1 Calculation of e^{At}

For discretization of continuous state-space model, we need to evaluate e^{At} where A is Jacobian matrix of the continuous state-space model and h is sample time. Recall we learned one technique to evaluate e^{At} from matrix diagonalization. In this seminar, we will learn a few more.

1.1 Series expansion

Note that

$$e^{At} = I + At + \frac{1}{2!}A^2t^2 + \cdots$$
$$\int_0^h e^{As}ds = Ih + \frac{Ah^2}{2!} + \frac{A^2h^3}{3!} + \cdots + \frac{A^i h^{i+1}}{(i+1)!} + \cdots$$

The infinite series can be simplified if higher-order terms A^n, A^{n+1}, are zeros. One simple theorem you can use is Caley-Hamilton's Therem for 2-by-2 matrix A:

$$A^2 - (a+d)A + (ad-bc)I = 0$$

Problem 1 Discretize the following continuous time SS model with the sample time of h.

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$
$$y = \begin{bmatrix} 1 & 0 \end{bmatrix} x$$

1.2 Diagonalization

We also learned that e^{At} can be found as

$$e^{At} = V \begin{bmatrix} e^{\lambda_1 t} & & & \\ & e^{\lambda_2 t} & & \\ & & \ddots & \\ & & & e^{\lambda_n t} \end{bmatrix} V^{-1}$$

where V is the eigenvector matrix.

It should be noted that this method cannot be used if there are “repeated" eigenvalues.

1.3 Inverse Laplace Transform

If there are repeated eignevalues, one can use the inverse Laplace transform.

Note that the solution of

$$\dot{x} = Ax \tag{1}$$

is

$$x(t) = e^{At}x(0) \tag{2}$$

One can also solve Eq. (1) using inverse Laplace transform as

$$\begin{aligned}
&sX(s) - x(0) = AX(s) \\
&(sI - A)X(s) = x(0) \\
&X(s) = (sI - A)^{-1}x(0) \\
&x(t) = \mathscr{L}^{-1}\left[(sI - A)^{-1}\right]x(0) = e^{At}x(0)
\end{aligned}$$

Hence, we have

$$\boxed{e^{At} = \mathscr{L}^{-1}\left[(sI - A)^{-1}\right]}$$

Problem 2 Discretize the following continuous time SS model with the sample time of h.

$$\begin{aligned}
\dot{x} &= \begin{bmatrix} -1 & 0 \\ 1 & 0 \end{bmatrix} x + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u \\
y &= \begin{bmatrix} 0 & 1 \end{bmatrix} x
\end{aligned}$$

Solutions

- Problem 1

  Note that $A^2 = 0$ from C-H Theorem (or you can find this easily by simple multiplication).

$$
\begin{aligned}
\Phi = e^{Ah} &= I + Ah + \frac{A^2h^2}{2!} + \cdots = I + Ah \\
&= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & h \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & h \\ 0 & 1 \end{bmatrix} \\
\Gamma &= \int_0^h \begin{bmatrix} 1 & v \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} dv = \int_0^h \begin{bmatrix} v \\ 1 \end{bmatrix} dv = \begin{bmatrix} \frac{h^2}{2} \\ h \end{bmatrix}
\end{aligned}
$$

  Hence,

$$
\begin{aligned}
x(kh+h) &= \begin{bmatrix} 1 & h \\ 0 & 1 \end{bmatrix} x(kh) + \begin{bmatrix} \frac{h^2}{2} \\ h \end{bmatrix} u(kh) \\
y(kh) &= \begin{bmatrix} 1 & 0 \end{bmatrix} x(kh)
\end{aligned}
$$

- Problem 2

  Note that we cannot simply ignore higher-order terms of A easily or cannot use eigenvectors for this system. We use

$$
e^{At} = \mathscr{L}^{-1}\left[(sI - A)^{-1}\right]
$$

$$
\begin{aligned}
sI - A &= \begin{bmatrix} s+1 & 0 \\ -1 & s \end{bmatrix} \\
(sI - A)^{-1} &= \frac{1}{s(s+1)} \begin{bmatrix} s & 0 \\ 1 & s+1 \end{bmatrix} = \Big[eginarraycc \frac{1}{s+1} \qquad 0 \\
\frac{1}{s(s+1)} \frac{1}{s}
\end{aligned}
$$

$$\mathscr{L}^{-1}\left[(sI-A)^{-1}\right] = e^{At} = \begin{bmatrix} e^{-t} & 0 \\ 1-e^{-t} & 1 \end{bmatrix}$$

Hence,

$$\Phi = e^{Ah} = \begin{bmatrix} e^{-h} & 0 \\ 1-e^{-h} & 1 \end{bmatrix}$$
$$\Gamma = \int_0^h \begin{bmatrix} e^{-v} \\ 1-e^{-v} \end{bmatrix} dv = \begin{bmatrix} 1-e^{-h} \\ h-1+e^{-h} \end{bmatrix}$$