_1\end{array}\right\} - \left\{\begin{array}{c}\delta\eta_4\\ \delta\eta_1\end{array}\right\}^T \left\{\begin{array}{c}Q_1\\ Q_2\end{array}\right\}_4
\end{aligned} \tag{2.107}$$

Add the individual integrals and set $I = 0$ to obtain

$$\left\{\begin{array}{c}\delta\eta_1\\ \delta\eta_2\\ \delta\eta_3\\ \delta\eta_4\end{array}\right\}^T A \left\{\begin{array}{c}\eta_1\\ \eta_2\\ \eta_3\\ \eta_4\end{array}\right\} = \left\{\begin{array}{c}\delta\eta_1\\ \delta\eta_2\\ \delta\eta_3\\ \delta\eta_4\end{array}\right\}^T Q \tag{2.108}$$

Because $\delta\underset{\sim}{\eta}$ is arbitrary we have

$$A\eta = \underset{\sim}{Q} \tag{2.109}$$

where A and Q are as follows

$$A = \begin{bmatrix} \boxed{\times} & \times & & \square \\ \times & \times\!\!\!\!\bullet & \bullet & \\ & \bullet & \bullet\!\!\!\!0 & 0 \\ \square & & 0 & \boxed{0} \end{bmatrix} \qquad \begin{array}{l} \times \rightarrow \text{from element 1} \\ \bullet \rightarrow \text{from element 2} \\ 0 \rightarrow \text{from element 3} \\ \square \rightarrow \text{from element 4} \end{array}$$

$$= \begin{bmatrix} (A_{11})_1 + (A_{22})_4 & (A_{12})_1 & 0 & (A_{21})_4 \\ (A_{21})_1 & (A_{22})_1 + (A_{11})_2 & (A_{12})_2 & 0 \\ 0 & (A_{21})_2 & (A_{22})_3 + (A_{11})_3 & (A_{12})_3 \\ (A_{12})_4 & 0 & (A_{21})_3 & (A_{12})_3 + (A_{11})_4 \end{bmatrix} \tag{2.110}$$

$$Q = \begin{Bmatrix} \boxed{\times} \\ \times\!\!\!\!\bullet \\ \bullet\!\!\!\!0 \\ \boxed{0} \end{Bmatrix} \qquad \begin{array}{l} \times \rightarrow \text{from element 1} \\ \bullet \rightarrow \text{from element 2} \\ 0 \rightarrow \text{from element 3} \\ \square \rightarrow \text{from element 4} \end{array} \qquad = \begin{Bmatrix} (Q_1)_1 + (Q_2)_4 \\ (Q_2)_1 + (Q_1)_2 \\ (Q_2)_2 + (Q_1)_3 \\ (Q_2)_3 + (Q_1)_4 \end{Bmatrix}$$

2.7.3 Time Integration Methods

A commonly used method for response solution of linear and non-linear equations is the time integration technique. There are many solution procedures used for time integration of equations. Some of these are the Runge-Kutta method, the Adams predictor corrector method, the Gear variable order method, the Newmark method, and the Energy-Momentum method.

2.8 Bending Moments and Stresses

Once the blade deformations in response to external loading are known, the bending moments and shear loads at any section can be determined. The stresses at a point in a section can then be calculated based on the bending moment and shear load at the section. The bending moment and shear loads at any section are determined using two methods: (1) Curvature method and (2) Force Summation method. The curvature method is also called the deflection method, as the curvature can be expressed as a function of deflection. If the deflection is calculated based on a modally reduced set of ODEs, the method is also called the modal method. For the purposes of illustration, assume that the deflection of the beam is of the following form

$$w(r,t) = \sum_{j=1}^{n} \phi_j(r)\, q_j(t) \tag{2.111}$$

where q_i due to the external loading have been solved for, and ϕ_j are known shape functions, either assumed as in the case of Galerkin or Rayleigh-Ritz, or determined using FEM.

2.8.1 Deflection and Force Summation methods

In the deflection method the resultant relation given in eqn.2.11 is used. The bending moment at a station r is given by

$$M(r) = EI_{\eta\eta}\kappa = \frac{EI_{\eta\eta}}{\rho} \cong EI_{\eta\eta}\frac{d^2w}{dr^2} = EI_{\eta\eta}\sum_{j=1}^{n} \phi_j''\, q_j$$

The bending stress, from eqn.2.12 is then simply

$$\sigma_{rr}(z) = \frac{M(r)}{I_{\eta\eta}} z = z\, E \sum_{j=1}^{n} \phi_j''\, q_j$$

where z is the distance from the beam centerline. The bending stress is proportional to the second derivative of displacement. Usually a large number of terms is needed to accurately calculate the bending stress. The method is simple but yields poor results for small n. The shear load at a station is given by

$$S(r) = \frac{\delta M}{\delta r} = \left(EI_{\eta\eta} \sum_{j=1}^{n} \phi_j'' \right)' q_j = EI_{\eta\eta} \sum_{j=1}^{n} \phi_j'''\, q_j \quad \text{for uniform beam}$$

The shear deformation was neglected in the analysis, thus the shear stress cannot be accurately calculated. For a rough estimate divide the shear load with the sectional area.

$$\tau_{rz} = S(r)/A$$

Again, a large number of terms is needed to calculate the shear load. The shear load is proportional to the third derivative of displacement. In general, error increases with the order of derivative. The error in shear load is greater than that in bending moment. By error, one refers to the difference in solution between using n terms and as many terms required for a converged solution.

The alternative to the deflection method, which relies on the derivative of the response solution, is to use the Force Summation method. See Fig.2.30. The bending moment at a station r is obtained

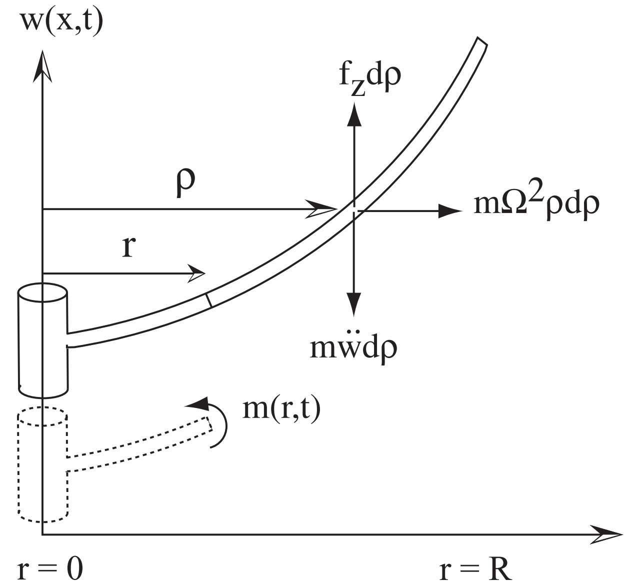


Figure 2.30: **Flap bending moment at a blade section using force summation method**

by integrating all of the elemental forces outboard of r.

$$M(r) = \int_{\rho=r}^{\rho=R} [(F_z - m\ddot{w})(\rho - r)]\, d\rho - \int_{\rho=r}^{\rho=R} m\Omega^2 \rho\, [w(\rho) - w(r)]\, d\rho \qquad (2.112)$$

Because integrations are involved with respect to spatial coordinate r, this method generally produces less error for smaller n. However, the method is more involved compared to the deflection method. The statement of equality between bending moments calculated using the deflection method and using force summation method reproduces the beam bending equation. To verify, substitute $M(r) = EI_{\eta\eta}w''$ on the left hand side of the above equation and differentiate twice with respect to r. Note that r occurs in the limits of integration on the right hand side, hence use the Leibnitz theorem. The Leibnitz theorem gives

$$\begin{aligned} \text{If} \quad & \phi(r) = \int_{u_1(r)}^{u_2(r)} F(r,\rho)d\rho \\ \text{then,} \quad & \frac{\partial \phi}{\partial r} = \int_{u_1(r)}^{u_2(r)} \frac{\partial F}{\partial r} d\rho - \frac{\partial u_1}{\partial r} F(r, u_1) - \frac{\partial u_1}{\partial r} F(r, u_2) \end{aligned} \tag{2.113}$$

Using the Leibnitz theorem twice it follows

$$\frac{\partial^2}{\partial r^2}\left(EI_{\eta\eta}\frac{\partial^2 w}{\partial r^2}\right) = -m\ddot{w} + \frac{\partial}{\partial r}\left(\int_r^R mr\Omega^2 dr \frac{\partial w}{\partial r}\right) \tag{2.114}$$

which is the flexible flap equation. Note that the equivalent expression for the rigid blade was given by eqn.2.5. There, the left hand side was the flap moment at the hinge via deflection method. The right hand side was the flap moment at the hinge via force summation method. Their equality generated the rigid flap equation.

2.8.2 Force summation vs. modal method

In the curvature method (also called modal method if the deflection is obtained using normal modes), the loads at a given section are determined by the elastic motion induced curvature and structural properties at that section. If there is a radial step change in structural properties, e.g. bending stiffness, or a concentrated loading, e.g. damper force, then there should be a corresponding step change in curvature, to keep the physical loads continuous. With a small number modes or shape functions this discontinuity cannot be captured. Moreover, the curvature method gives zero load on an element without elastic degrees of freedom. A force summation method rectifies the above deficiencies. It is a force balance method which obtains the section loads from the difference between the applied forces and the inertial forces acting on the blade on one side of the section. The forces used for this purpose must be exactly same as those used for solving the structural dynamic equations, otherwise inconsistent loads are obtained. For example, the bending moments at a pure hinge would not be identically zero. With lesser number of modes, the force summation method better captures the effects of concentrated loading and radial discontinuities of structural properties. However, with increase in number of modes the curvature method and the force summation method must approach the same solution.

2.9 Fourier Coordinate Transformation

Fourier Coordinate Transformation is also called Multi-blade Coordinate Transformation. Let $\beta^{(m)}(\psi)$ be the flapping motion of the m-th blade of a rotor with N_b blades, where $m = 1, 2, 3, \ldots, N_b$.

Then the forward Fourier Coordinate Transformation is defined as

$$
\begin{aligned}
B_0 &= \frac{1}{N_b}\sum_{m=1}^{N_b}\beta^{(m)} \\
B_{nc} &= \frac{2}{N_b}\sum_{m=1}^{N_b}\beta^{(m)}\cos n\psi_m \\
B_{ns} &= \frac{2}{N_b}\sum_{m=1}^{N_b}\beta^{(m)}\sin n\psi_m \\
B_d &= \frac{1}{N_b}\sum_{m=1}^{N_b}\beta^{(m)}(-1)^{(m)}
\end{aligned}
\tag{2.115}
$$

where ψ_m is the azimuthal angle for the m^{th} blade

$$\psi_m = \psi_1 + \frac{2\pi}{N_b}(m-1) = \psi + \frac{2\pi}{N_b}(m-1)$$

and ψ_1 is defined as ψ. n and d are defined as follows.

$$n = 1, 2, 3, \ldots, \frac{N_b - 2}{2} \quad \text{for } N_b \text{ even;} \quad \frac{N_b - 1}{2} \quad \text{for } N_b \text{ odd}$$

$$d = \frac{N_b}{2} \quad \text{for } N_b \text{ even; does not exist for } N_b \text{ odd}$$

For a 5-bladed rotor, the rotating coordinates are the flapping motion of the five blades, $\beta^1, \beta^2, \beta^3, \beta^4, \beta^5$. The fixed coordinates are also five in number, they are $B_0, B_{1c}, B_{1s}, B_{2c}, B_{2s}$. Note that N_b being odd, B_d does not exist. Similarly, for a 4-bladed rotor the rotating and fixed coordinates are $\beta^1, \beta^2, \beta^3, \beta^4$, and B_0, B_{1c}, B_{1s}, B_2 respectively. For a 3-bladed rotor they are $\beta^1, \beta^2, \beta^3$, and B_0, B_{1c}, B_{1s} respectively. For a 2-bladed rotor they are β^1, β^2, and B_0, B_1 respectively. In the last case there are no cosine or sine coordinates. Note that the transformation does not require that the flapping motion $\beta^{(m)}(\psi)$ be periodic.

For a physical feel, consider a rotor with 4 blades. For purposes of illustration assume that the blades undergo a periodic flapping motion. At any instant of time one of the blades, designated as say blade-1, occurs in the azimuth ψ_1. Define $\psi_1 = \psi$. Blade-2 at that instant occupies $\psi_2 = \psi + \pi/2$. Blade-3 occupies $\psi_3 = \psi + \pi$. Blade-4 occupies $\psi_4 = \psi + 3\pi/2$. Let $\beta^1(\psi)$, $\beta^2(\psi)$, $\beta^3(\psi)$, and $\beta^4(\psi)$, describe the flapping motion of the blades. If blade-1 exhibits the following flapping motion

$$\beta^1(\psi) = \beta_0 + \beta_{1c}\cos\psi + \beta_{1s}\sin\psi + \beta_{2c}\cos 2\psi + \beta_{2s}\sin 2\psi + \ldots\infty$$

then blades 2, 3 and 4 exhibit

$$
\begin{aligned}
\beta^2(\psi) = \beta^1(\psi_2) &= \beta_0 - \beta_{1c}\sin\psi + \beta_{1s}\cos\psi - \beta_{2c}\cos 2\psi - \beta_{2s}\sin 2\psi + \ldots\infty \\
\beta^3(\psi) = \beta^1(\psi_3) &= \beta_0 - \beta_{1c}\cos\psi - \beta_{1s}\sin\psi + \beta_{2c}\cos 2\psi + \beta_{2s}\sin 2\psi + \ldots\infty \\
\beta^4(\psi) = \beta^1(\psi_4) &= \beta_0 + \beta_{1c}\sin\psi - \beta_{1s}\cos\psi - \beta_{2c}\cos 2\psi - \beta_{2s}\sin 2\psi + \ldots\infty
\end{aligned}
$$

The fixed coordinates are then given by

$$
\begin{aligned}
B_0(\psi) &= \frac{1}{4}\left[\beta^1(\psi) + \beta^2(\psi) + \beta^3(\psi) + \beta^4(\psi)\right] \\
B_{1c}(\psi) &= \frac{2}{4}\left[\beta^1(\psi)\cos\psi_1 + \beta^2(\psi)\cos\psi_2 + \beta^3(\psi)\cos\psi_3 + \beta^4(\psi)\cos\psi_4\right] \\
B_{1s}(\psi) &= \frac{2}{4}\left[\beta^1(\psi)\sin\psi_1 + \beta^2(\psi)\sin\psi_2 + \beta^3(\psi)\sin\psi_3 + \beta^4(\psi)\sin\psi_4\right] \\
B_2(\psi) &= \frac{1}{4}\left[\beta^1(\psi)(-1)^1 + \beta^2(\psi)(-1)^2 + \beta^3(\psi)(-1)^3 + \beta^4(\psi)(-1)^4\right]
\end{aligned}
$$

The reverse Fourier Coordinate Transformation is given by

$$\beta^{(m)}(\psi) = B_0(\psi) + \sum_{n=1}^{(N_b-2)/2} [B_{nc}(\psi)\cos n\psi_m + B_{ns}(\psi)\sin n\psi_m] + B_d(-1)^m \qquad \text{for } N_b \text{ even}$$

$$\beta^{(m)}(\psi) = B_0(\psi) + \sum_{n=1}^{(N_b-1)/2} [B_{nc}(\psi)\cos n\psi_m + B_{ns}(\psi)\sin n\psi_m] \qquad \text{for } N_b \text{ odd} \tag{2.116}$$

The fourier coordinates B_0, B_{1c}, B_{1s}, ... etc are functions of ψ, and are different from fourier series coefficients which are constants. The forward and reverse transformations, eqns.2.115 and 2.116, are exact not approximate. As a result governing equations in fourier coordinates retain the same information as those in rotating coordinates. A complete description of rotor motion can be obtained by solving for the rotating coordinates $\beta^m(\psi), m = 1, 2, 3, \ldots, N_b$. Alternatively it can be obtained by solving for the fixed coordinates, equal in number, $B_0(\psi), B_{nc}(\psi), B_{ns}(\psi), B_d(\psi)$. The governing equations in rotating coordinates can be transformed into fixed coordinates in the following manner.

2.9.1 FCT of governing equations

To carry out FCT of governing equations the following expressions are required. We have by definition

$$B_{nc} = \frac{2}{N_b}\sum \beta^{(m)} \cos n\psi_m \tag{2.117}$$

Differentiate once to obtain

$$\overset{*}{B}_{nc} = \frac{2}{N_b}\sum \overset{*}{\beta}{}^{(m)} \cos n\psi_m - \frac{2}{N_b} n \sum \beta^{(m)} \sin n\psi_m = \frac{2}{N_b}\sum \overset{*}{\beta}{}^{(m)} \cos n\psi_m - nB_{ns}$$

where the definition of B_{ns} has been used in the second term on the right hand side. Hence we have

$$\frac{2}{N_b}\sum \overset{*}{\beta}{}^{(m)} \cos n\psi_m = \overset{*}{B}_{nc} + nB_{ns} \tag{2.118}$$

Similarly starting from the definition of B_{ns}, differentiating once, and using the definition of B_{nc} we have

$$\frac{2}{N_b}\sum \overset{*}{\beta}{}^{(m)} \sin n\psi_m = \overset{*}{B}_{ns} - nB_{nc} \tag{2.119}$$

Now differentiate eqn.2.118 to obtain

$$\frac{2}{N_b}\sum \overset{**}{\beta}{}^{(m)} \cos n\psi_m - \frac{2}{N_b} n \sum \overset{*}{\beta}{}^{(m)} \sin n\psi_m = \overset{**}{B}_{nc} + n\,\overset{*}{B}_{ns}$$

Use eqn.2.119 on the second term on the left hand side to obtain

$$\frac{2}{N_b}\sum \overset{**}{\beta}{}^{(m)} \cos n\psi_m = \overset{**}{B}_{nc} + 2n\,\overset{*}{B}_{ns} - n^2 B_{nc} \tag{2.120}$$

Similarly differentiating eqn.2.119 and using eqn.2.118 we have

$$\frac{2}{N_b}\sum \overset{**}{\beta}{}^{(m)} \sin n\psi_m = \overset{**}{B}_{ns} - 2n\,\overset{*}{B}_{nc} - n^2 B_{ns} \tag{2.121}$$

The derivatives of B_0 and B_d are straightforward as they do not involve sin or cosine harmonics. The final derivative expressions, necessary for FCT, are listed below. The B_0 expressions are

$$\begin{aligned}
\frac{1}{N_b}\sum_{m}^{N_b}\beta^{(m)} &= B_0 \\
\frac{1}{N_b}\sum_{m}^{N_b}\overset{*}{\beta}{}^{(m)} &= \overset{*}{B}_0 \\
\frac{1}{N_b}\sum_{m}^{N_b}\overset{**}{\beta}{}^{(m)} &= \overset{**}{B}_0
\end{aligned} \tag{2.122}$$

The B_d expressions are

$$\begin{aligned}
\frac{1}{N_b}\sum_{m}^{N_b}\beta^{(m)}(-1)^m &= B_d \\
\frac{1}{N_b}\sum_{m}^{N_b}\overset{*}{\beta}{}^{(m)}(-1)^m &= \overset{*}{B}_d \\
\frac{1}{N_b}\sum_{m}^{N_b}\overset{**}{\beta}{}^{(m)}(-1)^m &= \overset{**}{B}_d
\end{aligned} \tag{2.123}$$

The B_{nc} expressions are

$$\begin{aligned}
\frac{2}{N_b}\sum_{m}^{N_b}\beta^{(m)}\cos n\psi_m &= B_{nc} \\
\frac{2}{N_b}\sum_{m}^{N_b}\overset{*}{\beta}{}^{(m)}\cos n\psi_m &= \overset{*}{B}_{nc} + nB_{ns} \\
\frac{2}{N_b}\sum_{m}^{N_b}\overset{**}{\beta}{}^{(m)}\cos n\psi_m &= \overset{**}{B}_{nc} + 2n\,\overset{*}{B}_{ns} - n^2 B_{nc}
\end{aligned} \tag{2.124}$$

The B_{ns} expressions are

$$\begin{aligned}
\frac{2}{N_b}\sum_{m}^{N_b}\beta^{(m)}\sin n\psi_m &= B_{ns} \\
\frac{2}{N_b}\sum_{m}^{N_b}\overset{*}{\beta}{}^{(m)}\sin n\psi_m &= \overset{*}{B}_{ns} - nB_{nc} \\
\frac{2}{N_b}\sum_{m}^{N_b}\overset{**}{\beta}{}^{(m)}\sin n\psi_m &= \overset{**}{B}_{ns} - 2n\,\overset{*}{B}_{nc} - n^2 B_{ns}
\end{aligned} \tag{2.125}$$

The conversion of the governing equations to fixed coordinates is now carried out as follows

$$\begin{aligned}
&B_0 \;\text{ equation :}\; \frac{1}{N_b}\sum_{m=1}^{N_b} \text{(Equation of motion)} \\
&B_{nc} \;\text{ equation :}\; \frac{2}{N_b}\sum_{m=1}^{N_b} \text{(Equation of motion)}\; \cos n\psi_m \\
&B_{ns} \;\text{ equation :}\; \frac{2}{N_b}\sum_{m=1}^{N_b} \text{(Equation of motion)}\; \sin n\psi_m \\
&B_d \;\text{ equation :}\; \frac{1}{N_b}\sum_{m=1}^{N_b} \text{(Equation of motion)}\; (-1)^m
\end{aligned} \tag{2.126}$$

During this operation, certain expressions can arise which are not straightforward application of the above formulae and need to be substituted correctly. These are described below. For purposes of illustration consider $N_b = 4$. The fourier coordinates in this case are B_0, B_{1c}, B_{1s} and B_2. First consider summations over trigonometric functions.

$$\begin{aligned}
&\frac{1}{4}\sum_{m=1}^{4} \sin\psi_m = \sin\psi_1 + \sin\psi_2 + \sin\psi_3 + \sin\psi_4 = 0 \\
&\frac{1}{4}\sum_{m=1}^{4} \sin 2\psi_m = 0 \\
&\frac{1}{4}\sum_{m=1}^{4} \sin 3\psi_m = 0 \\
&\frac{1}{4}\sum_{m=1}^{4} \sin 4\psi_m = \sin 4\psi_1 = \sin 4\psi
\end{aligned} \tag{2.127}$$

In general

$$\frac{1}{N_b}\sum_{m=1}^{N_b} \cos n\psi_m = \cos n\psi \quad \text{and} \quad \frac{1}{N_b}\sum_{m=1}^{N_b} \sin n\psi_m = \sin n\psi$$

only when $n = pN_b$ where p is an integer, and zero otherwise. It follows that a harmonic which is an integral multiple of blade number can be taken outside the summation side. For example,

$$\frac{2}{4}\sum_{m=1}^{4} \beta^{(m)} \cos 4\psi_m \cos\psi_m = \cos 4\psi \frac{2}{4}\sum_{m=1}^{4} \beta^{(m)} \cos\psi_m = B_{1c}\cos 4\psi$$

Thus note the following treatment

$$\frac{2}{4}\sum_{m=1}^{4} \beta^{(m)} \cos 3\psi_m = \frac{2}{4}\sum_{m=1}^{4} \beta^{(m)} \cos(4\psi_m - \psi_m) = B_{1c}\cos 4\psi + B_{1s}\sin 4\psi$$

Just as harmonics which are integral multiples of blade number can be taken outside the summation, a special treatment is needed for harmonics which are integral multiples of half the blade number. For example

$$\begin{aligned}
\frac{2}{4}\sum_{m=1}^{4} \beta^{(m)} \cos 2\psi_m &= \frac{2}{4}\left[\beta^{(1)} \sin 2\psi_1 + \beta^{(1)} \sin 2\psi_2 + \beta^{(1)} \sin 2\psi_3 + \beta^{(1)} \sin 2\psi_4\right] \\
&= \frac{2}{4}\left[\beta^{(1)} \sin 2\psi - \beta^{(1)} \sin 2\psi + \beta^{(1)} \sin 2\psi - \beta^{(1)} \sin 2\psi\right] = -2B_2 \sin 2\psi
\end{aligned}$$

Another set of special cases arise during transformation of the B_d equation. They involve $(-1)^m$ multiplied with with sine and cosine terms. First consider the sum of harmonics

$$\begin{aligned}
&\frac{1}{4}\sum_{m=1}^{4}(-1)^m \sin\psi_m = -\sin\psi_1 + \sin\psi_2 - \sin\psi_3 + \sin\psi_4 = 0 \\
&\frac{1}{4}\sum_{m=1}^{4}(-1)^m \sin 2\psi_m = -\sin 2\psi \\
&\frac{1}{4}\sum_{m=1}^{4}(-1)^m \sin 3\psi_m = 0 \\
&\frac{1}{4}\sum_{m=1}^{4}(-1)^m \sin 4\psi_m = 0
\end{aligned} \tag{2.128}$$

In general

$$\frac{1}{N_b}\sum_{m=1}^{N_b}(-1)^m \cos n\psi_m = -\cos n\psi \quad \text{and} \quad \frac{1}{N}\sum_{m=1}^{N}(-1)^m \sin n\psi_m = -\sin n\psi$$

only when $n = pN_b + N_b/2$, where p is an integer and N_b is even, zero otherwise. It follows that a harmonic of frequency $n = pN_b + N_b/2$ can be taken outside the summation sign in the presence of the factor $(-1)^m$. Thus note the following treatment

$$\begin{aligned}
& \frac{1}{4}\sum_{m=1}^{4}\beta^{(m)}(-1)^m \sin\psi_m \\
= & \frac{1}{4}\sum_{m=1}^{4}\beta^{(m)}(-1)^m \sin(2\psi_m - \psi_m) \\
= & \frac{1}{4}\sum_{m=1}^{4}\beta^{(m)}(-1)^m \sin 2\psi_m \cos\psi_m - \frac{1}{4}\sum_{m=1}^{4}\beta^{(m)}(-1)^m \cos 2\psi_m \sin\psi_m \\
= & +\frac{1}{4}\left[-\beta^{(1)}\sin 2\psi\cos\psi_1 - \beta^{(2)}\sin 2\psi\cos\psi_2 - \beta^{(3)}\sin 2\psi\cos\psi_3 - \beta^{(4)}\sin 2\psi\cos\psi_4\right] \\
& -\frac{1}{4}\left[-\beta^{(1)}\cos 2\psi\sin\psi_1 - \beta^{(2)}\cos 2\psi\sin\psi_2 - \beta^{(3)}\cos 2\psi\sin\psi_3 - \beta^{(4)}\cos 2\psi\sin\psi_4\right] \\
= & -\frac{1}{2}B_{1c}\sin 2\psi + \frac{1}{2}B_{1s}\cos 2\psi
\end{aligned}$$

Consider the rigid blade flapping equation in forward flight. The blade twist and the cyclic control angles are assumed to be zero.

$$\begin{aligned}
&\overset{**}{\beta} + \left(\frac{\gamma}{8} + \mu\frac{\gamma}{6}\sin\psi\right)\overset{*}{\beta} + \left(\nu_\beta^2 + \mu\frac{\gamma}{6}\cos\psi + \mu^2\frac{\gamma}{8}\sin 2\psi\right)\beta = \\
&\qquad \gamma\theta_0\left(\frac{1}{8} + \frac{\mu}{3}\sin\psi + \frac{\mu^2}{4}\sin^2\psi\right) - \gamma\lambda\left(\frac{1}{6} + \frac{\mu}{4}\sin\psi\right)
\end{aligned} \tag{2.129}$$

Consider the transformation of the above equation for $N_b = 4$. The fixed coordinates are B_0, B_{1c}, B_{1s}, and B_2. Use the operators given by eqns.2.126, and the definitions given by eqns.2.122–2.125 Apply the first operator to obtain the B_0 equation.

$$\begin{aligned}
& \overset{**}{B_0} + \frac{\gamma}{8}\overset{*}{B_0} + \mu\frac{\gamma}{6}\frac{1}{2}\left(\overset{*}{B}_{1s} - B_{1c}\right) + \nu_\beta^2 B_0 + \mu\frac{\gamma}{6}\frac{1}{2}B_{1c} + \mu^2\frac{\gamma}{8}\underline{\frac{1}{4}\sum\beta\sin 2\psi} \\
& = \gamma\theta_0\left(\frac{1}{8} + \underline{\frac{1}{4}\sum\sin^2\psi}\right) - \gamma\lambda\frac{1}{6}
\end{aligned}$$

The underlined terms are to be replaced by

$$\frac{1}{4}\sum \beta \sin 2\psi = -B_2 \sin 2\psi$$

$$\frac{1}{4}\sum \sin^2\psi = \frac{1}{4}\sum \frac{1}{2}(1-\cos 2\psi) = \frac{1}{2}$$

Apply the second operator to obain the B_{1c} equation.

$$\overset{**}{B}_{1c} + 2\overset{*}{B}_{1s} - B_{1c} + \frac{\gamma}{8}\left(\overset{*}{B}_{1c} + B_{1s}\right) + \mu\frac{\gamma}{6}\underline{\frac{2}{4}\sum \overset{*}{\beta}\sin\psi\cos\psi} + \nu_\beta^2 B_{1c} + \mu\frac{\gamma}{6}\underline{\frac{2}{4}\sum \beta\cos^2\psi}$$
$$+\mu^2\frac{\gamma}{8}\underline{\frac{2}{4}\sum \beta\sin 2\psi\cos\psi} = \gamma\theta_0\left(\frac{\mu}{3}\underline{\frac{2}{4}\sum \sin\psi\cos\psi} + \frac{\mu^2}{4}\underline{\frac{2}{4}\sum \sin^2\psi\cos\psi}\right) - \gamma\lambda\frac{\mu}{4}\underline{\frac{2}{4}\sum \sin\psi\cos\psi}$$

The underlined terms are to be replaced by

$$\begin{aligned}
\frac{2}{4}\sum \overset{*}{\beta}\sin\psi\cos\psi &= \frac{2}{4}\sum \overset{*}{\beta}\,\frac{1}{2}\sin 2\psi = -\overset{*}{B}_2\sin 2\psi \\
\frac{2}{4}\sum \beta\cos^2\psi &= \frac{2}{4}\sum \beta\frac{1}{2}(1+\cos 2\psi) = B_0 - 2B_2\cos 2\psi \\
\frac{2}{4}\sum \beta\sin 2\psi\cos\psi &= \frac{2}{4}\sum \beta\frac{1}{2}(\sin 3\psi + \sin\psi) \\
&= \frac{2}{4}\sum \beta\frac{1}{2}(\sin 4\psi\cos\psi - \cos 4\psi\sin\psi + \sin\psi) \\
&= \frac{1}{2}B_{1c}\sin 4\psi - \frac{1}{2}B_{1s}\cos 4\psi + \frac{1}{2}B_{1s} \\
\frac{2}{4}\sum \sin^2\psi\cos\psi &= \frac{2}{4}\sum \frac{1}{2}(\cos\psi - \cos 2\psi\cos\psi) = 0 \\
\frac{2}{4}\sum \sin\psi\cos\psi &= 0
\end{aligned}$$

Apply the third operator to obain the B_{1s} equation.

$$\overset{**}{B}_{1s} - 2\overset{*}{B}_{1c} - B_{1s} + \frac{\gamma}{8}\left(\overset{*}{B}_{1s} + B_{1c}\right) + \mu\frac{\gamma}{6}\underline{\frac{2}{4}\sum \overset{*}{\beta}\sin^2\psi} + \nu_\beta^2 B_{1c} + \mu\frac{\gamma}{6}\underline{\frac{2}{4}\sum \beta\cos\psi\sin\psi}$$
$$+\mu^2\frac{\gamma}{8}\underline{\frac{2}{4}\sum \beta\sin 2\psi\sin\psi} = \gamma\theta_0\left(\frac{\mu}{3}\underline{\frac{2}{4}\sum \sin^2\psi} + \frac{\mu^2}{4}\underline{\frac{2}{4}\sum \sin^3\psi}\right) - \gamma\lambda\frac{\mu}{4}\underline{\frac{2}{4}\sum \sin^2\psi}$$

The underlined terms are to be replaced by

$$\begin{aligned}
\frac{2}{4}\sum \overset{*}{\beta}\sin^2\psi &= \frac{2}{4}\sum \overset{*}{\beta}\,\frac{1}{2}(1-\cos 2\psi) = \overset{*}{B}_0 + \frac{1}{2}\overset{*}{B}_2\cos 2\psi \\
\frac{2}{4}\sum \beta\cos\psi\sin\psi &= \frac{2}{4}\sum \beta\frac{1}{2}\sin 2\psi = -2B_2\sin 2\psi \\
\frac{2}{4}\sum \beta\sin 2\psi\sin\psi &= \frac{2}{4}\sum \beta\frac{1}{2}(\cos\psi - \cos 3\psi) \\
&= \frac{1}{2}B_{1c} - \frac{2}{4}\sum \beta\frac{1}{2}(\cos 4\psi\cos\psi + \sin 4\psi\sin\psi) \\
&= \frac{1}{2}B_{1c} - \frac{1}{2}B_{1c}\cos 4\psi - \frac{1}{2}B_{1s}\sin 4\psi \\
\frac{2}{4}\sum \sin^2\psi &= \frac{2}{4}\sum \frac{1}{2}(1-\cos 2\psi) = \frac{1}{4} \\
\frac{2}{4}\sum \sin^3\psi &= \frac{2}{4}\sum\left(\frac{3}{4}\sin\psi - \frac{1}{4}\sin 3\psi\right) = 0
\end{aligned}$$

Apply the fourth operator to obtain the B_2 equation.

$$\overset{**}{B}_2 + \frac{\gamma}{8}\overset{*}{B}_2 + \mu\frac{\gamma}{6}\underline{\frac{1}{4}\sum\overset{*}{\beta}(-1)^m\sin\psi} + \nu_\beta^2 B_2 + \mu\frac{\gamma}{6}\underline{\frac{1}{4}\sum\beta(-1)^m\cos\psi} + \mu^2\frac{\gamma}{8}\underline{\frac{1}{4}\sum\beta(-1)^m\sin 2\psi}$$
$$= \gamma\theta_0\frac{\mu^2}{4}\underline{\frac{1}{4}\sum(-1)^m\sin^2\psi}$$

The underlined terms are to be replaced by

$$\begin{aligned}
\frac{1}{4}\sum\overset{*}{\beta}(-1)^m\sin\psi &= \frac{1}{4}\sum\overset{*}{\beta}(-1)^m\sin(2\psi-\psi)\\
&= \frac{1}{4}\sum\overset{*}{\beta}(-1)^m(\sin 2\psi\cos\psi - \cos 2\psi\sin\psi)\\
&= \frac{1}{4}\left[-\overset{*}{\beta}{}^{(1)}\sin 2\psi\cos\psi_1 - \overset{*}{\beta}{}^{(2)}\sin 2\psi\cos\psi_2 - \overset{*}{\beta}{}^{(3)}\sin 2\psi\cos\psi_3 - \overset{*}{\beta}{}^{(4)}\sin 2\psi\cos\psi_4\right]\\
&- \frac{1}{4}\left[-\overset{*}{\beta}{}^{(1)}\cos 2\psi\sin\psi_1 - \overset{*}{\beta}{}^{(2)}\cos 2\psi\sin\psi_2 - \overset{*}{\beta}{}^{(3)}\cos 2\psi\sin\psi_3 - \overset{*}{\beta}{}^{(4)}\cos 2\psi\sin\psi_4\right]\\
&= -\frac{1}{2}\sin 2\psi\frac{2}{4}\sum\overset{*}{\beta}\cos\psi + \frac{1}{2}\cos 2\psi\frac{2}{4}\sum\overset{*}{\beta}\sin\psi\\
&= -\frac{1}{2}\sin 2\psi\left(\overset{*}{B}_{1c} + B_{1s}\right) + \frac{1}{2}\cos 2\psi\left(\overset{*}{B}_{1s} - B_{1c}\right)\\
\frac{1}{4}\sum\beta(-1)^m\cos\psi &= -\frac{1}{2}B_{1c}\cos 2\psi - \frac{1}{2}B_{1s}\sin 2\psi\\
\frac{1}{4}\sum\beta(-1)^m\sin 2\psi &= -B_0\sin 2\psi\\
\frac{1}{4}\sum(-1)^m\sin^2\psi &= \frac{1}{4}\sum(-1)^m\frac{1}{2}(1-\cos 2\psi) = \frac{1}{2}\cos 2\psi
\end{aligned}$$

Consider the flap equation in the rotating coordinates, eqn.2.129. The terms associated with forward speed, μ, and μ^2 terms, are all periodic in nature associated with sine and cosine harmonics. Now consider the equations in the fixed coordinates. Note that all the μ terms now occur as constants, not in association with sine and cosine harmonics. The μ^2 terms are still periodic in nature and occur as sine or cosine harmonics. The fact that the constant coefficients in the fixed coordinate equations retain the effect of forward speed can be utilized during the calculation of aeroelastic stability.

2.10 Aeroelastic Stability

Consider the rigid flap equation in forward flight. Assume that the twist $\theta_{tw} = 0$ for simplicity. Let $\beta_s(\psi)$ be the steady state flap solution. Then we have

$$\begin{aligned}
&\overset{**}{\beta}_s + \left(\frac{\gamma}{8} + \mu\frac{\gamma}{6}\sin\psi\right)\overset{*}{\beta}_s + \left(\nu_\beta^2 + \mu\frac{\gamma}{6}\cos\psi + \mu^2\frac{\gamma}{8}\sin 2\psi\right)\beta_s = \\
&\qquad \gamma\theta\left(\frac{1}{8} + \frac{\mu}{3}\sin\psi + \frac{\mu^2}{4}\sin^2\psi\right) - \gamma\lambda\left(\frac{1}{6} + \frac{\mu}{4}\sin\psi\right)
\end{aligned} \tag{2.130}$$

where $\theta = \theta(\psi) = \theta_0 + \theta_{1c}\cos\psi + \theta_{1s}\sin\psi$ in forward flight. Suppose a perturbation $\delta\beta$ is applied to the steady state flap motion at $\psi = \psi_0$. We seek the nature of its evolution with time $\delta\beta(\psi)$. At any instant $\beta(\psi) = \beta_s(\psi) + \delta\beta(\psi)$ must satisfy the governing eqn.2.129. Substituting $\beta(\psi)$ in the governing equation and noting that the steady state solution $\beta_s(\psi)$ must satisfy eqn.2.130, we have the governing equation for the perturbation

$$\overset{**}{\delta\beta} + \left(\frac{\gamma}{8} + \mu\frac{\gamma}{6}\sin\psi\right)\overset{*}{\delta\beta} + \left(\nu_\beta^2 + \mu\frac{\gamma}{6}\cos\psi + \mu^2\frac{\gamma}{8}\sin 2\psi\right)\delta\beta = 0 \tag{2.131}$$

The 'δ's are dropped.

$$\overset{**}{\beta} + \left(\frac{\gamma}{8} + \mu\frac{\gamma}{6}\sin\psi\right)\overset{*}{\beta} + \left(\nu_\beta^2 + \mu\frac{\gamma}{6}\cos\psi + \mu^2\frac{\gamma}{8}\sin 2\psi\right)\beta = 0 \tag{2.132}$$

Note that for a linear system, the perturbation equation is identical to the main equation with zero forcing.

2.10.1 Stability roots in hover

In hover, the perturbation equation becomes

$$\overset{**}{\beta} + \frac{\gamma}{8}\overset{*}{\beta} + \nu_\beta^2\beta = 0 \tag{2.133}$$

Seek a perturbation solution of the form

$$\beta = \beta_0 e^{s\psi}$$

The nature of s determines whether the perturbation grows or dies down with time. Substitute in the governing equation. For non-trivial β_0, i.e., $\beta_0 \neq 0$, s must satisfy the following equation

$$s^2 + \frac{\gamma}{8}s + \nu_\beta^2 = 0$$

which leads to the following eigenvalues

$$s_R = -\frac{\gamma}{16} \pm i\sqrt{\nu_\beta^2 - \left(\frac{\gamma}{16}\right)^2} \tag{2.134}$$

where s_R denotes the stability roots of the rotating coordinates. These are complex conjugate pairs. There are four pairs of such roots, one for each blade. Note that in the absence of aerodynamics, and structural damping if any, the stability roots are simply the natural frequencies of the system. For example, here they would be simply $\pm\nu_\beta$. The evolution of the perturbation with time is then

$$\beta = \beta_0 e^{s_R\psi} = \beta_0 e^{-\frac{\gamma}{16}}\psi e^{\pm i\sqrt{\nu_\beta^2 - \left(\frac{\gamma}{16}\right)^2}\psi}$$

The real part of the eigenvalue represents damping of the perturbation. Ihe imaginary part represents the frequency of oscillation of the perturbation. The stability roots in eqn.2.134 can be written in standard notation as

$$s_R = -\text{ decay rate} \pm i \text{ damped frequency} = -\zeta\nu_n \pm i\nu_n\sqrt{1-\zeta^2}$$

where $\nu_n\sqrt{1-\zeta^2} = \nu_d$ is the damped frequency in /rev and ξ is the damping ratio as a percentage of critical. These are the frequency of oscillation of the perturbation and its decay rate. They are expressed in terms of ν_n, the natural frequency of the system. The natural frequency is the frequency of oscillation of the perturbation in the case of zero damping and no aerodyanmics. In general, these parameters can be extracted from stability roots by making use of the above standard notation as follows

$$\begin{aligned}
\nu_d &= \text{Im}(s) \\
\zeta &= -\frac{\text{Re}(s)}{|s|} \\
\nu_n &= |s|
\end{aligned} \tag{2.135}$$

A positive ζ means stable system, negative ζ means unstable system, and $\zeta = 0$ means neutral stability. In the last case, the perturbation once introduced neither grows nor decays with time.

To get a feel of these numbers, consider a typical Lock number $\gamma = 8$ and an articulated rotor with flap frequency $\nu_\beta = 1$.

$$\text{critical damping ratio} \;\; \zeta = \frac{\gamma}{16\nu_\beta} = 0.5 \implies 50\% \text{ damping}$$

$$\text{damped frequency} \;\; \nu_d = \sqrt{\nu_\beta^2 - \left(\frac{\gamma}{16}\right)^2} = 0.87/\text{rev}$$

Therefore, the flap mode is highly damped and the frequency of oscillation as a perturbation dies down is less than 1/rev. The damped frequency in Hz and the time period are given by

$$f_d = \frac{\nu_d\,\Omega}{2\pi} \quad \text{in cycles/sec or Hertz}$$

$$T_d = \frac{1}{f_d} \quad \text{in sec/cycle}$$

Consider now the stability roots in the fixed coordinates. The flap perturbation eqn.2.133 in fixed coordinates produce the following equations for a N_b bladed rotor. The B_0 and B_d perturbation equations (where B_d exists only in the case of even N_b) are

$$\overset{**}{B}_0 + \frac{\gamma}{8} B_0 + \nu_\beta^2 B_0 = 0$$

$$\overset{**}{B}_d + \frac{\gamma}{8}\overset{*}{B}_d + \nu_\beta^2 B_d = 0$$

These equations are identical to the rotating frame equation. Therefore, for B_0 and B_d the fixed coordinate eigenvalues are the same as the rotating coordinate eigenvalue.

$$s_F = s_R$$

The B_{nc} and B_{ns} perturbation equations (B_{nc} and B_{ns} exist only in the case of $N_b > 2$) are

$$\left\{ \begin{array}{c} \overset{**}{B}_{nc} \\ \overset{**}{B}_{ns} \end{array} \right\} + \begin{bmatrix} \frac{\gamma}{8} & 2n \\ -2n & \frac{\gamma}{8} \end{bmatrix} \left\{ \begin{array}{c} \overset{*}{B}_{nc} \\ \overset{*}{B}_{ns} \end{array} \right\} + \begin{bmatrix} \nu_\beta^2 - n^2 & \frac{\gamma}{8}n \\ -\frac{\gamma}{8}n & \nu_\beta^2 - n^2 \end{bmatrix} \left\{ \begin{array}{c} B_{nc} \\ B_{ns} \end{array} \right\} = \left\{ \begin{array}{c} 0 \\ 0 \end{array} \right\}$$

For a particular n, seek solution of the type

$$\left\{ \begin{array}{c} B_{nc}(\psi) \\ B_{ns}(\psi) \end{array} \right\} = \left\{ \begin{array}{c} B_{nc0} \\ B_{ns0} \end{array} \right\} e^{s\psi}$$

where s is an eigenvalue in the fixed coordinate which determines the azimuthal evolution of perturbations in B_{nc} and B_{ns}. Substitute the solution type into the governing equation.

$$\begin{bmatrix} s^2 + \frac{\gamma}{8}s + \nu_\beta^2 - n^2 & 2ns + \frac{\gamma}{8}n \\ -(2ns + \frac{\gamma}{8}n) & s^2 + \frac{\gamma}{8}s + \nu_\beta^2 - n^2 \end{bmatrix} \begin{bmatrix} B_{nc0} \\ B_{ns0} \end{bmatrix} = 0 \tag{2.136}$$

For nontrivial $[B_{nc0}\, B_{ns0}]^T$ set the determinant of the left hand side matrix to zero. It follows

$$(s^2 + \frac{\gamma}{8}s + \nu_\beta^2 - n^2)^2 = -\left(2ns + \frac{\gamma}{8}n\right)^2 \tag{2.137}$$

This leads to two complex conjugate pairs of eigenvalues, i.e. four eigenvalues in total

$$\begin{aligned} s_F &= -\frac{\gamma}{16} \mp i\sqrt{\nu_\beta^2 - \left(\frac{\gamma}{16}\right)^2} \pm in \\ &= s_R \pm in \end{aligned} \tag{2.138}$$

Thus the eigenvalues for B_{nc} and B_{ns} are shifted by $\pm$ n/rev from the rotating eigenvalue. The exponential decay rate is the same in both the fixed frame and the rotating frame. The corresponding eigenvectors, using eqn.2.137 are given by

$$\frac{B_{nc0}}{B_{ns0}} = -\frac{2ns_F + \frac{\gamma}{8}n}{s_F^2 + \frac{\gamma}{8}s_F + \nu_\beta^2 - n^2} = \pm i = e^{\pm i\frac{\pi}{2}} \tag{2.139}$$

The eigenvalues and their corresponding eigenvectors are tabulated below. Out of the two complex conjugate pairs, one has higher frequency compared to the other. These are noted as high frequency and low frequency eigenvalues.

$$s_F = -\frac{\gamma}{16} + i\left(\sqrt{\ldots} + 1\right) \quad \text{High Frequency,} \quad \text{obtained using:} \quad s_F = s_R + i \implies \frac{B_{nc0}}{B_{ns0}} = +i$$

$$s_F = -\frac{\gamma}{16} - i\left(\sqrt{\ldots} + 1\right) \quad \text{High Frequency,} \quad \text{obtained using:} \quad s_F = s_R - i \implies \frac{B_{nc0}}{B_{ns0}} = -i$$

$$s_F = -\frac{\gamma}{16} + i\left(\sqrt{\ldots} - 1\right) \quad \text{Low Frequency,} \quad \text{obtained using:} \quad s_F = s_R - i \implies \frac{B_{nc0}}{B_{ns0}} = -i$$

$$s_F = -\frac{\gamma}{16} - i\left(\sqrt{\ldots} - 1\right) \quad \text{Low Frequency,} \quad \text{obtained using:} \quad s_F = s_R + i \implies \frac{B_{nc0}}{B_{ns0}} = +i$$

where the entry within the square root, $\nu_\beta^2 - (\gamma/16)^2$ has been replaced with $\ldots$ for brevity. The first line means s_F as given is a high frequency. It has been obtained by using $s_R + i$, which implies that the corresponding eigenvector $B_{nc0}/B_{ns0} = +i$. Each frequency with its associated eigenvector is referred to as a mode. Note that $s_R + i$ is not necessarily the high frequency mode. Similarly $s_R - i$ is not necessarily the low frequency mode. Consider each mode one by one. The first high frequency mode is given by

$$\begin{aligned}
B_{nc}(\psi) &= B_{nc0}\; e^{s_F\psi} \\
&= B_{ns0}\; (+i)e^{s_F\psi} \\
&= B_{ns0}\; e^{i\frac{\pi}{2}}e^{s_F\psi} \\
&= B_{ns0}\; e^{-\frac{\gamma}{16}\psi}e^{i(\sqrt{\ldots}+1)\psi}e^{i\frac{\pi}{2}} \\
&\approx B_{ns0}\; e^{-\frac{\gamma}{16}\psi}\cos\left[\left(\sqrt{\ldots}+1\right)\psi + \frac{\pi}{2}\right]
\end{aligned} \tag{2.140}$$

Thus $B_{nc}(\psi)$ has the same magnitude as $B_{ns}(\psi)$, same decay rate $-\gamma/16$, same frequency $\sqrt{\ldots}+1$, except that it is ahead of $B_{ns}(\psi)$ by $\pi/2$. Consider the second high frequency mode

$$\begin{aligned}
B_{nc}(\psi) &= B_{nc0}\; e^{s_F\psi} \\
&= B_{ns0}\; (-i)e^{s_F\psi} \\
&= B_{ns0}\; e^{-i\frac{\pi}{2}}e^{s_F\psi} \\
&= B_{ns0}\; e^{-\frac{\gamma}{16}\psi}e^{i(\sqrt{\ldots}+1)\psi}e^{-i\frac{\pi}{2}} \\
&\approx B_{ns0}\; e^{-\frac{\gamma}{16}\psi}\cos\left[\left(\sqrt{\ldots}+1\right)\psi + \frac{\pi}{2}\right]
\end{aligned} \tag{2.141}$$

Again, $B_{nc}(\psi)$ has the same magnitude as $B_{ns}(\psi)$, same decay rate $-\gamma/16$, same frequency $\sqrt{\ldots}+1$, except that it is ahead of $B_{ns}(\psi)$ by $\pi/2$. In both the high frequency modes $B_{nc}(\psi)$ leads $B_{ns}(\psi)$ by $\pi/2$. This is defined as a 'Progressive Mode'. Figure 2.31 shows the fixed coordinate perturbation variations for $B_{nc}(\psi)$ and $B_{ns}(\psi)$ for a Progressive Mode.

Now consider the low frequency modes. From the first mode, we have

$$\begin{aligned}
B_{nc}(\psi) &= B_{nc0}\ e^{s_F}\psi \\
&= B_{ns0}\ (-i)e^{s_F}\psi \\
&= B_{ns0}\ e^{-i\frac{\pi}{2}}e^{s_F}\psi \\
&= B_{ns0}\ e^{-\frac{\gamma}{16}\psi}e^{i(\sqrt{\cdots}-1)\psi}e^{-i\frac{\pi}{2}} \\
&\approx B_{ns0}\ e^{-\frac{\gamma}{16}\psi}\cos\left[\left(\sqrt{\ldots}-1\right)\psi-\frac{\pi}{2}\right]
\end{aligned} \tag{2.142}$$

Similarly, from the second mode, we have

$$\begin{aligned}
B_{nc}(\psi) &= B_{nc0}\ e^{s_F}\psi \\
&= B_{ns0}\ (+i)e^{s_F}\psi \\
&= B_{ns0}\ e^{i\frac{\pi}{2}}e^{s_F}\psi \\
&= B_{ns0}\ e^{-\frac{\gamma}{16}\psi}e^{i(\sqrt{\cdots}-1)\psi}e^{i\frac{\pi}{2}} \\
&\approx B_{ns0}\ e^{-\frac{\gamma}{16}\psi}\cos\left[\left(\sqrt{\ldots}-1\right)\psi-\frac{\pi}{2}\right]
\end{aligned} \tag{2.143}$$

Here $B_{nc}(\psi)$ can again lead $B_{ns}(\psi)$ by $\pi/2$, but only if

$$\sqrt{\nu_\beta^2-\left(\frac{\gamma}{16}\right)^2}-1<0 \quad \text{i.e., if} \quad \sqrt{\nu_\beta^2-\left(\frac{\gamma}{16}\right)^2}<1$$

In this case the low frequency mode is again a 'Progressive Mode'. Otherwise, if

$$\sqrt{\nu_\beta^2-\left(\frac{\gamma}{16}\right)^2}-1>0 \quad \text{i.e., if} \quad \sqrt{\nu_\beta^2-\left(\frac{\gamma}{16}\right)^2}>1$$

$B_{nc}(\psi)$ lags $B_{ns}(\psi)$ by $\pi/2$. This is defined as a 'Regressive Mode'.

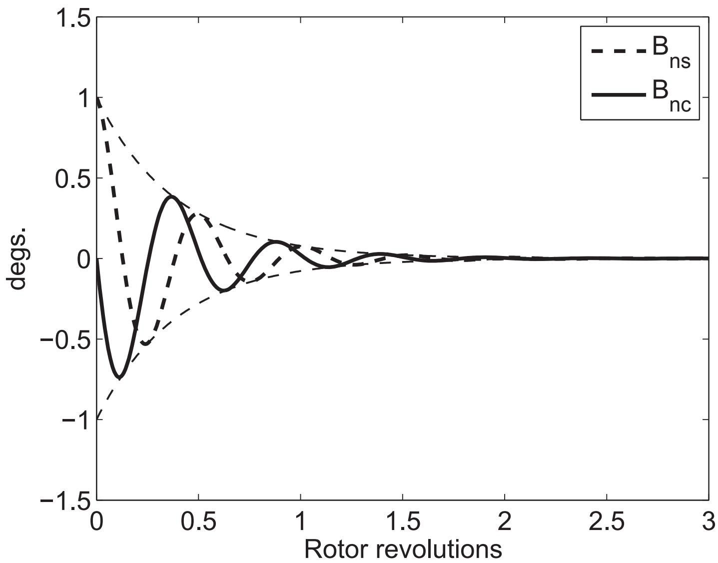


Figure 2.31: **Progressive mode of flapping: Fixed coordinate B_{nc} leads B_{ns} by $\pi/2$**

<u>Example: 2.9</u>

A four bladed rotor has a fundamental flap frequency of 1.12 /rev and a Lock number of 8. Calculate the hover eigenvalues in the rotating and the fixed coordinates. Discuss the nature of the modes in the fixed coordinates.

In the rotating coordinates, we have

$$\nu_\beta = 1.12 \qquad \gamma = 8$$

$$s_R = -\frac{\gamma}{16} \pm i\sqrt{\nu_\beta^2 - \left(\frac{\gamma}{16}\right)^2} = -0.5 \pm i\,1.002$$

Four blades have four identical pairs of rotating stability roots. In the fixed coordinates, we have $n = 1$, four complex conjugate pairs of roots. For collective B_0 and differential B_2, the eigenvalues are the same as the rotating roots

$$s_F = s_R = -0.5 \pm i\,1.002$$

For B_{1c} and B_{1s}

$$s_F = s_R \pm i$$

Thus the high frequency mode is

$$s_F = -0.5 \pm i\,2.002$$

The low frequency mode is

$$s_F = -0.5 \pm i\,0.002$$

The high frequency modes are always progressive. The low frequency mode can be either progressive or regressive. Here

$$\sqrt{\nu_\beta^2 - \left(\frac{\gamma}{16}\right)^2} = 1.002 \ \text{ i.e. } \ > 1.0$$

Therefore the low frequency mode is regressive.

Example: 2.10

For the flutter testing of a helicopter blade, the rotor was excited by wobbling the swash plate and the response was measured from the pick-ups mounted on all the four blades. The response of the lowest mode in the fixed system was analyzed using the Moving Block method. The frequency of oscillations and the damping coefficient were obtained as 1.25 Hz and 0.5 respectively. Calculate the corresponding blade frequency and damping coefficient (in rotating system) for a rotor rpm of 350.

We have in the fixed coordinates

$$\omega_d = 1.25Hz = 7.854 \text{ rad/sec}$$

$$\zeta = .5$$

$$\Omega = 350 \text{ RPM } = 36.65 \text{ rad/sec}$$

$$\nu_d = \frac{\omega_d}{\Omega} = \frac{7.854}{36.65} = 0.2143/\text{rev}$$

$$\text{Natural} \qquad \omega_n = \frac{\omega_d}{\sqrt{1-\zeta^2}}$$

$$\frac{\omega_n}{\Omega} = .2475$$

$$\zeta\frac{\omega_n}{\Omega} = .1238$$

Eigenvalue $\quad S = \zeta\frac{\omega_n}{\Omega} + i\frac{\omega_d}{\Omega} = .1238 \pm i\,.2143$

Rotating frame: $\quad S_R = S + i$

$$= .1238 + i\,1.2143$$

$$\frac{\zeta}{\sqrt{1-\zeta^2}} = \frac{.1238}{1.2143} = .1019$$

$$\zeta = .1010$$

Frequency $\quad = 1.2143/\text{rev} \ = 7.08Hz$

2.11 Stability Analysis in Forward flight

In dynamic analysis of rotating systems, one frequently encounters with the equations of motion with periodic coefficients. For example, the equation of motion expressing the dynamic response of a flapping blade in forward flight contains many periodic terms. For some dynamic problems, one also gets equations with constant coefficients. Here, the example is the flapping motion of a blade in hovering flight. The analysis techniques for constant coefficient systems are simple and familiar whereas analysis for periodic systems is more involved and the analysis techniques are less familiar.

2.11.1 Constant Coefficient System

Let us consider N linear differential equations with constant coefficients,

$$\underbrace{\widetilde{M}\ddot{\widetilde{q}}}_{\text{inertia}} + \underbrace{\widetilde{C}\dot{\widetilde{q}}}_{\text{damping}} + \underbrace{\widetilde{K}\widetilde{q}}_{\text{stiffness}} = \underbrace{\widetilde{F}(t)}_{\text{force}} \tag{2.144}$$

where $\widetilde{M}$, $\widetilde{C}$ and $\widetilde{K}$ are square matrices of order N x N while displacement vector $\widetilde{q}$ and force vector $\widetilde{F}$ are of order N x 1.

These equations can be rewritten as

$$\begin{bmatrix} \widetilde{I} & \widetilde{0} \\ \widetilde{0} & \widetilde{M} \end{bmatrix} \begin{bmatrix} \dot{\widetilde{q}} \\ \ddot{\widetilde{q}} \end{bmatrix} - \begin{bmatrix} \widetilde{0} & \widetilde{I} \\ -\widetilde{K} & -\widetilde{C} \end{bmatrix} \begin{bmatrix} \widetilde{q} \\ \dot{\widetilde{q}} \end{bmatrix} = \begin{bmatrix} \widetilde{0} \\ \widetilde{F} \end{bmatrix}$$

where $\widetilde{I}$ = identity matrix (unity on diagonal)
order N x N
$\widetilde{0}$ = null matrix (zeros)
order N x N

Let us define

$$\widetilde{y} = \begin{bmatrix} \dot{\widetilde{q}} \\ \widetilde{q} \end{bmatrix}_{2N\times 1}$$

The above equations can be rearranged as

$$\dot{\widetilde{y}} = \widetilde{A}\widetilde{y} + \widetilde{G} \tag{2.145}$$

This results into 2N first order equations.

$$\widetilde{A} = \begin{bmatrix} \widetilde{0} & \widetilde{I} \\ -\widetilde{M}^{-1}\widetilde{K} & -\widetilde{M}^{-1}\widetilde{C} \end{bmatrix}_{2N\times 2N}$$

$$\widetilde{G} = \begin{bmatrix} \widetilde{0} \\ -\widetilde{M}^{-1}\widetilde{F} \end{bmatrix}_{2N\times 1}$$

The above arrangements are valid provided $\widetilde{M}$ is not singular.

Stability

To examine stability of the system, set $\widetilde{F} = 0$ i.e., $\widetilde{G} = 0$. This results into a set of homogeneous equations and then seek the solution as

$$\widetilde{y}(t) = \widetilde{y}e^{\lambda t}$$

The equation (78) becomes

$$\widetilde{A}\widetilde{y} = \lambda\widetilde{y} \tag{2.146}$$

This results into a standard algebraic eigenvalue problem and can be solved using any standard eigenvalue routine. This gives 2N eigenvalues, complex in nature.

$$\lambda_k = \underset{\text{real}}{\alpha_k} + \underset{\text{imaginary}}{i\,\omega_k}$$

The real part of the eigenvalue represents the damping of the mode whereas the imaginary part represents the frequency of the mode. If any one of the eigenvalues has a positive real part, the system is unstable.

Forced response

Under steady conditions, the external forces F(t) in a rotating system are generally periodic, in multiples of the rotation frequency Ω. Let us say the forcing function if m^{th} harmonic, frequency $\omega_m = m\Omega$.

$$\widetilde{q}(t) = Re(\widetilde{q}e^{i\omega_m t}) \tag{2.147}$$

$$= \widetilde{F}_R \cos\omega_m t - \widetilde{F}_I \sin\omega_m t$$

where $\widetilde{F}_R$ and $\widetilde{F}_I$ are real and imaginary parts of $\widetilde{F}$. Assuming the steady response to be m^{th} harmonic

$$\widetilde{q}(t) = \text{Re}(\widetilde{q}e^{i\omega_m t})$$

$$= \widetilde{q}_R \cos\omega_m t - \widetilde{q}_I \sin\omega_m t$$

Placing this in basic equation (78) and using the harmonic balance method (discussed earlier). Comparing $\sin\omega_m t$ and $\cos\omega_m t$ terms, one gets,

$$\underset{2N\times 2N}{\begin{bmatrix} \widetilde{G} & \widetilde{H} \\ -\widetilde{H} & \widetilde{G} \end{bmatrix}} \underset{2N\times 1}{\begin{Bmatrix} \widetilde{q}_R \\ \widetilde{q}_I \end{Bmatrix}} = \underset{2N\times 1}{\begin{Bmatrix} \widetilde{F}_R \\ \widetilde{F}_I \end{Bmatrix}} \tag{2.148}$$

where

$$\widetilde{G} = \widetilde{K} - \omega_m^2 \widetilde{M}$$

$$\widetilde{H} = \omega_m \widetilde{C}$$

For the known external forces, these equations can be solved to calculate $\widetilde{q}_R$ and $\widetilde{q}_I$. The total response can be calculated by summing up the response components from all harmonics.

$$\widetilde{q}(t) = \sum_{m=0}^{N} q_R^{(m)} \cos \omega_m t - \sum_{m=0}^{N} q_I^{(m)} \sin \omega_m t \tag{2.149}$$

This method is more physical than the direct numerical integration because individual harmonic components are calculated and assessed. One can also use the finite difference method to calculate response but it generally results into more involved analysis for rotor problems.

2.11.2 Periodic coefficient systems

The governing equations are

$$\widetilde{M}(t)\ddot{\widetilde{q}} + \widetilde{C}(t)\dot{\widetilde{q}} + \widetilde{K}\widetilde{q} = \widetilde{F}(t) \tag{2.150}$$

where matrices $\widetilde{M}$, $\widetilde{C}$ and $\widetilde{K}$ contain periodic terms. These equations can be rearranged as first order equations,

$$\dot{\widetilde{y}} - \widetilde{A}(t)\widetilde{y} = \widetilde{G}(t) \tag{2.151}$$

where $\widetilde{A}(t)$ and $\widetilde{G}(t)$ are periodic over an interval T.

2.11.3 Floquet stability solution

To investigate stability, set $\widetilde{G}(t)$. Seek solution of the form

$$\underset{2N \times 1}{\widetilde{y}(t)} = \underset{2N \times 2N}{\widetilde{B}(t)} \ \underset{2N \times 1}{\{c_k e^{p_k t}\}} \tag{2.152}$$

The square matrix $\widetilde{B}(t)$ is periodic over period T.

$$\widetilde{B}(T) = \widetilde{B}(0)$$

$$\widetilde{y}(0) = \widetilde{B}(0)\{c_k\}$$

$$\widetilde{y}(T) = \widetilde{B}(T)\{c_k e^{p_k t}\}$$

$$= \widetilde{B}(0)\{c_k e^{p_k t}\}$$

Also, one can express y(T) as,

$$\widetilde{y}(T) = \begin{bmatrix} \widetilde{y}^{(1)} & \widetilde{y}^{(2)} & \dots \end{bmatrix} \begin{Bmatrix} y_1(0) \\ y_1(0) \\ \vdots \end{Bmatrix}$$

where $\widetilde{y}^{(1)}$ is the solution at t = T of the basic equation with $\widetilde{G}(t) = 0$ for the initial condition $y_1(0) = 1$ and all remaining $y_i(0) = 0$, etc.

$$[Q] = \begin{bmatrix} \widetilde{y}^{(1)} & \widetilde{y}^{(2)} & \dots \end{bmatrix} \tag{2.153}$$

This is a square matrix of order 2N and is called as "transition matrix". Thus

$$\begin{aligned}\{y(T)\} &= [Q]\{y(0)\} \\ &= [Q][B(0)]\{c_k\} \\ &= [Q](\{B(0)\}_1\, c_1 + \{B(0)\}_2\, c_2 + \ldots)\end{aligned} \tag{2.154}$$

Another form is

$$\begin{aligned}\{y(T)\} &= [B(0)]\{c_k e^{p_k T}\} \\ &= \{B(0)\}_1\, c_1 e^{p_1 T} + \{B(0)\}_2\, c_2 e^{p_2 T} + \ldots\end{aligned} \tag{2.155}$$

Comparing Eqs. (86) and (87) one gets

$$[Q]\{B(0)\} = \lambda_k \{B(0)\}_k \tag{2.156}$$

where

$$\lambda_k = e^{p_k T}$$

This is a standard eigenvalue problem, where λ_k are the eigenvalues of the transition matrix [Q].

$$p_k = \frac{1}{T} ln(\lambda_k) = \alpha_k + i\,\omega_k$$

The real and imaginary parts of stability exponent p_k are

$$\begin{aligned}\alpha_k &= \frac{1}{T} ln|\lambda_k| \qquad (2.157) \\ &= \frac{1}{27} ln[(\lambda_k)_R^2 + (\lambda_k)_I^2] \\ \omega_k &= \frac{1}{T} \tan^{-1}[(\lambda_k)_I/(\lambda_k)_R] \qquad (2.158)\end{aligned}$$

The α_k measures the growth or decay of the response. The α_k positive (¿0) or λ_k greater than one indicates instability of the mode. The ω_k represents frequency of vibration. However, the $\tan^{-1}$ is multivalued, one will get multivalues for ω_k. Taking physical consideration one can choose the right value of ω_k.

2.11.4 Floquet response solution

The governing equation is

$$\widetilde{\dot{y}} = \widetilde{A}\widetilde{y} + \widetilde{G} \tag{2.159}$$

The solution of this equation can be obtained by direct numerical integration using some standard time integration techniques. With arbitrary initial conditions one needs many cycles of integration before a converged solution is obtained. Through a proper choice of initial conditions, one can however eliminate all transients from the response and the steady dynamic response can be calculated by integrating through only one period T. The objective of the Floquet method is to calculate the proper initial conditions.

Let us assume a general solution

$$\widetilde{y}(t) = \widetilde{y}_H(t) + \widetilde{y}_p(t) \tag{2.160}$$

where $\widetilde{y}_H$ is the homogeneous solution and $\widetilde{y}_p$ is the particular solution. Let us say $\widetilde{y}_E(t)$ is the complete solution of the governing equation for a given set of initial conditions. One can add any