Mid-term Exam

- 4월26일(월)
- 15:30~17:00
- 33동 225,226,327,328,330,331

재료의 기계적 거동 (Mechanical Behavior of Materials)

VISCOELASTICITY (점탄성)

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Viscoelasticity

- **Elastic materials** deform with stress and quickly return to their original state if the stress is removed due to the bond stretching along crystallographic planes in an ordered solid

- **Viscous materials**, like honey, resist shear flow and strain with time when a stress is applied due to the diffusion of atoms or molecules inside an amorphous material.

- **Viscoelasticity** is the property of materials that exhibit both viscosity and elasticity during deformation and time-dependent strain.

Viscoelasticity

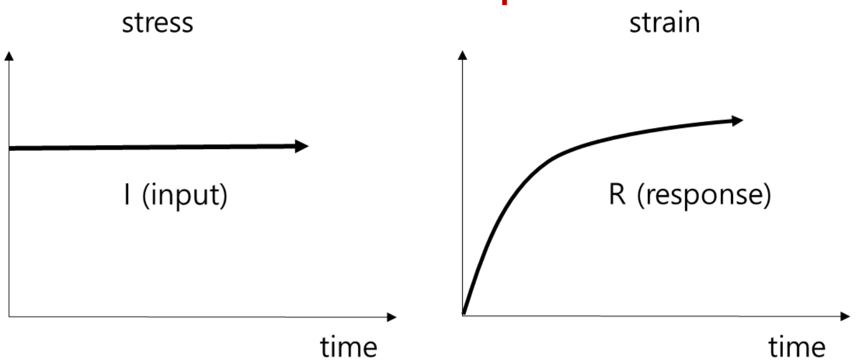
- **Creep:** Increase in strain with time as the stress or load is kept constant. Typical creep behavior shows that strain increases with time at a decreasing rate followed by a constant rate and finally increasing rate.

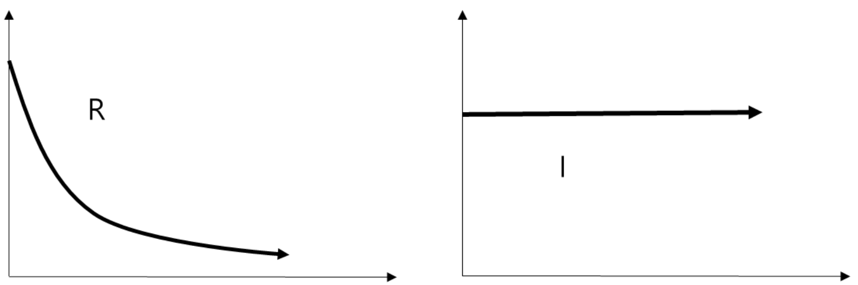
- **Recovery:** When the applied load is reduced (or instantly decreased), the strain decreases with time, partially or completely. i.e., anelastic, inelastic, elastic aftereffect

- **Relaxation:** Stress decreases with time when a strain is kept constant

Viscoelasticity

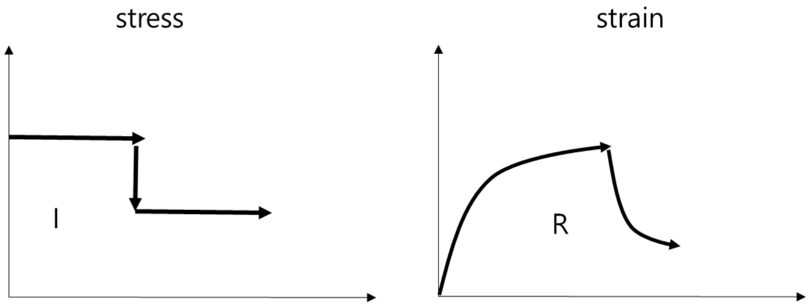
Creep



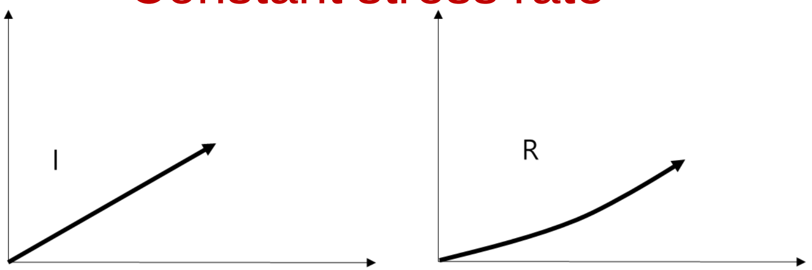


Relaxation

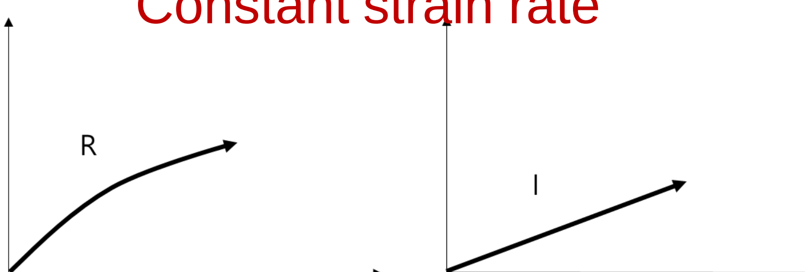
Recovery



Constant stress rate

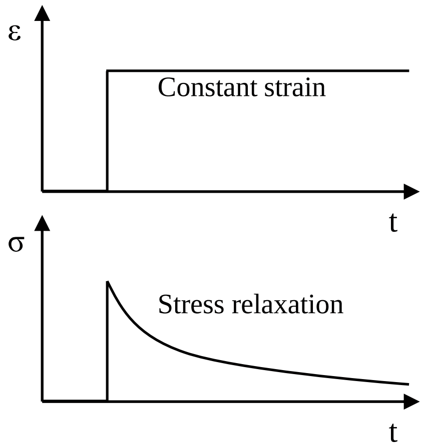


Constant strain rate

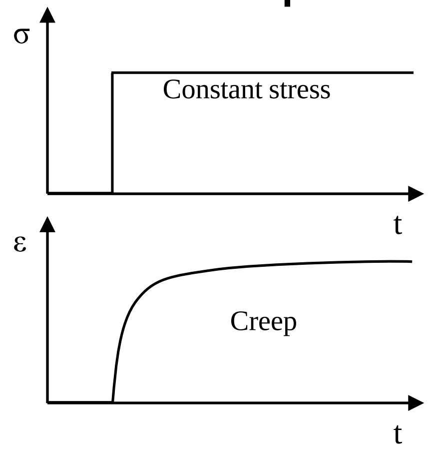


Phenomenon of Viscoelastic Materials

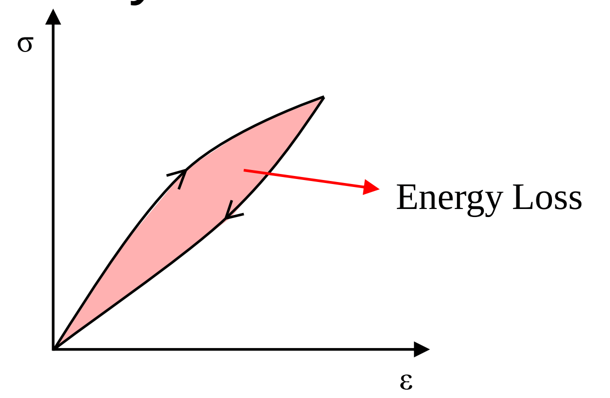
Stress Relaxation



Creep



Hysteresis



- If the stress is held constant, the strain increases with time **(creep)**
- If the strain is held constant, the stress decreases with time **(stress relaxation)**
- If a cyclic loading is applied, **hysteresis occurs**, leading to a dissipation of mechanical energy $\oint \boldsymbol{\sigma d\varepsilon}$

Constitutive models for linear viscoelasticity

$$\sigma = \sigma(t) \qquad\qquad \varepsilon = \varepsilon(t)$$

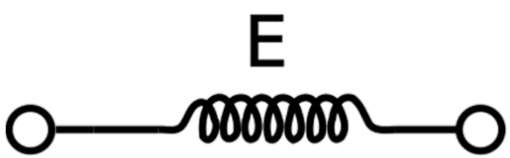
Since its viscous component,

the stress-strain relation of viscoelastic materials is

time-dependent!

Constitutive models for linear viscoelasticity

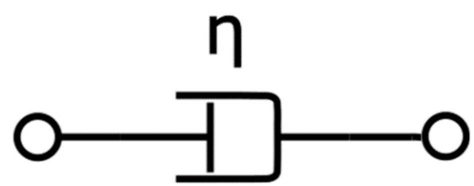
Viscoelasticity can be divided to elastic components and viscous components. We can model viscoelastic materials as **linear combinations** of ***springs*** and ***dashpots***.



The ***springs*** represent the ***elastic*** components.

$$\sigma = E\varepsilon$$

where E is the elastic modulus of the material.



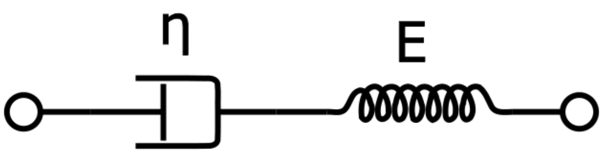
The ***dashpots*** represent the ***viscous*** components (perfect viscous fluid).

$$\sigma = \eta \frac{d\varepsilon}{dt} = \eta \dot{\varepsilon}$$

where η is the viscosity of the material and $d\varepsilon/dt$ is the strain rate.

* No immediate extension takes place at zero time when a sudden load is applied (like a rigid body)

Maxwell Model



$$\varepsilon = \varepsilon_s + \varepsilon_d \qquad \sigma = \sigma_s = \sigma_d$$

$$\varepsilon_s = \frac{\sigma}{E} \qquad \dot{\varepsilon}_d = \frac{\sigma}{\eta}$$

A purely viscous damper and purely elastic spring connected *in series.*

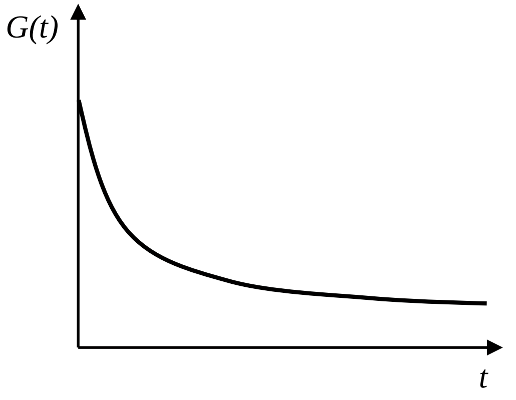
$$\boxed{\dot{\varepsilon} = \dot{\varepsilon}_s + \dot{\varepsilon}_d = \frac{\dot{\sigma}}{E} + \frac{\sigma}{\eta}}$$

Maxwell Model

$$\dot{\varepsilon} = \dot{\varepsilon}_s + \dot{\varepsilon}_d = \frac{\dot{\sigma}}{E} + \frac{\sigma}{\eta}$$

Stress Relaxation

$(\varepsilon = const.\,(\dot{\varepsilon} = 0))$

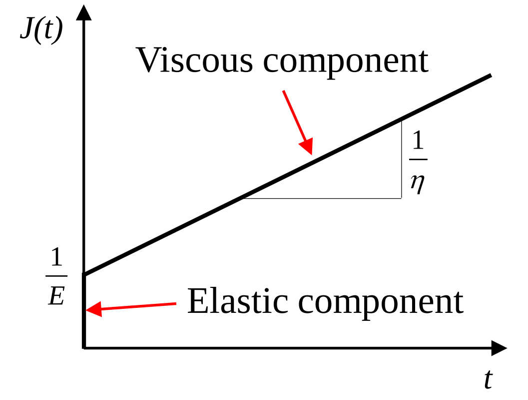


Relaxation Modulus *G(t)*

$$G_M(t) = \frac{\sigma(t)}{\varepsilon} = E \cdot \mathrm{e}^{-\frac{E}{\eta}t}$$

Creep

$(\sigma = const.\,(\dot{\sigma} = 0))$



Creep Compliance *J(t)*

$$J_M(t) = \frac{\varepsilon(t)}{\sigma} = \frac{1}{E} + \frac{1}{\eta}t$$

In creep, actual strain rate decreases with time!

Maxwell Model

Creep Compliance *J(t)*

$$d\varepsilon = \frac{\sigma_0}{\eta} dt$$

$$\varepsilon(t) = \frac{\sigma_0}{\eta} t + C = \frac{\sigma_0}{\eta} t + \frac{\sigma_0}{E}$$

$$\boxed{G_M(t) = \frac{\sigma(t)}{\varepsilon} = E \cdot \mathrm{e}^{-\frac{E}{\eta}t}}$$

Relaxation Modulus *G(t)*

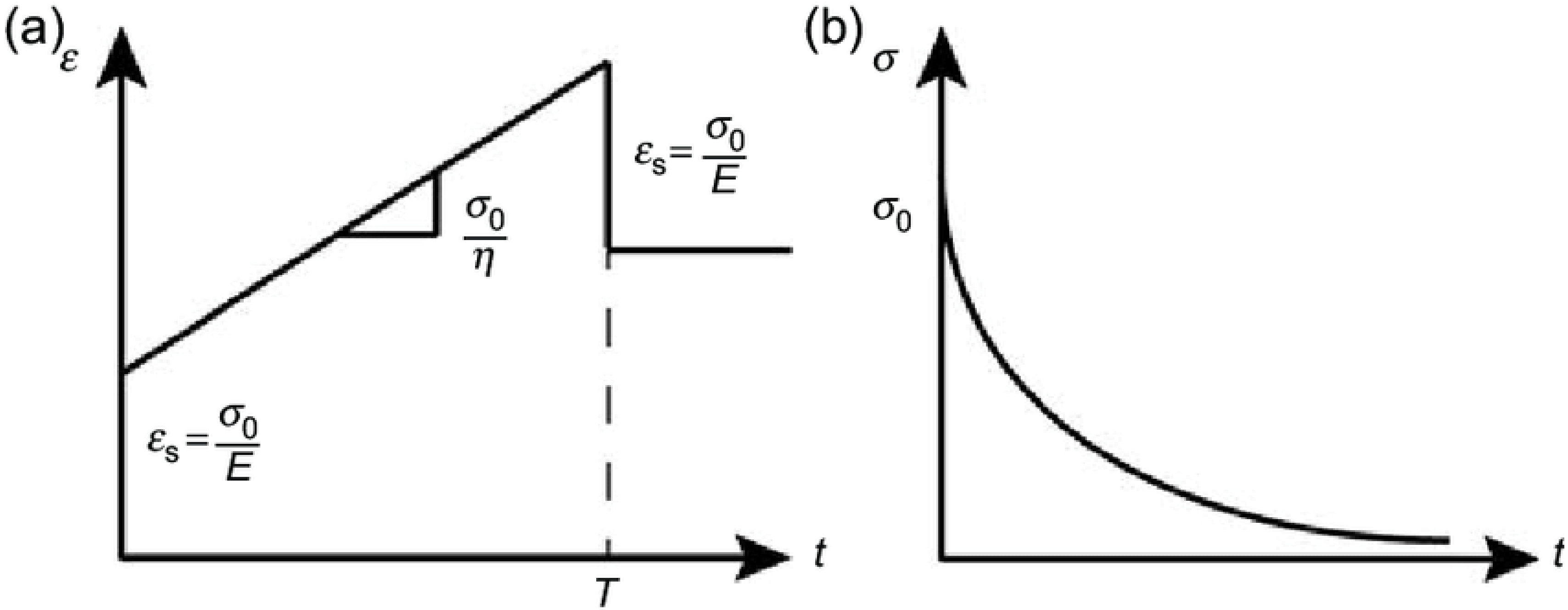
$$\frac{d\sigma}{\sigma} = -\frac{E}{\eta} dt$$

$$\ln\sigma = -\frac{E}{\eta} t + C = -\frac{E}{\eta} t + \ln\sigma_0$$

$$\sigma = \sigma_0 \exp(-\frac{E}{\eta} t) = E\varepsilon_0 \exp(-\frac{E}{\eta} t)$$

$$\boxed{G_M(t) = \frac{\sigma(t)}{\varepsilon_0} = E \cdot \mathrm{e}^{-\frac{E}{\eta}t}}$$

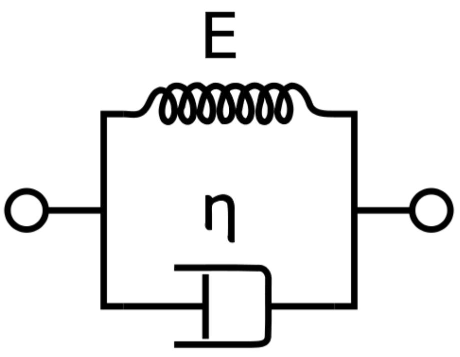
Maxwell Model



Creep & recovery

Relaxation

Voigt-Kelvin (V-K) Model



$$\varepsilon = \varepsilon_s = \varepsilon_d \qquad \sigma = \sigma_s + \sigma_d$$

$$\sigma_s = E \cdot \varepsilon \qquad \sigma_d = \eta \cdot \dot{\varepsilon}$$

A purely viscous damper and purely elastic spring connected *in parallel.*

$$\boxed{\sigma = \sigma_s + \sigma_d = E\varepsilon + \eta\dot{\varepsilon}}$$

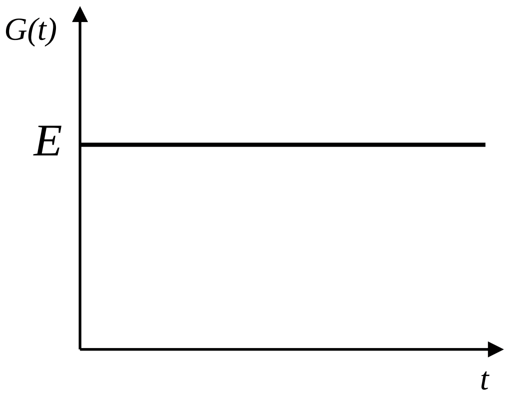
* When a constant stress is applied, the dashpot prevent instantaneous extension of spring and each component supports a portion of applied stress

V-K Model

$$\sigma = \sigma_s + \sigma_d = E\varepsilon + \eta\dot{\varepsilon}$$

Stress Relaxation

$(\varepsilon = const.\,(\dot{\varepsilon} = 0))$



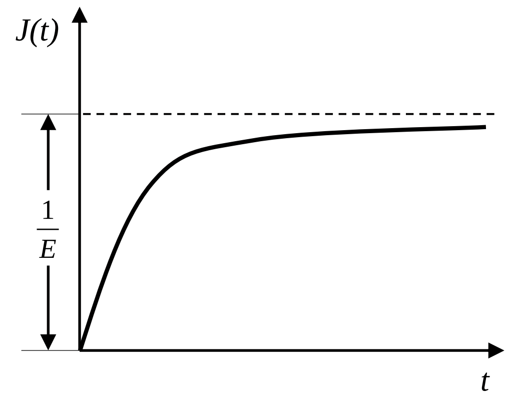
Relaxation Modulus *G(t)*

$$G_{V-K}(t) = \frac{\sigma(t)}{\varepsilon} = E$$

Actual stress is not **constant** in viscoelastic materials.

Creep

$(\sigma = const.\,(\dot{\sigma} = 0))$



Creep Compliance Function *J(t)*

$$J_{V-K}(t) = \frac{\varepsilon(t)}{\sigma} = \frac{1}{E}(1 - \mathrm{e}^{-\frac{E}{\eta}t})$$

V-K Model

Creep Compliance *J(t)*

$$d\varepsilon + \frac{E}{\eta}\varepsilon dt = \frac{\sigma_0}{\eta}dt$$

$$\varepsilon(t) = \frac{\sigma_0}{E}\left[1 - \exp(-\frac{E}{\eta}t)\right]$$

$$\boxed{J_{V-K}(t) = \frac{\varepsilon(t)}{\sigma} = \frac{1}{E}(1 - \mathrm{e}^{-\frac{E}{\eta}t})}$$

Relaxation Modulus *G(t)*

$$\sigma_0 = E\varepsilon_0 = const$$

$$\boxed{G_{V-K}(t) = \frac{\sigma(t)}{\varepsilon} = E}$$

No relaxation is predicted by the Kelvin model

V-K Model

Recovery

At time t=t1, stress is suddenly reduced to zero!!

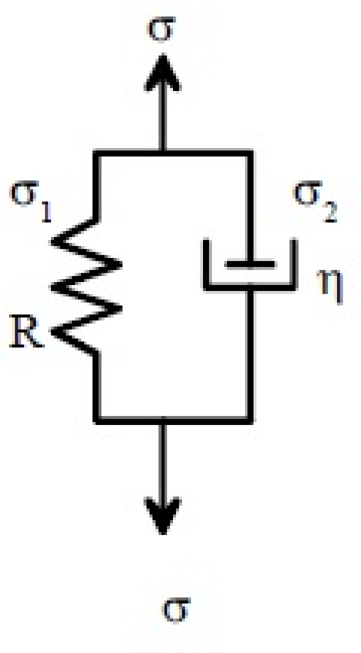
$$\varepsilon(t)=\frac{\sigma_0}{E}\left[1-\exp(-\frac{E}{\eta}t)\right] \qquad 0\le t\le t_1$$

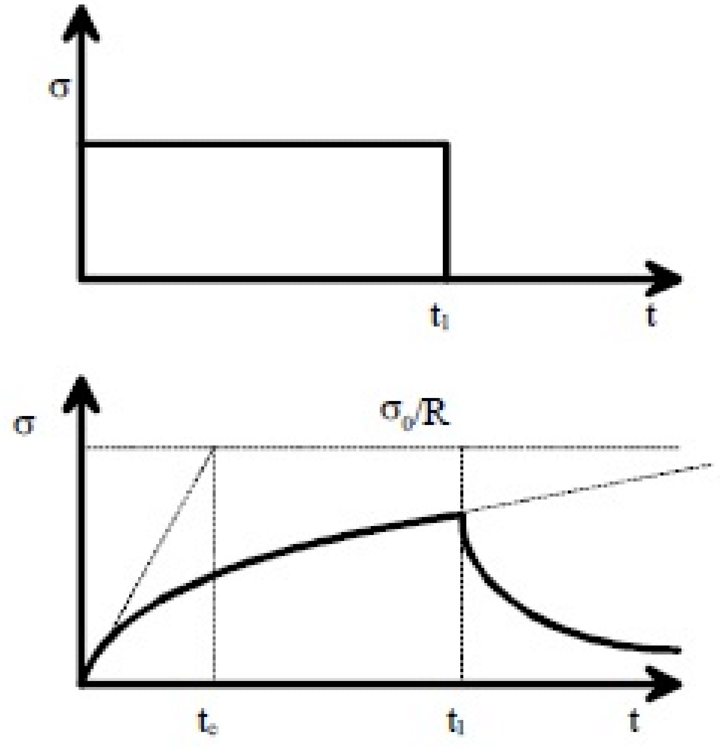
Let, $\frac{E}{\eta}=\frac{1}{\tau} \qquad t'=t-t_1$

$$d\varepsilon+\frac{\varepsilon}{\tau}dt'=0 \qquad t\ge t_1$$

$$\varepsilon(t)=\varepsilon_1\exp(-\frac{1}{\tau}t') \quad \text{or} \quad \varepsilon(t)=\frac{\sigma_0}{E}\left[\exp(\frac{t_1}{\tau})-1\right]\exp(-\frac{1}{\tau}t) \qquad t\ge t_1$$

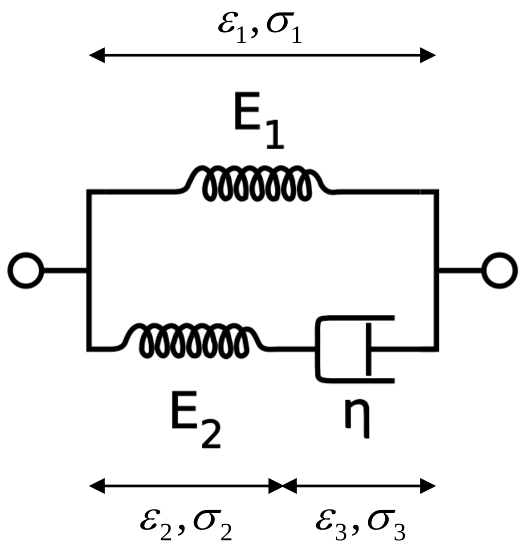
V-K Model





Creep & recovery

Standard Linear Solid(Zener) Model



A Maxwell model and a purely elastic spring connected in parallel (three-parameter standard model)

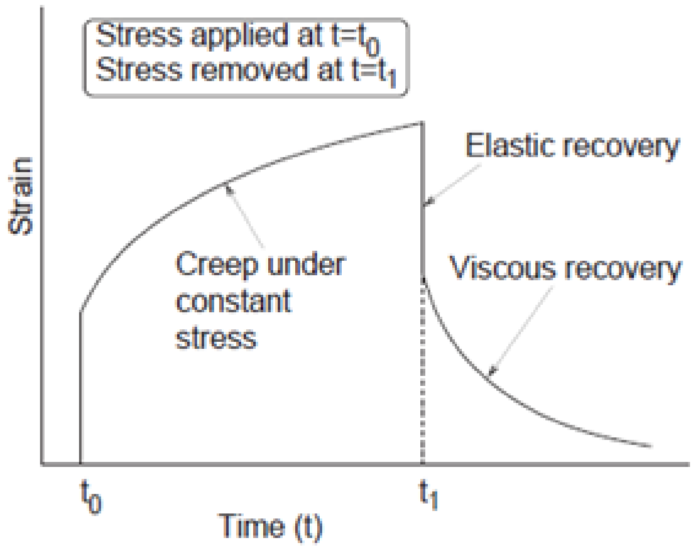
$$\varepsilon = \varepsilon_1 = \varepsilon_2 + \varepsilon_3$$
$$\sigma = \sigma_1 + \sigma_2$$
$$\sigma_2 = \sigma_3$$

$$\boxed{\eta(E_1 + E_2)\dot{\varepsilon} + E_1 E_2 \varepsilon = \eta\dot{\sigma} + E_2\sigma}$$

SLS Model

$$\eta(E_1+E_2)\dot{\varepsilon}+E_1E_2\varepsilon=\eta\dot{\sigma}+E_2\sigma$$

Homework #1 – Derive the following two equations



Creep and recovery response

Creep Compliance Function *J(t)*

$$J_{SLS}(t)=\frac{1}{E_1}\left(1-\frac{E_2}{E_1+E_2}\,\mathrm{e}^{-\frac{E_1E_2}{\eta(E_1+E_2)}t}\right)$$

Relaxation Modulus *G(t)*

$$G_{SLS}(t)=E_1+E_2\,\mathrm{e}^{-\frac{E_2}{\eta}t}$$

It matches well to real linear viscoelastic behaviors!

Comparison of Several Models

Model	Creep compliance function *J(t)*	Relaxation modulus *G(t)*
Maxwell	$\frac{1}{E}\left(1+\frac{E}{\eta}t\right)$	$E\,\mathrm{e}^{-\frac{E}{\eta}t}$
Voigt-Kelvin	$\frac{1}{E}\left(1-\mathrm{e}^{-\frac{E}{\eta}t}\right)$	$E+\eta\delta(t)$
Standard Linear Solid (Zener)	$\frac{1}{E_1}\left(1-\frac{E_2}{E_1+E_2}\mathrm{e}^{-\frac{E_1E_2}{\eta(E_1+E_2)}t}\right)$	$E_1+E_2\,\mathrm{e}^{-\frac{E_2}{\eta}t}$

Comparison of Several Models

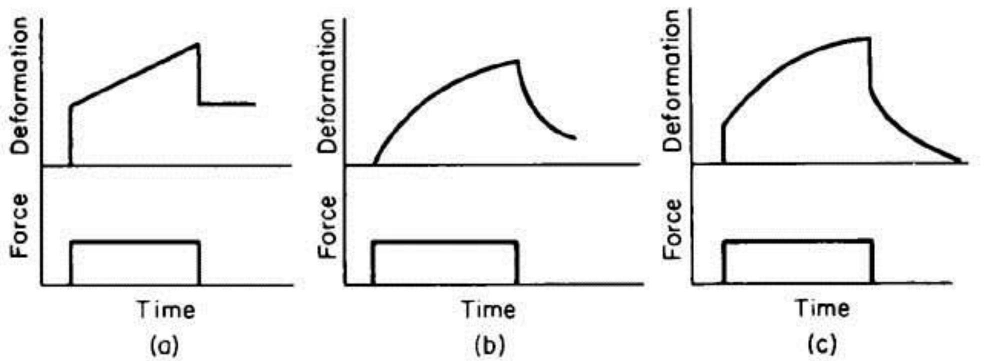


Figure 2.11:3 Creep functions of (a) a Maxwell, (b) a Voigt, and (c) a standard linear solid. A negative phase is superposed at the time of unloading.

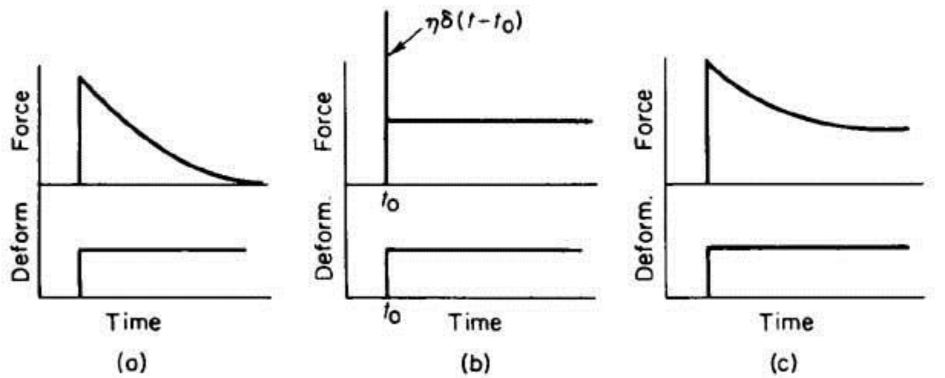
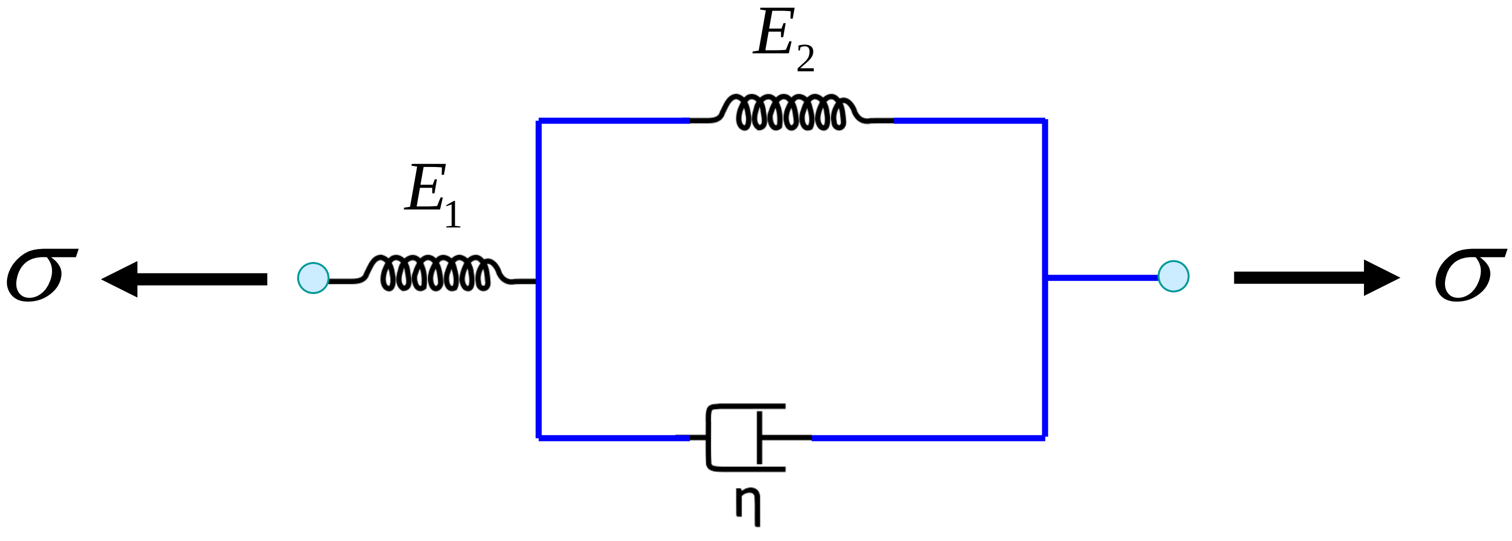


Figure 2.11:4 Relaxation functions of (a) a Maxwell, (b) a Voigt, and (c) a standard linear solid.

Y.C. Fung, *"Biomechanics : mechanical properties of living tissues"*, Second edition, New York : Springer-Verlag, 1993.

Homework #2



Homework #1

1) Derive relaxation modulus and creep compliance

2) Discuss the recovery response of the unit

Due on April ??