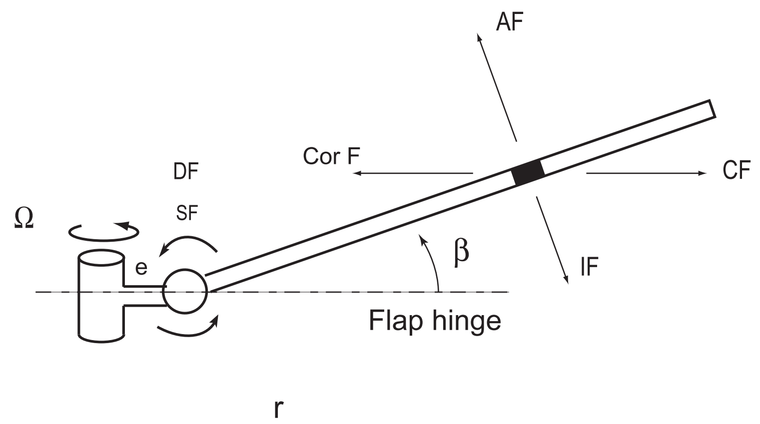
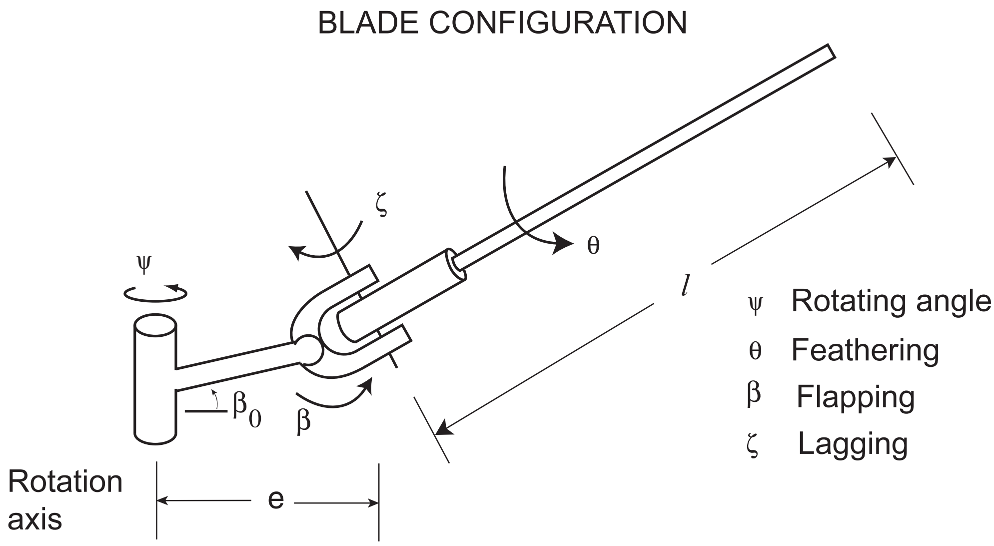
3.5 Rigid Flap-Lag-Torsion

The blade is assumed rigid and it undergoes three degrees of motion, flap, lag and feather rotations about three hinges. The hinge sequence assumed here is, from the rotation axis the flap hinge is followed by lag hinge and then at the outboard is pitch bearing. with the changed hinge sequence, there will be modification in some of the nonlinear terms. This model not only represents articulated blades but also can be a good approximation of hingeless blades for dynamic analysis. For simplicity of analysis, it is assumed that all the hinges are located at the same place. The important nonlinear terms up to second order are retained. Let us examine the element forces in each mode of vibration.

Flap Mode



(a) IF: inertia force $m(r-e)\ddot{\beta}\,dr$ arm $(r-e)$ about flap hinge
$-mx_I\ddot{\theta}\,drm$ arm $(r-e)$
(b) CF: centrifugal force $m\Omega^2 r\,dr$ arm $(r-e)\beta$
$-m\Omega^2 r\,dr$ arm $x_I\theta$



(c) CorF: coriolis force $2m\Omega\dot{\zeta}$ arm $(r-e)\beta$
(d) AF: aerodynamic force $F_\beta\,dr$ arm $(r-e)$
(e) SF: spring force $k_\beta(\beta-\beta_p)$ moment
(f) DF: damping force $c_\beta\dot{\beta}$ moment

where

x_I = chordwise offset of cg behind feathering axis

β_p = precone angle
k_β = flap bending spring at hinge
c_β = damping constant
$= 2\zeta\omega_{\beta 0} I_\beta$
ζ_β = viscous damping ratio in flap mode
$\omega_{\beta 0}$ = non-rotating flap frequency, rad/sec
I_β = mass moment of inertia about flap hinge

Taking moment of forces about flap hinge

$$\int_e^R \{m(r-e)^2\ddot{\beta} - mx_I(r-e)\ddot{\theta} + m\Omega^2 r(r-e)\beta - m\Omega^2 r x_I \theta - 2m\Omega(r-e)\beta\dot{\zeta}$$

$$-F_\beta(r-e)\}dr + k_\beta(\beta-\beta_p) + 2\zeta_\beta\omega_{\beta 0} I_\beta \dot{\beta} = 0$$

Assuming $I_\beta \simeq I_\beta$ where I_b is the total flap inertia. Dividing the above equation by $I_b\Omega^2$ gives the flap equation in nondimensional form.

$$\overset{**}{\beta} + \nu_\beta^2 \beta + 2\omega_{\frac{\beta 0}{\Omega}} \zeta_\beta \overset{*}{\beta} - 2\beta \overset{*}{\zeta} - I_{\overset{*}{x}}(\overset{**}{\theta} + \theta) = \gamma \overline{M}_\beta + \frac{w_{\beta 0}^2}{}\beta_p \quad (3.45)$$

where ν_β is the rotating flap frequency,

$$\nu_\beta^2 = 1 + \frac{e\int_e^R m(r-e)\,dr}{I_\beta} + \frac{\omega_{\beta 0}^2}{\Omega^2} \qquad \text{per rev.}$$

and

$$I_{\overset{*}{x}} = \frac{I_x}{I_b} = \frac{\int_e^R m x_I r\,dr}{I_b}$$

For uniform blades

$$\nu_\beta^2 = 1 + \frac{3}{2}\frac{e}{R-e} + \frac{\omega_{\beta 0}^2}{I_b}$$

and

$$I_{\overset{*}{x}} = \frac{3}{2} x_I/R$$

The γ is the Lock number,

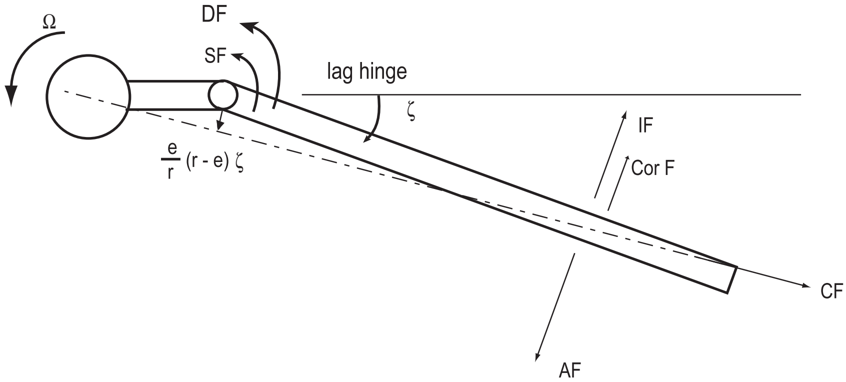
$$\frac{\rho a c R^4}{I_b}$$

and

$$\overline{M}_\beta = \frac{1}{\rho a c R^4 \Omega^2} \int_e^R F_\beta(r-e)\,dr$$

Also note that $\overset{*}{\beta} = \frac{\partial \beta}{\partial \psi}$ and $\psi = \Omega t$.

II. Lag Mode



(a) IF: $m(r-e)\ddot{\zeta}\,dr$ arm $(r-e)$ about lag hinge

(b) CF: $m\Omega^2 r\,dr$ arm $\frac{e}{r}(r-e)\zeta$

(c) CorF: $2m\Omega(r-e)\dot{\beta}\beta\,dr$ arm $(r-e)$ $\quad -mx_I\dot{\theta}\beta\,dr$ arm $(r-e)$

(d) AF: $F_\zeta\,dr$ arm $(r-e)$

(e) SF: $k_\zeta\zeta$ moment

(f) DF: $2I_\zeta\omega_{\zeta 0}\zeta_L\dot{\zeta}$ moment

where

ζ_L = viscous damping ratio in lag mode

$\omega_{\zeta 0}$ = non-rotating lag frequency

Taking moment of forces about lag hinge

$$\int_e^R \{m(r-e)^2\ddot{\zeta} + m\Omega^2 e(r-e)\zeta + 2m\Omega(r-e)^2\beta\dot{\beta} - mx_I(r-e)\beta\dot{\theta} - F_\zeta\}dr$$
$$+k_\zeta\zeta + 2I_\zeta\omega_{\zeta 0}\zeta_L\dot{\zeta} = 0$$

Assuming $I_\zeta \simeq I_b$ and dividing through $I_b\Omega^2$ gives the lag equation in nondimensional form

$$\overset{**}{\zeta} + \nu_\zeta^2\zeta + 2\frac{\omega_{\zeta 0}}{\Omega}\zeta_L \overset{*}{\zeta} + 2\beta\overset{*}{\beta} - I_{\overset{*}{x}}(2\beta\overset{*}{\theta}) = \gamma\overline{M}_\zeta \tag{3.46}$$

where ν_ζ is the rotating frequency

$$\nu_\zeta^2 = \frac{e\int_e^R m(r-e)\,dr}{I_b} + \frac{\omega_{\zeta 0}^2}{\Omega^2}$$

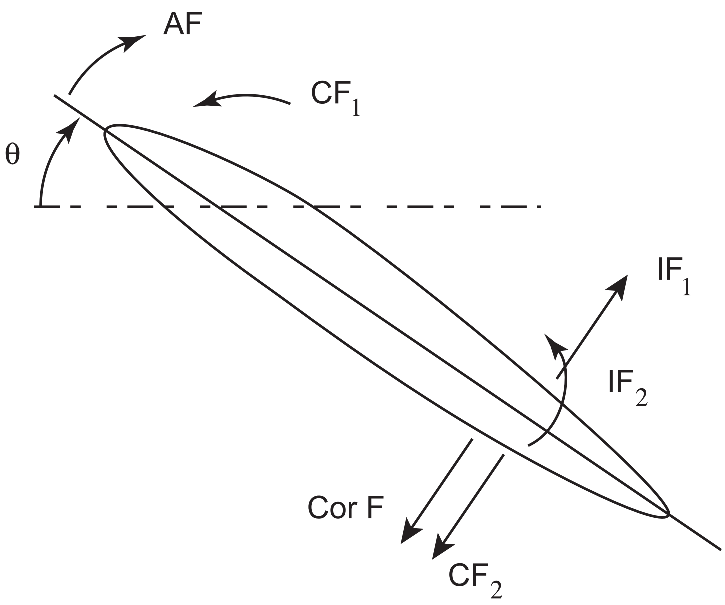
For uniform blades

$$\nu_\zeta^2 = \frac{3}{2}\frac{e}{R-e} + \frac{\omega_{\zeta 0}^2}{\Omega^2}$$

and also $I_{\overset{*}{x}} = \frac{3}{2}x_I/R$. The aerodynamic moment is

$$\overline{M}_\zeta = \frac{1}{\rho ac\Omega^2 R^4}\int_e^R F_\zeta(r-e)\,dr$$

III. Torsion Mode



(a) IF_1: $mx_I\ddot{\theta}\,dr$ arm x_I about feathering axis
$-m(r-e)\ddot{\beta}\,dr$ arm x_I
IF_2: $I_0\ddot{\theta}\,dr$ moment
(b) CF_1: $I_\theta\Omega^2\theta\,dr$ moment
CF_2: $mr\Omega^2\beta\,dr$ arm x_I
(c) CorF: $2m\Omega r\dot{\zeta}\beta$ arm x_I
(d) AF: $M_\theta\,dr$ moment
(e) SF: $k_\theta(\theta-\theta_{\text{con}})$ moment
(f) DF: $2I_f\omega_{\theta 0}\zeta_\theta\dot{\theta}$ moment
where

$$\overline{M}_\theta = \frac{1}{\rho a c\Omega^2 R^4}\int_e^R M_\theta\,dr$$

and the ν_θ is the rotating torsion frequency.

$$\nu_\theta^2 = 1 + \frac{\omega_{\theta 0}^2}{\Omega^2}$$

The flap, lag and torsion equations can be rewritten as

$$\underbrace{\begin{bmatrix} 1 & 0 & -I_{\overset{*}{x}} \\ 0 & 1 & 0 \\ -I_{\overset{*}{x}} & 0 & I_{\overset{*}{f}} \end{bmatrix}}_{\text{inertia}} \begin{bmatrix} \overset{**}{\beta} \\ \overset{**}{\zeta} \\ \overset{**}{\theta} \end{bmatrix} + \underbrace{\begin{bmatrix} 2\frac{\omega_{\beta 0}}{\Omega}\zeta_\beta & -2\beta & 0 \\ 2\beta & 2\frac{\omega_{\zeta 0}}{\Omega}\zeta_L & -2\beta I_{\overset{*}{x}} \\ 0 & 2\beta I_{\overset{*}{x}} & 2\frac{\omega_{\theta 0}}{\Omega}\zeta_\theta I_{\overset{*}{f}} \end{bmatrix}}_{\text{damping}} \begin{bmatrix} \overset{*}{\beta} \\ \overset{*}{\zeta} \\ \overset{*}{\theta} \end{bmatrix}$$

$$+ \underbrace{\begin{bmatrix} \nu_\beta^2 & 0 & -I_{\overset{*}{x}} \\ 0 & \nu_\zeta^2 & 0 \\ -I_{\overset{*}{x}} & 0 & -I_{\overset{*}{f}}\nu_\theta^2 \end{bmatrix}}_{\text{stiffness}} \begin{bmatrix} \beta \\ \zeta \\ \theta \end{bmatrix} = \underbrace{\gamma \begin{bmatrix} \overline{M}_\beta \\ \overline{M}_\zeta \\ \overline{M}_\theta \end{bmatrix}}_{\text{force}} + \begin{bmatrix} \frac{\omega_{\beta 0}^2}{\Omega^2}\beta_p \\ 0 \\ I_{\overset{*}{f}}\frac{\omega_{\theta 0}^2}{\Omega^2}\theta_{\text{con}} \end{bmatrix} \tag{3.47}$$

These equations are coupled inertially. The inertia and stiffness matrices are symmetric. The damping matrix consists of two parts. The viscous damping terms are diagonal terms whereas coriolis force terms are antisymmetric. Also, the coriolis force terms are nonlinear in nature, but these are important coupling terms.

The complete nonlinear equations for this hinge sequency are also available (Chopra (83)). With changed hinge sequence one will get a new set of equations with different coupling terms. For example, Chopra and Dugundji (1979) derived nonlinear equations for a blade with pitch bearing inboard, followed by flap hinge and the lag hinge outboard.

3.6 Flexible Flap-Lag-Torsion-extension

An appropriate model for rotor blade is to assume it as an elastic beam undergoing flap bending, lead-lag bending and elastic torsion. These motions are coupled through inertial and aerodynamic forces. The derivation of the equations of motion for the coupled flap-lag-torsion blade is lengthy and involved. Many authors have derived these equations with different approximations in mind. Among notable works are Houbolt and Brooks (1958), Hodges and Dowell (1974), and Johnson (1977).

3.6.1 Second order non-linear beam model

The blade is idealized into a twisted beam. Due to pitch and twist distribution, there is a structural coupling between the out of plane bending and inplane bending. The derivation details are not given here. The equations of motion are given here for uniform blades.

u axial deflection, in
v lead-lag deflection, in
w flap deflection, in
ϕ elastic twist, rad
θ blade pitch, rad

3.6.2 Equations for uniform beams

Assumptions

1. Uniform blade
2. Slender beam
3. Moderate slopes (terms 2nd order retained)
4. No droop, sweep or torque offset
5. Tension axis lies on elastic axis

x_I = chordwise offset of cg from ea (+ ve aft)
EI_y = flapwise stiffness, lb - in^2
EI_z = chordwise stiffness, lb-in^2
GJ = torsional stiffness, lb-in^2
k_A = polar radius of gyration, in
m = mass per unit length, lb-sec^2/in^2
$mk_{m_1}^2$ = flapwise principal mass moment of inertia
$mk_{m_2}^2$ = chordwise principal mass moment of inertia
mk_m^2 = torsional mass moment of inertia

Flap Equation:

$$
\begin{aligned}
&[EI_y + (EI_z - EI_y)\sin^2\theta]w^{IV} + \frac{1}{2}(EI_z - EI_y)\sin 2\theta v^{IV} \\
&\quad +(EI_z - EI_y)[\cos 2\theta(\phi v'')'' + \sin 2\theta(\phi w'')''] - \tfrac{1}{2}m\Omega^2[w'(R^2 - r^2)]' \\
&\quad +m\ddot{w} + 2m\Omega\beta_p\dot{v} - 2m\Omega(w' \textstyle\int_r^R \dot{v}\,dx) - mx_I\ddot{\phi} \\
&\quad +\{mx_I[\Omega^2 r\phi\cos\theta + 2\Omega\dot{v}\sin\theta]\}' = L_w - m\Omega^2 r\beta_p
\end{aligned}
$$

Lag Equation:

$$
\begin{aligned}
&[EI_z - (EI_z - EI_y)\sin^2\theta]v^{IV} + \frac{1}{2}(EI_z - EI_y)\sin 2\theta w^{IV} \\
&\quad +(EI_z - EI_y)[-sin2\theta(\phi v'')'' + \cos 2\theta(\phi w'')''] - \tfrac{1}{2}m\Omega^2[v'(R^2 - r^2)]' \\
&\quad +m\ddot{v} \;-\; m\Omega^2 v \;-\; 2m\Omega\beta_p\dot{w} - 2m\Omega\int_0^r (v'\dot{v}' + w'\dot{w}')\,dr \\
&\quad -2m\Omega(v'\int_r^R \dot{v}\,dx)' + mx_I\ddot{\phi}\sin\theta + 2m\Omega x_I(\dot{v}'\cos\theta + \dot{w}'\sin\theta) \\
&\quad -m\Omega^2 x_I\sin\theta\,\phi = L_v
\end{aligned}
$$

Torsion Equation:

$$
\begin{aligned}
&-GJ\phi'' + \frac{1}{2}(EI_z - EI_y)[(w''^2 - v''^2)\sin 2\theta + v''w''\cos 2\theta] \\
&\quad -\tfrac{1}{2}m\Omega^2 k_A^2[\phi'(R^2 - r^2)]' + mk_m^2\ddot{\phi} + m\Omega^2(k_{m_2}^2 - k_{m_1}^2)\phi\cos 2\theta \\
&\quad -mx_I[\Omega^2 r(w'\cos\theta - v'\sin\theta) - (\ddot{v} - \Omega^2 v)\sin\theta + w\cos\theta
\end{aligned}
$$

$$
= M_\phi - \frac{1}{2}m\Omega^2(k_{m_2}^2 - k_{m_1}^2)\sin 2\theta
$$

3.6.3 Detailed model for non-uniform beams

The rotor blades are modeled as long, slender, homogeneous, isotropic beams undergoing axial, flap, lag and torsion deformations. The deformations can be moderate as the model includes geometric non-linearities up-to second order. Radial non-uniformities of mass, stiffness, twist, etc., chordwise offsets of mass centroid (center of gravity) and area centroid (tension axis) from the elastic axis, precone, and warp of the cross section are included. The model follows the Hodges and Dowell formulation (1974) while treating elastic torsion and elastic axial deformation as quasi-coordinates based on Ormiston (1980). The model assumes a straight blade. Modeling refinements required to incorporate structural sweep and droop were first treated by Celi and Friedmann (1992), and Kim and Chopra (1995). The governing equations and their derivations can remain same, the swept and drooped elements can be formulated using additional coordinate transformations and a modified finite element assembly procedure. The following derivation is taken from Datta (2004). Details of the validation can be found in Datta and Chopra (2006).

The equations of motion are developed using Hamilton's Principle, a statement of the Principle of Least Action. The governing partial differential equations can be solved using finite element method in time and space. The finite element method provides flexibility in the implementation of boundary conditions for modern helicopter rotors. For example, specialized details like blade root pitch flexibility (pitch link stiffness), pitch damping, elastomeric bearing stiffness and damping can be incorporated within a finite element model.

3.6.4 Blade Coordinate Systems

There are 4 coordinate systems of interest, the hub-fixed system, (X_H, Y_H, Z_H) with unit vectors $\hat{I_H}, \hat{J_H}, \hat{K_H}$, the hub-rotating system, (X, Y, Z) with unit vectors $\hat{I}, \hat{J}, \hat{K}$, the undeformed blade coordinate system, (x, y, z) with unit vectors $\hat{i}, \hat{j}, \hat{k}$ and the deformed blade coordinate system, (ξ, η, ζ) with the unit vectors $\hat{i_\xi}, \hat{j_\eta}, \hat{k_\zeta}$. These frames of references are denoted as H, R, U and D respectively. The hub-rotating coordinate system is rotating at a constant angular velocity $\Omega\hat{K}$ with respect to the hub-fixed coordinate system. The transformation between the hub-fixed system and the hub-rotating system is defined as

$$\left\{ \begin{array}{c} \hat{I} \\ \hat{J} \\ \hat{K} \end{array} \right\} = \left[\begin{array}{ccc} \cos\psi & \sin\psi & 0 \\ -\sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{array} \right] \left\{ \begin{array}{c} \hat{I_H} \\ \hat{J_H} \\ \hat{K_H} \end{array} \right\} = \mathbf{T_{RH}} \left\{ \begin{array}{c} \hat{I_H} \\ \hat{J_H} \\ \hat{K_H} \end{array} \right\} \tag{3.48}$$

where the azimuth angle, ψ, equals $\Omega\, t$. The undeformed blade coordinate system is at a precone angle of β_p with respect to the hub-fixed system. The transformation between the undeformed blade coordinate system and the hub-fixed system is defined as

$$\left\{ \begin{array}{c} \hat{i} \\ \hat{j} \\ \hat{k} \end{array} \right\} = \left[\begin{array}{ccc} \cos\beta_p & 0 & \sin\beta_p \\ 0 & 1 & 0 \\ -\sin\beta_p & 0 & \cos\beta_p \end{array} \right] \left\{ \begin{array}{c} \hat{I} \\ \hat{J} \\ \hat{K} \end{array} \right\} = \mathbf{T_{UR}} \left\{ \begin{array}{c} \hat{I} \\ \hat{J} \\ \hat{K} \end{array} \right\} \tag{3.49}$$

The transformation between the undeformed blade coordinate system and the deformed blade coordinate system remains to be determined.

3.6.5 Blade Deformation Geometry

Consider a generic point P on the undeformed blade elastic axis. The orientation of a frame consisting of the axes normal to and along principle axes for the cross section at P defines the undeformed coordinate system (x, y, z). When the blade deforms, P reaches P'. The orientation of a frame consisting of the axes normal to and along principle axes for the cross section at P' defines the deformed coordinate system (ξ, η, ζ). Figure 3.1 shows the undeformed and deformed coordinate systems. Adequate description of the deformed blade requires in general a total to six variables : three translational variables from P to P', u, v, w along x, y, z, and three rotational variables from (x, y, z) system to (ξ, η, ζ) system, and any out of plane deformations of the cross section, e.g., warp. These out of plane deformations are neglected, which results in plane sections remaining plane after deformation i.e., the Euler-Bernoulli beam assumption. The Euler-Bernoulli assumption leads to a further simplification - two of the three angles can be expressed as derivatives of the deflection variables. Thus four deformation variables - three deflections u, v, w and one rotational angle, completely determine the deformed geometry. The definition of this rotation angle - the angle of elastic twist is described below.

The coordinate transformation matrix between the undeformed system and the deformed system is defined by the direction cosines of (ξ, η, ζ) with respect to (x, y, z), where x is tangent to the elastic axis of the undeformed blade and ξ is tangent to the elastic axis of the deformed blade. The transformation matrix can be written as

$$\left\{ \begin{array}{c} \hat{i_\xi} \\ \hat{j_\eta} \\ \hat{k_\zeta} \end{array} \right\} = \mathbf{T_{DU}} \left\{ \begin{array}{c} \hat{i} \\ \hat{j} \\ \hat{k} \end{array} \right\} \tag{3.50}$$

where $\mathbf{T_{DU}}$ can be described as a function of three successive angular rotations in space required to align (x, y, z) along (ξ, η, ζ). The two intermediate orientations can be described as (x_1, y_1, z_1) and (x_2, y_2, z_2) with unit vectors $(\hat{i_1}, \hat{j_1}, \hat{k_1})$ and $(\hat{i_2}, \hat{j_2}, \hat{k_2})$. Classical Euler angles use rotation ψ about z, θ about x_1 and ϕ about z_2 to orient (x, y, z) along (ξ, η, ζ). Singularities result when the second angle is zero because the first and third transformations are then about the same axis. Small angle rotations are important for a rotor problem, zero rotations being a special case. Therefore, instead of Euler angles, modified Euler angles are used where the axes do not approach one another for rotations in the neighborhood of zero. The unit vectors $\hat{i}, \hat{j}, \hat{k}$, initially coincident with (x, y, z), can be made to align with $\hat{i_\xi}, \hat{j_\eta}, \hat{k_\zeta}$ by rotation through three orientation angles in space ξ, β, θ. Depending on the choice of their sequence, six combinations are possible. The

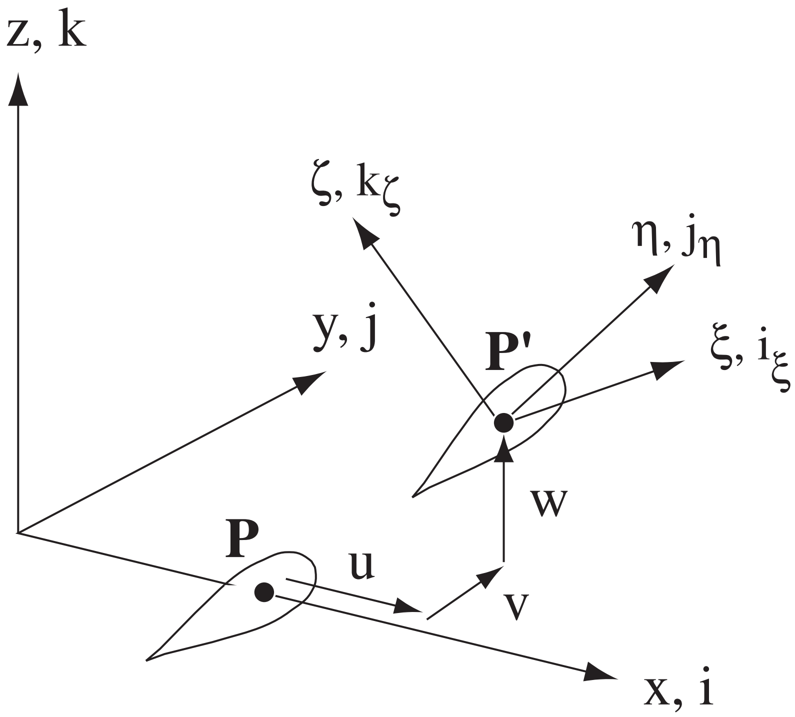


Figure 3.1: **Beam cross-section before and after deformation showing undeformed and deformed coordinate systems**

resulting transformation matrix takes a different form depending on the selected combination. The numerical values of the matrix elements, i.e. the direction cosines of the unit vectors $\hat{i_\xi}, \hat{j_\eta}, \hat{k_\zeta}$ with respect to $\hat{i}, \hat{j}, \hat{k}$, are offcourse independant of the choice of combination. The direction cosines are intrinsic properties, i.e., they are uniquely determined by the loading and boundary conditions. Here, we consider a rotation sequence ξ_1, β_1, θ_1 about $\hat{k}, -\hat{j_1}, \hat{i_2}$ respectively, in that order. That is, the first rotation ξ_1 is about z resulting in the new set (x_1, y_1, z_1), β_1 about $-y_1$ resulting in (x_2, y_2, z_2) and θ_1 about x_2 resulting in (ξ, η, ζ). This produces

$$\mathbf{T_{DU}} = \begin{bmatrix} c_{\beta_1} c_{\xi_1} & c_{\beta_1} s_{\xi_1} & s_{\beta_1} \\ -c_{\xi_1} s_{\beta_1} s_{\theta_1} - c_{\theta_1} s_{\xi_1} & c_{\xi_1} c_{\theta_1} - s_{\xi_1} s_{\beta_1} s_{\theta_1} & c_{\beta_1} s_{\theta_1} \\ -c_{\xi_1} s_{\beta_1} c_{\theta_1} + s_{\theta_1} s_{\xi_1} & -c_{\xi_1} s_{\theta_1} - s_{\xi_1} s_{\beta_1} c_{\theta_1} & c_{\beta_1} c_{\theta_1} \end{bmatrix} \tag{3.51}$$

where $c_{()} = \cos(\ )$, $s_{()} = \sin(\ )$ and $\mathbf{T_{DU}}^{\mathbf{-1}} = \mathbf{T_{DU}}^{\mathbf{T}}$. The goal is to express this transformation as a function of blade deflections and one rotation angle.

The position vector of any point on the deformed-blade elastic axis can be written as

$$\bar{\mathbf{r}} = (x + u)\hat{i} + v\hat{j} + w\hat{k} \tag{3.52}$$

and the unit vector tangent to the elastic axis of the deformed blade is

$$\frac{\partial \bar{\mathbf{r}}}{\partial r} = (x + u)^+\hat{i} + v^+\hat{j} + w^+\hat{k} \tag{3.53}$$

where r is the curvilinear distance coordinate along the deformed-beam elastic axis and $()^+ = \partial/\partial r()$. Assuming pure bending and the cross sections remain normal to the elastic axis during deformation

$$\frac{\partial \bar{\mathbf{r}}}{\partial r} = \hat{i_\xi} = T_{11}\hat{i} + T_{12}\hat{j} + T_{13}\hat{k} \tag{3.54}$$

where T_{ij} is the element on the ith row and $j$the column of $\mathbf{T_{DU}}$. Thus

$$\left.\begin{array}{l} T_{11} = (x+u)^+ \\ T_{12} = v^+ \\ T_{13} = w^+ \end{array}\right\} \tag{3.55}$$

In the case of a pure elastic axial elongation, u_e, in addition to pure bending, it is subtracted from total axial elongation to calculate the unit vector tangent to the elastic axis of the deformed blade.

$$\hat{i_\xi} = (x+u-u_e)^+\hat{i} + v^+\hat{j} + w^+\hat{k} \tag{3.56}$$

and then

$$\left.\begin{array}{l} T_{11} = (x+u-u_e)^+ \\ T_{12} = v^+ \\ T_{13} = w^+ \end{array}\right\} \tag{3.57}$$

Because $\mathbf{T_{DU}}$ is orthonormal

$$T_{11}{}^2 + T_{12}{}^2 + T_{13}{}^2 = 1 \tag{3.58}$$

and therefore

$$(x+u-u_e)^+ = \sqrt{1 - v^{+\,2} - w^{+\,2}} \tag{3.59}$$

Using equations (3.51) and (3.57) it can be deduced

$$\left.\begin{array}{l} s_{\beta_1} = w^+ \\ c_{\beta_1} = \sqrt{1 - w^{+\,2}} \\ s_{\xi_1} = \frac{v^+}{\sqrt{1-w^{+\,2}}} \\ c_{\xi_1} = \frac{\sqrt{1-v^{+\,2}-w^{+\,2}}}{\sqrt{1-w^{+\,2}}} \end{array}\right\} \tag{3.60}$$

c_{θ_1} and s_{θ_1} remain to be expressed in terms of the blade deflections and some appropriate measure of elastic torsion. The angular velocity of the frame (x, y, z) as it moves to (ξ, η, ζ) is

$$\begin{aligned} \omega &= \dot{\xi}_1\hat{k} - \dot{\beta}_1\hat{j}_1 + \dot{\theta}_1\hat{i}_2 \\ &= \omega_\xi\hat{i_\xi} + \omega_\eta\hat{i_\eta} + \omega_\zeta\hat{i_\zeta} \end{aligned} \tag{3.61}$$

where $\hat{j_1}$ and $\hat{i_2}$ are unit vectors of the intermediate frames (x_1, y_1, z_1) and (x_2, y_2, z_2), and $\dot{(\ )} = \partial/\partial t(\ )$. The components of the angular velocity are

$$\left.\begin{array}{l} \omega_\xi = \dot{\theta}_1 + \dot{\xi}_1\, s_{\beta_1} \\ \omega_\eta = -\dot{\beta}_1\, c_{\theta_1} + \dot{\xi}_1\, c_{\beta_1} s_{\theta_1} \\ \omega_\zeta = \dot{\xi}_1\, c_{\beta_1}\, c_{\theta_1} + \dot{\beta}_1\, s_{\theta_1} \end{array}\right\} \tag{3.62}$$

The bending curvatures and torsion (or angle of twist per unit length) can be deduced with the use of Kirchhoff's kinetic analog by replacing $\dot{(\ )}$ with $(\ )^+$. Thus,

$$\left.\begin{array}{l} \kappa_\xi = \theta_1^+ + \xi_1^+\, s_{\beta_1} \\ \kappa_\eta = -\beta_1^+\, c_{\theta_1} + \xi_1^+\, c_{\beta_1} s_{\theta_1} \\ \kappa_\zeta = \xi_1^+\, c_{\beta_1} c_{\theta_1} + \beta_1^+\, s_{\theta_1} \end{array}\right\} \tag{3.63}$$

where κ_ξ, κ_η and κ_ζ are the components of bending curvatures in the deformed blade ξ, η, ζ directions. κ_ξ is the torsion. The angle of elastic twist, ϕ is defined such that

$$(\theta_t + \phi)^+ = \kappa_\xi \tag{3.64}$$

where $\theta_t^+ = \theta_t' \, x^+$. θ_t is the rigid pretwist of the blade. From (3.63a) and (3.64) we have

$$\theta_1^+ = (\theta_t + \phi)^+ - \xi^+ \, w^+ \tag{3.65}$$

ξ can be expressed as a function of blade deflections. Using (3.51), (3.57b) and (3.60b) we have

$$s_\xi = \frac{v^+}{\sqrt{(1 - w^+)}} \tag{3.66}$$

Differentiating equation (3.66) and substituting equation (3.60d) we have

$$\xi_1^+ = \frac{v^{++}}{\sqrt{1 - v^{+2} - w^{+2}}} + \frac{v^+ w^+ w^{++}}{(1 - w^{+2})\sqrt{1 - v^{+2} - w^{+2}}} \tag{3.67}$$

From (3.67) and (3.65) we have

$$\theta_1^+ = (\theta_t + \phi)^+ - \frac{w^+}{\sqrt{1 - v^{+2} - w^{+2}}} \left(v^{++} + \frac{v^+ w^+ w^{++}}{1 - w^{+2}} \right) \tag{3.68}$$

or

$$\theta_1 = \theta_t + \phi - \int_0^r \frac{w^+}{\sqrt{1 - v^{+2} - w^{+2}}} \left(v^{++} + \frac{v^+ w^+ w^{++}}{1 - w^{+2}} \right) dr \tag{3.69}$$

and

$$\theta_1 = \theta_t + \hat{\phi} \tag{3.70}$$

where θ_t is the blade rigid twist arising from pre-twist and control angles. ϕ is the blade elastic twist which is used in Hodges(1974) as the rotation variable. $\hat{\phi}$ is the blade elastic twist including the kinematic integral component and is used in the present work as the rotation variable. The T_{DU} matrix can now be expressed as a function of the unknown blade deflections and one rotation angle θ_1 related to the unknown blade twist $\hat{\phi}$ via equation (3.70).

$$\mathbf{T_{DU}} = \begin{bmatrix} \sqrt{(1 - v^{+2} - w^{+2})} & v^+ & w^+ \\ \frac{-v^+ c_{\theta_1} - w^+ s_{\theta_1} \sqrt{(1 - v^{+2} - w^{+2})}}{\sqrt{(1 - w^{+2})}} & \frac{c_{\theta_1} \sqrt{(1 - v^{+2} - w^{+2})} - v^+ w^+ s_{\theta_1}}{\sqrt{(1 - w^{+2})}} & s_{\theta_1} \sqrt{(1 - w^{+2})} \\ \frac{v^+ s_{\theta_1} - w^+ c_{\theta_1} \sqrt{(1 - v^{+2} - w^{+2})}}{\sqrt{(1 - w^{+2})}} & \frac{-s_{\theta_1} \sqrt{(1 - v^{+2} - w^{+2})} - v^+ w^+ c_{\theta_1}}{\sqrt{(1 - w^{+2})}} & c_{\theta_1} \sqrt{(1 - w^{+2})} \end{bmatrix} \tag{3.71}$$

The above expressions and coordinate transformation T_{DU} are exact. Now they are reduced to second order. To second order $(\;)^+ = (\;)'$. To second order

$$\begin{aligned} \frac{\sqrt{1 - v^{+2} - w^{+2}}}{\sqrt{1 - w^{+2}}} &= \frac{1 - \frac{1}{2}(v'^2 + w'^2)}{1 - \frac{1}{2}w'^2} \\ &= \frac{1 - \frac{1}{2}w'^2}{1 - \frac{1}{2}w'^2} - \frac{\frac{1}{2}v'^2}{1 - \frac{1}{2}w'^2} \\ &= 1 - \frac{1}{2}v'^2 \end{aligned} \tag{3.72}$$

Finally we have

$$\mathbf{T_{DU}} = \begin{bmatrix} 1 - \frac{v'^2}{2} - \frac{w'^2}{2} & v' & w' \\ -v' c_{\theta_1} - w' s_{\theta_1} & (1 - \frac{v'^2}{2}) c_{\theta_1} - v' w' s_{\theta_1} & (1 - \frac{w'^2}{2}) s_{\theta_1} \\ v' s_{\theta_1} - w' c_{\theta_1} & -(1 - \frac{v'^2}{2}) s_{\theta_1} - v' w' c_{\theta_1} & (1 - \frac{w'^2}{2}) c_{\theta_1} \end{bmatrix} \tag{3.73}$$

where θ is expressed as

$$\begin{aligned}\theta_1 &= \theta_0 + \theta_{1C}\,\cos(\psi) + \theta_{1S}\,\sin(\psi) + \theta_{tw} + \hat{\phi} \\ &= \theta + \hat{\phi}\end{aligned} \tag{3.74}$$

$\theta_0, \theta_{1C}, \theta_{1S}$ are the collective, lateral and longitudinal cyclic angles respectively, ψ is the blade azimuth location, θ_{tw} is the rigid twist angle and $\hat{\phi}$ is the elastic rotation angle. From equation (3.69), the elastic rotation is related to the blade elastic twist as follows

$$\hat{\phi} = \phi - \int_0^r w'v''dr \tag{3.75}$$

where r denotes a blade radial station. Now the blade equations can be formulated using Hamilton's Principle. The equations are formulated in a non-dimensional form.

3.6.6 Nondimensionalization and Ordering scheme

The entire analysis has been done in a nondimensional form. This avoids scaling problems while computing results and increases the generality of the analysis. Table 3.1 shows the reference parameters used to nondimensionalize the relevant physical quantities.

Physical Quantity	**Reference Parameter**
Length	R
Time	$1/\Omega$
Mass/Length	m_0
Velocity	ΩR
Acceleration	$\Omega^2 R$
Force	$m_0\Omega^2 R^2$
Moment	$m_0\Omega^2 R^3$
Energy or Work	$m_0\Omega^2 R^3$

Table 3.1: Nondimensionalization of Physical Quantities

In deriving a nonlinear system of equations, it is necessary to neglect higher-order terms to avoid over-complicating the equations of motion. A systematic and consistent set of guidelines has been adopted for determining which terms to retain and which to ignore. The ordering scheme is same as that in Hodges (1974). It is based on a parameter ε which is of the order of nondimensional flap deflection w or lag deflection v (nondimesionalized with respect to radius, R, as described in table 3.1). u is of the same order as the square of w or v. The elastic twist ϕ is a small angle in the sense that $\sin\phi \approx \phi$ and $\cos\phi \approx 1$. The axial coordinate x is of order R and the lateral coordinates are of order chord, c, and thickness, t. Chord, c, thickness, t and rigid blade twist θ_t are all of same order as v and w. The warp function λ_T is of the same order of magnitude as u so that the warp displacement, which is λ_T multiplied with twist is one order of magnitude less than u. Thus,

$$\left.\begin{array}{ll} \frac{u}{R} = O(\varepsilon^2) & \frac{\lambda}{R} = O(\varepsilon^2) \\ \frac{v}{R} = O(\varepsilon) & \frac{w}{R} = O(\varepsilon) \\ \frac{\eta}{R} = O(\varepsilon) & \frac{\zeta}{R} = O(\varepsilon) \\ \phi = O(\varepsilon) & \frac{\delta\lambda/\delta\eta}{R} = O(\varepsilon) \\ \frac{x}{R} = O(\varepsilon) & \frac{\delta\lambda/\delta\zeta}{R} = O(\varepsilon) \end{array}\right\} \tag{3.76}$$

The order of magnitude of the other nondimensional physical quantities are are as follows.

$$\left.\begin{array}{c}
\frac{EA}{m_0\Omega^2R^2} = O(\varepsilon^{-2}) \\
\frac{x}{R}, \frac{h}{R}, \frac{x_{cg}}{R}, \frac{y_{cg}}{R}, \frac{m}{m_0}, \frac{\delta}{\delta\psi}, \frac{\delta}{\delta x} = O(1) \\
\mu, \cos\psi, \sin\psi, \theta, \theta_t, \frac{c_1}{a}, \frac{d_2}{a} = O(1) \\
\frac{EI_y}{m_0\Omega^2R^4}, \frac{EI_z}{m_0\Omega^2R^4}, \frac{GJ}{m_0\Omega^2R^4} = O(1) \\
\beta_p, \frac{k_A}{R}, \frac{k_{m1}}{R}, \frac{k_{m2}}{R} = O(\varepsilon) \\
\alpha_s, \phi_s = 0(\varepsilon) \\
\lambda, \frac{\eta_c}{R}, \frac{c_0}{a}, \frac{d_1}{a}, \frac{f_0}{a} = O(\varepsilon) \\
\frac{EB_2}{m_0\Omega^2R^5}, \frac{EC_2}{m_0\Omega^2R^5} = O(\varepsilon) \\
\frac{e_d}{R}, \frac{e_g}{R}, \frac{e_a}{R} = O(\varepsilon^{\frac{3}{2}}) \\
\dot{\alpha_s}, \dot{\phi_s} = 0(\varepsilon^{\frac{3}{2}}) \\
\frac{EB_1}{m_0\Omega^2R^6}, \frac{EC_1}{m_0\Omega^2R^6} = O(\varepsilon^2) \\
\frac{d_0}{a}, \frac{f_1}{a} = O(\varepsilon^2)
\end{array}\right\} \tag{3.77}$$

R is the rotor radius, Ω is the rotational speed, E is the Young's Modulus, G is the shear modulus, I_y and I_z are cross-section moment of inertia from the y and z axis in the undeformed blade frame, J is the torsional rigidity constant, a is the lift curve slope and m_0 is mass per unit length of the blade. Rest of the symbols are defined in the beginning and later on as they appear. m_0 is defined as the mass per unit length of an uniform beam which has the same flap moment of inertia as the actual beam. Therefore

$$m_0 = \frac{3I_\beta}{R^3} \approx \frac{3\int_0^R mr^2dr}{R^3} \tag{3.78}$$

Azimuth angle is considered as nondimensional time, therefore

$$\left.\begin{array}{c}
\dot{(\,)} = \frac{\delta(\,)}{\delta t} = \frac{\delta(\,)}{\delta\psi}\frac{\delta\psi}{\delta t} = \Omega\frac{\delta(\,)}{\delta\psi} \\
\ddot{(\,)} = \frac{\delta^2(\,)}{\delta^2 t} = \frac{\delta^2(\,)}{\delta^2\psi}\frac{\delta^2\psi}{\delta^2 t} = \Omega^2\frac{\delta^2(\,)}{\delta^2\psi}
\end{array}\right\} \tag{3.79}$$

The ordering scheme is systematically and consistently adopted within the total energy context as is explained during the calculation of the energy terms. However, while following the scheme, terms are lost, which destroy the symmetric nature of the mass and stiffness matrix of the system, or, the antisymmetric gyroscopic nature of the modal equations, then those terms must be retained in violation to the ordering scheme.

3.6.7 Formulation Using Hamilton's Principle

Hamilton's variational principle is used to derive the blade equations of motion. For a conservative system, Hamilton's principle states that the true motion of a system, between prescribed initial conditions at time t_1 and final conditions at time t_2, is that particular motion for which the time integral of the difference between the potential and kinetic energies is a minimum. For an aeroelastic system, e.g., the rotor, there are nonconservative forces which are not derived from a potential function. The generalized Hamilton's Principle, applicable to nonconservative systems, is expressed as

$$\delta\Pi_b = \int_{t_1}^{t_2} (\delta U - \delta T - \delta W)\, dt = 0 \tag{3.80}$$

where δU is the virtual variation of strain energy and δT is the virtual variation of kinetic energy. The δW is the virtual work done by the external forces. These virtual variations have contributions from the rotor blades and the fuselage.

The variations can be written as

$$\delta U = \delta U_R + \delta U_F = \left(\sum_{b=1}^{N_b} \delta U_b \right) + \delta U_F \tag{3.81}$$

$$\delta T = \delta T_R + \delta T_F = \left(\sum_{b=1}^{N_b} \delta T_b \right) + \delta T_F \tag{3.82}$$

$$\delta W = \delta W_R + \delta W_F = \left(\sum_{b=1}^{N_b} \delta W_b \right) + \delta W_F \tag{3.83}$$

where the subscript R denotes the contribution from the rotor, which is the sum of individual contributions from the N_b blades, and F denotes the contribution from the fuselage. In the present study, only the rotor contribution is considered. Strain energy variation from the flexible pitch links are included in the blade energy terms. The expression for the virtual work δW has been dealt with in the chapter on Aerodynamic Modeling.

3.6.8 Derivation of Strain Energy

Because each blade is assumed to be a long slender isotropic beam, the uniaxial stress assumption ($\sigma_{yy} = \sigma_{yz} = \sigma_{zz} = 0$) can be used. The relation between stresses and classical engineering strains are

$$\sigma_{xx} = E\epsilon_{xx} \tag{3.84}$$

$$\sigma_{x\eta} = G\epsilon_{x\eta} \tag{3.85}$$

$$\sigma_{x\zeta} = G\epsilon_{x\zeta} \tag{3.86}$$

where ϵ_{xx} is axial strain, and $\epsilon_{x\eta}$ and $\epsilon_{x\zeta}$ are engineering shear strains. The expression for strain energy of the bth blade is

$$U_b = \frac{1}{2} \int_0^R \int \int_A (\sigma_{xx}\epsilon_{xx} + \sigma_{x\eta}\epsilon_{x\eta} + \sigma_{x\zeta}\epsilon_{x\zeta}) d\eta d\zeta dx \tag{3.87}$$

Using the stress-strain relations the variation of strain energy becomes

$$\delta U_b = \int_0^R \int \int_A (E\epsilon_{xx}\delta\epsilon_{xx} + G\epsilon_{x\eta}\delta\epsilon_{x\eta} + G\epsilon_{x\zeta}\delta\epsilon_{x\zeta}) d\eta d\zeta dx \tag{3.88}$$

The general non-linear strain displacement equations to second order are

$$\begin{aligned} \epsilon_{xx} = {} & u' + \frac{v'^2}{2} + \frac{w'^2}{2} - \lambda_T \phi'' + (\eta^2 + \zeta^2)(\theta'\phi' + \frac{\phi'^2}{2}) \\ & - v'' \left[\eta \cos(\theta + \phi) - \zeta \sin(\theta + \phi) \right] \\ & - w'' \left[\eta \sin(\theta + \phi) + \zeta \cos(\theta + \phi) \right] \end{aligned} \tag{3.89}$$

$$\epsilon_{x\eta} = -\left(\zeta + \frac{\partial \lambda_T}{\partial \eta} \right) \phi' = -\hat{\zeta}\phi' \tag{3.90}$$

$$\epsilon_{x\zeta} = -\left(\eta - \frac{\partial \lambda_T}{\partial \zeta} \right) \phi' = \hat{\eta}\phi' \tag{3.91}$$

where λ_T is the cross-sectional warping function. From equation (3.75) we have the relations between the deformation variable ϕ and quasi-coordinate $\hat{\phi}$.

$$\left.\begin{aligned} \phi' &= \hat{\phi}' + w'v'' \\ \delta\phi' &= \delta\hat{\phi}' + w'\delta v'' + v''\delta w' \end{aligned}\right\} \tag{3.92}$$

From equation (3.59) we have the relations between the deformation variable u and the quasi-coordinate u_e.

$$\left.\begin{aligned} u' &= u_e' - \tfrac{1}{2}(v'^2 + w'^2) \\ u &= u_e - \tfrac{1}{2}\textstyle\int_0^x (v'^2 + w'^2) \\ \delta u' &= \delta u_e' - v'\delta v' - w'\delta w' \\ \delta u &= \delta u_e - \textstyle\int_0^x (v'\delta v' + w'\delta w')dx \end{aligned}\right\} \tag{3.93}$$

Using equations (3.92) and (3.93) we obtain the strains as follows.

$$\begin{aligned} \epsilon_{xx} =& u_e' - \lambda_T(\hat{\phi}'' + w'v''' + v''w'') + (\eta^2 + \zeta^2)(\theta'\hat{\phi}' + \theta'w'v'' + \frac{\hat{\phi}'^2}{2} + \frac{w'^2v''^2}{2} + \hat{\phi}'w'v'') \\ & - v''\left[\eta\cos(\theta + \hat{\phi}) - \zeta\sin(\theta + \hat{\phi})\right] - w''\left[\eta\sin(\theta + \hat{\phi}) + \zeta\cos(\theta + \hat{\phi})\right] \end{aligned} \tag{3.94}$$

$$\epsilon_{x\eta} = -\hat{\zeta}(\hat{\phi}' + w'v'') \tag{3.95}$$

$$\epsilon_{x\zeta} = \hat{\eta}(\hat{\phi}' + w'v'') \tag{3.96}$$

The variation of the strains are

$$\begin{aligned} \delta\epsilon_{xx} =& \delta u_e' + \lambda_T(\delta\hat{\phi}'' + w'\delta v'' + v''\delta w'' + v'''\delta w' + w''\delta v'') \\ & + (\eta^2 + \zeta^2)[\theta'\delta\hat{\phi}' + \theta'w'\delta v'' + \theta'v''\delta w' + (\hat{\phi}' + w'v'')(\delta\hat{\phi}' + w'\delta v'' + v''\delta w')] \\ & - [\eta\cos(\theta + \hat{\phi}) - \zeta\sin(\theta + \hat{\phi})]\delta v'' + [\eta\sin(\theta + \hat{\phi}) + \zeta\cos(\theta + \hat{\phi})]v''\delta\hat{\phi} \\ & - [\eta\sin(\theta + \hat{\phi}) + \zeta\cos(\theta + \hat{\phi})]\delta w'' - [\eta\cos(\theta + \hat{\phi}) - \zeta\sin(\theta + \hat{\phi})]w''\delta\hat{\phi} \end{aligned} \tag{3.97}$$

$$\delta\epsilon_{x\eta} = -\hat{\zeta}(\delta\phi' + w'\delta v'' + v''\delta w') \tag{3.98}$$

$$\delta\epsilon_{x\zeta} = \hat{\eta}(\delta\phi' + w'\delta v'' + v''\delta w') \tag{3.99}$$

Substituting equations (3.97), (3.98) and (3.99) in equation (3.88) gives the variation of strain energy as function of the deformation variables. It can be expressed in nondimensional form as follows.

$$\begin{aligned} \delta U = \frac{\delta U_b}{m_0\Omega^2R^3} = \int_0^1 & (U_{u_e'}\delta u_e' + U_{v'}\delta v' + U_{w'}\delta w' + U_{v''}\delta v'' + U_{w''}\delta w'' \\ & + U_{\hat{\phi}}\delta\hat{\phi} + U_{\hat{\phi}'}\delta\hat{\phi}' + U_{\hat{\phi}''}\delta\hat{\phi}'')dx \end{aligned} \tag{3.100}$$

In deriving the expressions the following section properties are used.

$$\left.\begin{aligned} \textstyle\int\int_A d\eta d\zeta &= A \\ \textstyle\int\int_A \eta d\eta d\zeta &= Ae_A \\ \textstyle\int\int_A \zeta d\eta d\zeta &= 0 \\ \textstyle\int\int_A \lambda_T d\eta d\zeta &= 0 \\ \textstyle\int\int_A (\eta^2 + \zeta^2) d\eta d\zeta &= AK_A^2 \\ \textstyle\int\int_A (\eta^2 + \zeta^2)^2 d\eta d\zeta &= B_1 \\ \textstyle\int\int_A \eta(\eta^2 + \zeta^2)^2 d\eta d\zeta &= B_2 \\ \textstyle\int\int_A \eta^2 d\eta d\zeta &= I_Z \\ \textstyle\int\int_A \zeta^2 d\eta d\zeta &= I_Y \\ \textstyle\int\int_A \lambda_T^2 d\eta d\zeta &= EC_1 \\ \textstyle\int\int_A \zeta\lambda_T d\eta d\zeta &= EC_2 \end{aligned}\right\} \tag{3.101}$$

The coefficients, up to second order of non-linearities are given below.

$$\begin{aligned} U_{u'_e} =& EA\left[u'_e + K_A^2(\theta'\hat{\phi}' + \theta' w' v'' + \frac{\hat{\phi}'^2}{2})\right] \\ &- EAe_A\left[v''(\cos\theta - \hat{\phi}\sin\theta) + w''(\sin\theta + \hat{\phi}\cos\theta)\right] \end{aligned} \tag{3.102}$$

$$U_{v'} = 0 \tag{3.103}$$

$$U_{w'} = (GJ + EB_1\theta'^2)\hat{\phi}'v'' + EAK_A^2\theta' v'' u'_e \tag{3.104}$$

$$\begin{aligned} U_{v''} =& v''[EI_Z\cos^2(\theta+\hat{\phi}) + EI_Y\sin^2(\theta+\hat{\phi})] \\ &+ w''(EI_Z - EI_Y)\cos(\theta+\hat{\phi})\sin(\theta+\hat{\phi}) \\ &- EB_2\theta'\hat{\phi}'\cos\theta - EAe_A u'_e(\cos\theta - \hat{\phi}\sin\theta) + EAK_A^2 u'_e w'\theta' \\ &+ (GJ + EB_1\theta'^2)\hat{\phi}'w' - EC_2\hat{\phi}''\sin\theta \end{aligned} \tag{3.105}$$

$$\begin{aligned} U_{w''} =& w''[EI_Z\sin^2(\theta+\hat{\phi}) + EI_Y\cos^2(\theta+\hat{\phi})] \\ &+ v''[EI_Z - EI_Y]\cos(\theta+\hat{\phi})\sin(\theta+\hat{\phi}) \\ &- EAe_A u'_e(\sin\theta + \hat{\phi}\cos\theta) - EB_2\hat{\phi}'\theta'\sin\theta + EC_2\hat{\phi}''\cos\theta \end{aligned} \tag{3.106}$$

$$\begin{aligned} U_{\hat{\phi}} =& w''^2(EI_Z - EI_Y)\cos(\theta+\hat{\phi})\sin(\theta+\hat{\phi}) - v''^2(EI_Z - EI_Y)\cos(\theta+\hat{\phi})\sin(\theta+\hat{\phi}) \\ &+ v''w''(EI_Z - EI_Y)\cos 2(\theta+\hat{\phi}) \end{aligned} \tag{3.107}$$

$$\begin{aligned} U_{\hat{\phi}'} =& GJ(\hat{\phi}' + w'v'') + EAK_A^2(\theta' + \phi')u'_e \\ &+ EB_1\theta'^2\hat{\phi}' - EB_2\theta'(v''\cos\theta + w''\sin\theta) \end{aligned} \tag{3.108}$$

$$U_{\hat{\phi}''} = EC_1\hat{\phi}'' + EC_2(w''\cos\theta - v''\sin\theta) \tag{3.109}$$

Note that in the above expressions, the $\cos(\theta+\hat{\phi})$ and $\sin(\theta+\hat{\phi})$ terms associated with bending curvature, i.e., with EI_Z and EI_Y, have been retained. These terms are expanded to second order as

$$\left.\begin{aligned} \sin(\theta+\hat{\phi}) &= (1 - \tfrac{\hat{\phi}^2}{2})\sin\theta + \hat{\phi}\cos\theta \\ \cos(\theta+\hat{\phi}) &= (1 - \tfrac{\hat{\phi}^2}{2})\cos\theta - \hat{\phi}\sin\theta \end{aligned}\right\} \tag{3.110}$$

This expansion introduces third order terms in $U_{v''}$, $U_{w''}$ and $U_{\hat{\phi}}$ which are retained in violation of the ordering scheme. This is to maintain consistency between the force-summation and modal methods of blade loads calculation. Thus we have the following

$$\begin{aligned} U_{v''} =& v''(EI_Z\cos^2\theta + EI_Y\sin^2\theta) + w''(EI_Z - EI_Y)\cos\theta\sin\theta \\ &- v''\hat{\phi}\sin 2\theta(EI_Z - EI_Y) + w''\hat{\phi}\cos 2\theta(EI_Z - EI_Y) \\ &- v''\hat{\phi}^2\cos 2\theta(EI_Z - EI_Y) - w''\hat{\phi}^2\sin 2\theta(EI_Z - EI_Y) \\ &- EB_2\theta'\hat{\phi}'\cos\theta - EAe_A u'_e(\cos\theta - \hat{\phi}\sin\theta) + EAK_A^2 u'_e w'\theta' \\ &+ (GJ + EB_1\theta'^2)\hat{\phi}'w' - EC_2\hat{\phi}''\sin\theta \end{aligned} \tag{3.111}$$

$$\begin{aligned} U_{w''} =& w''(EI_Z\sin^2\theta + EI_Y\cos^2\theta) + v''(EI_Z - EI_Y)\cos\theta\sin\theta \\ &+ w''\hat{\phi}\sin 2\theta(EI_Z - EI_Y) + v''\hat{\phi}\cos 2\theta(EI_Z - EI_Y) \\ &+ w''\hat{\phi}^2\cos 2\theta(EI_Z - EI_Y) - v''\hat{\phi}^2\sin 2\theta(EI_Z - EI_Y) \\ &- EAe_A u'_e(\sin\theta + \hat{\phi}\cos\theta) - EB_2\hat{\phi}'\theta'\sin\theta + EC_2\hat{\phi}''\cos\theta \end{aligned} \tag{3.112}$$

$$\begin{aligned} U_{\hat{\phi}} =& (w''^2 - v''^2)\cos\theta\sin\theta(EI_Z - EI_Y) + v''w''\cos 2\theta \\ &\hat{\phi}(w''^2 - v''^2)\cos 2\theta(EI_Z - EI_Y) - 2\hat{\phi}v''w''\sin 2\theta \end{aligned} \tag{3.113}$$

3.6.9 Derivation of Kinetic Energy

The kinetic energy of the bth blade, δT_b depends on the blade velocity relative to the hub and the velocity of the hub itself. The velocity of the hub originates from fuselage dynamics and is neglected in the present analysis.

Let the position of an arbitrary point after the beam has deformed is given by (x_1, y_1, z_1) where

$$\begin{aligned}\bar{\mathbf{r}} &= \begin{bmatrix} x_1 & y_1 & z_1 \end{bmatrix} \begin{Bmatrix} \hat{i} \\ \hat{j} \\ \hat{k} \end{Bmatrix} = \begin{bmatrix} x+u & v & w \end{bmatrix} \begin{Bmatrix} \hat{i} \\ \hat{j} \\ \hat{k} \end{Bmatrix} + \begin{bmatrix} -\lambda\phi' & \eta & \zeta \end{bmatrix} \begin{Bmatrix} \hat{i_\xi} \\ \hat{j_\eta} \\ \hat{k_\zeta} \end{Bmatrix} \\ &= \left\{ \begin{bmatrix} x+u & v & w \end{bmatrix} + \begin{bmatrix} -\lambda\phi' & \eta & \zeta \end{bmatrix} \mathbf{T_{DU}} \right\} \begin{Bmatrix} \hat{i} \\ \hat{j} \\ \hat{k} \end{Bmatrix}\end{aligned} \tag{3.114}$$

Using equation (3.73) we obtain

$$\left.\begin{aligned} x_1 &= x + u - \lambda\phi' - v'(y_1 - v) - w'(z_1 - w) \\ y_1 &= v + (y_1 - v) \\ z_1 &= w + (z_1 - w) \end{aligned}\right\} \tag{3.115}$$

where

$$\left.\begin{aligned} y_1 - v &= \eta\cos(\theta + \hat{\phi}) - \zeta\sin(\theta + \hat{\phi}) \\ z_1 - w &= \eta\sin(\theta + \hat{\phi}) + \zeta\cos(\theta + \hat{\phi}) \end{aligned}\right\} \tag{3.116}$$

Now,

$$\bar{V}_b = \frac{\partial \bar{\mathbf{r}}}{\partial t} + \bar{\Omega} \times \bar{\mathbf{r}} \tag{3.117}$$

where using equation (3.49) we have

$$\bar{\Omega} = \Omega\hat{K} = \Omega\sin\beta_p\hat{i} + \Omega\cos\beta_p\hat{k} \tag{3.118}$$

and

$$\frac{\partial \bar{\mathbf{r}}}{\partial t} = \dot{x}_1\hat{i} + \dot{y}_1\hat{j} + \dot{z}_1\hat{k} \tag{3.119}$$

Using equations (3.119) and (3.49) in equation (3.117) we have

$$\bar{V}_b = V_{bx}\hat{i} + V_{by}\hat{j} + V_{bz}\hat{k} \tag{3.120}$$

where all velocities are non-dimensionalized with respect to ΩR and $\dot{(\ )} = \partial(\ )/\partial\psi$.

$$V_{bx} = \dot{x}_1 - y_1\cos\beta_p \tag{3.121}$$

$$V_{by} = \dot{y}_1 + x_1\cos\beta_p - z_1\sin\beta_p \tag{3.122}$$

$$V_{bz} = \dot{z}_1 + y_1\sin\beta_p \tag{3.123}$$

Taking variations of the velocities we have

$$\begin{aligned}\bar{V}.d\bar{V} = {} & \dot{x}_1\delta\dot{x}_1 - y_1\cos\beta_p\delta\dot{x}_1 - \dot{x}_1\cos\beta_p\delta y_1 + y_1\cos^2\beta_p\delta y_1 \\ & \dot{y}_1\delta\dot{y}_1 + x_1\cos\beta_p\delta\dot{y}_1 - z_1\sin\beta_p\delta\dot{y}_1 + \dot{y}_1\cos\beta_p\delta x_1 \\ & x_1\cos^2\beta_p\delta x_1 - z_1\sin\beta_p\cos\beta_p\delta x_1 - \dot{y}_1\sin\beta_p\delta z_1 - x_1\cos\beta_p\sin\beta_p\delta z_1 \\ & + \dot{z}_1\sin^2\beta_p\delta z_1 + \dot{z}_1\delta\dot{z}_1 + y_1\sin\beta_p\delta\dot{z}_1 + \dot{z}_1\sin\beta_p\delta y_1 + y_1\sin^2\beta_p\delta y_1\end{aligned} \tag{3.124}$$

According to variational method, this equation must be integrated in time between two arbitrary points in time, t_1 and t_2. The initial and final values (e.g.,$\dot{x}_1\delta x_1|_{t1}^{t2}$) are taken as zero. Anticipating integration by parts the various terms can be combined in equation (3.124) to obtain

$$\begin{aligned}\bar{V}.d\bar{V} = &-\ddot{x}_1\delta x_1 + 2\dot{y}_1\cos\beta_p\delta x_1 + y_1\cos^2\beta_p\delta y_1 - \ddot{y}_1\delta y_1\\ &- 2\dot{x}_1\cos\beta_p\delta y_1 + 2\dot{z}_1\sin\beta_p\delta y_1 + x_1\cos^2\beta_p\delta x_1\\ &- z_1\sin\beta_p\cos\beta_p\delta x_1 - 2\dot{y}_1\sin\beta_p\delta z_1 - x_1\cos\beta_p\sin\beta_p\delta z_1 + z_1\sin^2\beta_p\delta z_1\\ &- \ddot{z}_1\delta z_1 + y_1\sin^2\beta_p\delta y_1\end{aligned} \tag{3.125}$$

For the bth, the resultant kinetic energy expression in non-dimensional form is given by

$$\frac{\delta T_b}{m_0\Omega^2 R^3} = \int_0^1 \iint_A \rho\bar{V}.\delta\bar{V}\, d\eta\, d\zeta\;\; dx \tag{3.126}$$

where ρ is the structural mass density. Substituting the velocity expressions as given before we have

$$\frac{\delta T_b}{m_0\Omega^2 R^3} = \int_0^1 \left[\iint_A \rho\left(T_{x1}\delta x_1 + T_{y1}\delta y_1 + T_{z1}\delta z_1\right)\, d\eta\, d\zeta\right] dx \tag{3.127}$$

where

$$T_{x1} = -\ddot{x}_1 + 2\dot{y}_1\cos\beta_p + x_1\cos^2\beta_p - z_1\sin\beta_p\cos\beta_p \tag{3.128}$$

$$T_{y1} = y_1\cos^2\beta_p - \ddot{y}_1 - 2\dot{x}_1\cos\beta_p + y_1\sin^2\beta_p + 2\dot{z}_1\sin\beta_p \tag{3.129}$$

$$T_{z1} = -2\dot{y}_1\sin\beta_p - \ddot{z}_1 + z_1\sin^2\beta_p - x_1\cos\beta_p\sin\beta_p \tag{3.130}$$

Now, using equations (8.32) and (3.116) we have

$$\left.\begin{aligned}\dot{y_1} &= \dot{v} - (z_1 - w)\dot{\theta}_1\\ \dot{z_1} &= \dot{w} + (y_1 - v)\dot{\theta}_1\\ \dot{x_1} &= \dot{u} - \lambda_T\dot{\phi}' - (\dot{v}' + w'\dot{\theta}_1)(y - 1 - v) - (\dot{w}' - v'\dot{\theta}_1)(z_1 - w)\end{aligned}\right\} \tag{3.131}$$

and

$$\left.\begin{aligned}\ddot{y}_1 &= \ddot{v} - (z_1 - w)\ddot{\theta}_1 - (y_1 - v)\dot{\theta}^2\\ \ddot{z}_1 &= \ddot{w} + (y_1 - v)\ddot{\theta}_1 - (z_1 - w)\dot{\theta}^2\\ \ddot{x}_1 &= \ddot{u} - \lambda_T\ddot{\theta}_1' - (y_1 - v)(\ddot{v}' + w'\ddot{\theta}' - v'\dot{\theta}^2 + 2\dot{w}'\dot{\theta}_1)\\ &\quad -(z_1 - w)(\ddot{w}' - v'\ddot{\theta}_1 - w'\dot{\theta}_1^2 - 2\dot{v}'\dot{\theta}_1)\end{aligned}\right\} \tag{3.132}$$

The variations are as follows

$$\left.\begin{aligned}\delta y_1 &= \delta v - \delta\hat{\phi}(z_1 - w)\\ \delta z_1 &= \delta w + \delta\hat{\phi}(y_1 - v)\\ \delta x_1 &= \delta u - \lambda_T\delta\hat{\phi}' - (y_1 - v)(\delta v' + w'\delta\hat{\phi})\\ &\quad -(z_1 - w)(\delta w' - v'\delta\hat{\phi})\end{aligned}\right\} \tag{3.133}$$

Using equations (3.133), (3.132), (3.131), (8.32) in (3.126) we obtain

$$\delta T = \frac{\delta T_b}{m_0\Omega^2 R^3} = \int_0^1 m(T_{u_e}\delta u_e + T_v\delta v + T_w\delta w + T_{w'}\delta w' + T_{v'}\delta v' + T_\phi\delta\phi + T_F)dx \tag{3.134}$$

In deriving the expressions the following section properties are used.

$$\left.\begin{array}{c}\int\int_A \rho d\eta d\zeta = m \\ \int\int_A \rho\eta d\eta d\zeta = me_g \\ \int\int_A \rho\zeta^2 d\eta d\zeta = mk_{m_1}^2 \\ \int\int_A \rho\eta^2 d\eta d\zeta = mk_{m_2}^2 \\ k_{m_1}^2 + k_{m_1}^2 = k_m^2 \\ \int\int_A \rho\zeta d\eta d\zeta = 0 \\ \int\int_A \rho\eta\zeta d\eta d\zeta = 0 \\ \int\int_A \rho\lambda_T d\eta d\zeta = 0\end{array}\right\} \tag{3.135}$$

assuming cross-section symmetry about the η axis and an antisymmetric warp function λ_T. The terms involving $(y_1 - v)$ and $(z_1 - w)$ are given by

$$\left.\begin{array}{c}\int\int_A \rho(y_1 - v)d\eta d\zeta = me_g\cos(\theta + \hat{\phi}) \\ \int\int_A \rho(z_1 - w)d\eta d\zeta = me_g\sin(\theta + \hat{\phi}) \\ \int\int_A \rho(z_1 - w)(y_1 - v)d\eta d\zeta = m(k_{m_2}^2 - k_{m_1}^2)\sin(\theta + \hat{\phi})\cos(\theta + \hat{\phi}) \\ \int\int_A \rho[(y_1 - v)^2 - (z_1 - w)^2]d\eta d\zeta = mk_m^2\end{array}\right\} \tag{3.136}$$

The coefficients in equation (3.134) are written up to second order, $O(\epsilon^2)$, as follows

$$T_{U_e} = -\ddot{u} + u + x + 2\dot{v} \tag{3.137}$$

where

$$\left.\begin{array}{c}u = u_e - \frac{1}{2}\int_0^x (v'^2 + w'^2)dx \\ \ddot{u} = \ddot{u}_e - \int_0^x (\dot{v}'^2 + v'\ddot{v}' + \dot{w}'^2 + w'\ddot{w}')dx\end{array}\right\} \tag{3.138}$$

$$\begin{aligned}T_v =& -\ddot{v} + e_g\ddot{\theta}\sin\theta + e_g\cos\theta + v - \hat{\phi}\sin\theta + 2\dot{w}\beta_p + 2e_g\dot{v}'\cos\theta \\ &+ 2e_g\dot{w}'\sin\theta + \ddot{\hat{\phi}}e_g\sin\theta - 2\dot{u}_e + 2\int_0^x (v'\dot{v}' + w'\dot{w}')dx\end{aligned} \tag{3.139}$$

$$T_{v'} = -e_g(x\cos\theta - \hat{\phi}x\sin\theta + 2\dot{v}\cos\theta) \tag{3.140}$$

$$\begin{aligned}T_w =& -\ddot{w} - e_g\ddot{\theta}\cos\theta - e_g\ddot{\hat{\phi}}\cos\theta \\ &- 2\dot{v}\beta_p - x\beta_p\end{aligned} \tag{3.141}$$

$$T_{w'} = -e_g(x\sin\theta + \hat{x}\cos\theta + 2\dot{v}\sin\theta) \tag{3.142}$$

$$\begin{aligned}T_{\hat{\phi}} =& -k_m^2\ddot{\hat{\phi}} - \hat{\phi}(k_{m_2}^2 - k_{m_1}^2)\cos 2\theta - (k_{m_2}^2 - k_{m_1}^2)\cos\theta\sin\theta - x\beta_p e_g\cos\theta \\ &- ve_g\sin\theta + xv'e_g\sin\theta - xw'e_g\cos\theta + \ddot{v}e_g\sin\theta - \ddot{w}e_g\cos\theta - k_m^2\ddot{\theta}\end{aligned} \tag{3.143}$$

The non-variation term T_F is given by

$$\begin{aligned}T_F =& -(-\ddot{u} + u + x + 2\dot{v})\int_0^x (v'\delta v' + w'\delta w') \\ &= -T_{U_e}\int_0^x (v'\delta v' + w'\delta w')\end{aligned} \tag{3.144}$$

Note that the ordering scheme is violated in equation (3.144). It is important to keep the entire T_{U_e} in the non-variation form for articulated rotors where the bending moments at the hinge must go to zero. For hingeless rotors with large bending moments at the blade root the error is negligible.

3.6.10 Virtual Work

For each degree of freedom, there is a corresponding external force (or moment) which contribute to virtual work on the system. The general expression is given by

$$\frac{\delta W_b}{m_0 \Omega^2 R^3} = \int_0^1 (L_u^A \delta u + L_v^A \delta v + L_w^A \delta w + M_{\hat{\phi}}^A \delta \hat{\phi}) dx \tag{3.145}$$

where L_u^A, L_v^A, L_w^A, $M_{\hat{\phi}}^A$ are the distributed air loads in the x, y and z directions and $M_{\hat{\phi}}^A$ is the aerodynamic pitching moment about the undeformed elastic axis. Calculated air loads are motion dependent. Measured air loads are not motion dependent.

In addition to distributed air loads, there can be concentrated forces and moments acting on locations over the blade span, e.g. a prescribed damper force. They can be included as follows.

$$\begin{aligned}\frac{\delta W_b}{m_0 \Omega^2 R^3} &= \int_0^1 (L_u^A \delta u + L_v^A \delta v + L_w^A \delta w + M_{\hat{\phi}}^A \delta \hat{\phi}) dx \\ &+ \int_0^1 (F_x \delta u + F_y \delta v + F_z \delta w + M_x \delta \hat{\phi} - M_y \delta w' + M_z \delta v') \delta(x - x_f) dx\end{aligned} \tag{3.146}$$

where F_x, F_y, F_z, M_x, M_y, M_z are the concentrated forces and moments acting at $x = x_f$ along the blade span. The calculated forces and moments are described in Chapter 3.

3.6.11 Equations of Motion

Integrating the strain energy, kinetic energy and virtual work expressions (3.100), (3.134) and (3.145) by parts we obtain

$$\left.\begin{aligned}\delta U &= \textstyle\int_0^1 (Y_{u_e} \delta u_e + Y_v \delta v + Y_w \delta w + Y_{\hat{\phi}} \delta_{\hat{\phi}}) dx + b(U) \\ \delta T &= \textstyle\int_0^1 (Z_{u_e} \delta u_e + Z_v \delta v + Z_w \delta w + Z_{\hat{\phi}} \delta_{\hat{\phi}}) dx + b(T) \\ \delta W &= \textstyle\int_0^1 (W_{u_e} \delta u_e + W_v \delta v + W_w \delta w + W_{\hat{\phi}} \delta_{\hat{\phi}}) dx + b(W)\end{aligned}\right\} \tag{3.147}$$

where $b(U)$, $b(T)$ and $b(W)$ are the force and displacement boundary conditions. Using equation (3.80) and collecting terms associated with δu, δv, δw and $\delta \hat{\phi}$ we obtain the blade equations as follows.

u_e **equation :**

$$\begin{aligned}&\Bigg[EAu_e' + EAK_A^2 \left(\theta' \hat{\phi}' + \theta' w' v'' + \frac{\hat{\phi}'^2}{2}\right) \\ &- EAe_A v'' (\cos\theta - \hat{\phi}\sin\theta) + EAw''(\sin\theta + \hat{\phi}\cos\theta)\Bigg]' \\ &+ m(\ddot{u}_e - u_e - x - 2\dot{v}) = L_u\end{aligned} \tag{3.148}$$

v **equation :**

$$
\begin{aligned}
&\big[v''(EI_Z\cos^2\theta + EI_Y\sin^2\theta) + w''(EI_Z - EI_Y)\cos\theta\sin\theta \\
&-v''\hat{\phi}\sin 2\theta(EI_Z - EI_Y) + w''\hat{\phi}\cos 2\theta(EI_Z - EI_Y) \\
&-v''\hat{\phi}^2\cos 2\theta(EI_Z - EI_Y) - w''\hat{\phi}^2\sin 2\theta(EI_Z - EI_Y) \\
&-EB_2\theta'\hat{\phi}'\cos\theta - EAe_A u_e'(\cos\theta - \hat{\phi}\sin\theta) + EAK_A^2 u_e' w'\theta' \\
&+(GJ + EB_1\theta'^2)\hat{\phi}'w' - EC_2\hat{\phi}''\sin\theta\big]'' \\
&- m\Big[-\ddot{v} + e_g\ddot{\theta}\sin\theta + e_g\cos\theta + v - \hat{\phi}\sin\theta + 2\dot{w}\beta_p + 2e_g\dot{v}'\cos\theta \\
&+2e_g\dot{w}'\sin\theta + \ddot{\hat{\phi}}e_g\sin\theta - 2\dot{u}_e + 2\int_0^x(v'\dot{v}' + w'\dot{w}')dx\Big] \\
&- me_g\Big(x\cos\theta - \hat{\phi}x\sin\theta + 2\dot{v}\cos\theta\Big)' + \Big\{mv'\int_x^1(-\ddot{u}_e + u_e + x + 2\dot{v})\Big\}' = L_v
\end{aligned}
\qquad (3.149)
$$

w **equation :**

$$
\begin{aligned}
&\big[w''(EI_Z\sin^2\theta + EI_Y\cos^2\theta) + v''(EI_Z - EI_Y)\cos\theta\sin\theta \\
&+w''\hat{\phi}\sin 2\theta(EI_Z - EI_Y) + v''\hat{\phi}\cos 2\theta(EI_Z - EI_Y) \\
&+w''\hat{\phi}^2\cos 2\theta(EI_Z - EI_Y) - v''\hat{\phi}^2\sin 2\theta(EI_Z - EI_Y) \\
&-EAe_A u_e'(\sin\theta + \hat{\phi}\cos\theta) - EB_2\hat{\phi}'\theta'\sin\theta + EC_2\hat{\phi}''\cos\theta\big]'' \\
&- m\Big(-\ddot{w} - e_g\ddot{\theta}\cos\theta - e_g\ddot{\hat{\phi}}\cos\theta - 2\dot{v}\beta_p - x\beta_p\Big) \\
&- me_g\left(x\sin\theta + \hat{x}\cos\theta + 2\dot{v}\sin\theta\right)' + \Big\{mw'\int_x^1(-\ddot{u}_e + u_e + x + 2\dot{v})\Big\}' = L_v
\end{aligned}
\qquad (3.150)
$$

$\hat{\phi}$ **equation :**

$$
\begin{aligned}
&(w''^2 - v''^2)\cos\theta\sin\theta(EI_Z - EI_Y) + v''w''\cos 2\theta \\
&\hat{\phi}(w''^2 - v''^2)\cos 2\theta(EI_Z - EI_Y) - 2\hat{\phi}v''w''\sin 2\theta \\
&+\Big[GJ(\hat{\phi}' + w'v'') + EAK_A^2(\theta' + \phi')u_e' \\
&+EB_1\theta'^2\hat{\phi}' - EB_2\theta'(v''\cos\theta + w''\sin\theta)\Big]' \\
&-\Big[-k_m^2\ddot{\hat{\phi}} - \hat{\phi}(k_{m_2}^2 - k_{m_1}^2)\cos 2\theta - (k_{m_2}^2 - k_{m_1}^2)\cos\theta\sin\theta - x\beta_p e_g\cos\theta \\
&-ve_g\sin\theta + xv'e_g\sin\theta - xw'e_g\cos\theta + \ddot{v}e_g\sin\theta - \ddot{w}e_g\cos\theta - k_m^2\ddot{\theta}\Big] = L_{\hat{\phi}}
\end{aligned}
\qquad (3.151)
$$

3.7 Structural loads

The blade sectional loads, i.e. the flap, lag and torsion bending moments, are calculated using two methods - (1) Modal Curvature and (2) Force Summation Method. For converged blade response, i.e. when the response does not change with increase in the number of blade normal modes, both methods should produce identical loads. In the immediate vicinity of a concentrated loading, e.g., lag damper force, the force summation method captures the blade loads with lesser number of modes.

To obtain the same loads using force summation and modal curvature methods, the response equations must be consistent with loads calculations. Consistency is specially important for articulated rotors where the bending loads must reduce to zero at the hinge.

3.7.1 Modal Curvature Method

The flap and lag bending moments, M_η and M_ζ are obtained as follows.

$$\begin{aligned} M_\eta &= \int\int_A \zeta\sigma d\eta d\zeta = \int\int_A E\zeta\epsilon_{xx} d\eta d\zeta \\ &= EI_\eta[v''\sin(\theta+\hat{\phi}) - w''\cos(\theta+\hat{\phi})] - EC_1\hat{\phi}'' \end{aligned} \tag{3.152}$$

$$\begin{aligned} M_\zeta &= -\int\int_A \eta\sigma d\eta d\zeta = -\int\int_A E\eta\epsilon_{xx} d\eta d\zeta \\ &= EI_\zeta[v''\cos(\theta+\hat{\phi}) + w''\sin(\theta+\hat{\phi})] - EAe_A u_e' - EB_2\theta'\hat{\phi}' \end{aligned} \tag{3.153}$$

The expression for torsion bending moment is given by

$$\begin{aligned} M_\xi &= \int\int_A \left[\eta\sigma_{x\zeta} - \zeta\sigma_{x\eta} + \lambda_T(\frac{\partial\sigma_{x\eta}}{\partial\eta} + \frac{\partial\sigma_{x\zeta}}{\partial\zeta})\right] d\eta d\zeta \\ &+ \frac{\partial}{\partial x}\int\int_A \lambda_T\sigma_{xx} d\eta d\zeta \\ &+ (\theta+\hat{\phi})'\int\int_A (\eta^2+\zeta^2)\sigma_{xx} d\eta d\zeta \\ &= EAk_A^2(\theta+\hat{\phi})'u_e' + EB_1\theta'^2\hat{\phi}' - EB_2\theta'(v''\cos\theta + w''\sin\theta) \\ &+ GJ(\hat{\phi}' + w'v'') \\ &- [EC_1\hat{\phi}'' + EC_2(w''\cos\theta - v''\sin\theta)]' \end{aligned} \tag{3.154}$$

3.7.2 Force Summation Method

The loads occurring at a blade section are the reaction forces (and moments) to those occurring outboard. It is equal (and opposite) to the integrated air loads and inertial loads from blade tip to the desired section. The inertial forces and moments at each blade section are given by the following.

$$\left.\begin{aligned} F^I &= -\int\int_A \rho\bar{a} d\eta d\zeta \\ M^I &= -\int\int_A \bar{s}\times\rho d\eta d\zeta \end{aligned}\right\} \tag{3.155}$$

The acceleration of the section, $\bar{a}$ is given by

$$\bar{a} = \ddot{\bar{r}} + \Omega\times(\Omega\times\bar{r}) + 2(\Omega\times\dot{\bar{r}}) \tag{3.156}$$

The moment arm of a point on the blade section measured from the deformed shear center, $\bar{s}$ is obtained from equation(8.32) as

$$\bar{s} = -[v'(y_1 - v) + w'(z_1 - w)]\hat{i} + (y_1 - v)\hat{j} + (z_1 - w)\hat{k} \tag{3.157}$$

Using equations (8.32), (3.118), (3.131), and (3.132) we obtain

$$\bar{a} = a_x\hat{i} + a_y\hat{j} + a_z\hat{k} \tag{3.158}$$

where

$$\begin{aligned} a_x =& \ddot{u} - \lambda_T\ddot{\theta}_1' - (y_1 - v)(\ddot{v}' + w'\ddot{\theta}_1 + 2\dot{w}'\dot{\theta}_1 - v'\dot{\theta}_1^2) \\ &- (z_1 - w)(\ddot{w}' - 2\dot{v}'\dot{\theta}_1 - v'\ddot{\theta}_1 - w'\dot{\theta}_1^2) - 2[\dot{v} - \dot{\theta}_1(z_1 - w)] \\ &+ \beta_p[w + (z_1 - w)] - [x + u - v'(y_1 - v) - w'(z_1 - w)] \end{aligned} \tag{3.159}$$

$$\begin{aligned} a_y =& \ddot{v} - \ddot{\theta}_1(z_1 - w) - \dot{\theta}_1^2(y_1 - v) - 2\beta_p[\dot{w} + \dot{\theta}_1(y_1 - v)] \\ & 2[\dot{u} - \lambda_T\dot{\theta}_1' - (y_1 - v)(\dot{v}' + w'\dot{\theta}_1) - (z_1 - w)(\dot{w}' - v'\dot{\theta}_1)] - \\ & \beta_p[v + (y_1 - v)] - [v + (y_1 - v)] \end{aligned} \tag{3.160}$$

$$\begin{aligned} a_z =& \ddot{w} + (y_1 - v)\ddot{\theta}_1 - (z_1 - w)\dot{\theta}_1^2 + 2\beta_p[\dot{v} - \dot{\theta}_1(z_1 - w)] \\ &+ \beta_p[x + u - v'(y_1 - v) - w'(z_1 - w)] \end{aligned} \tag{3.161}$$

Let L_u^I, L_v^I, L_w^I and M_u^I, M_v^I, M_w^I be the inertial forces and moments in the undeformed frame x, y, z directions. Then, to second order, we have the following.

$$L_u^I = -\int\int_A \rho a_x d\eta d\zeta = T_{U_e} \tag{3.162}$$

$$L_v^I = -\int\int_A \rho a_y d\eta d\zeta = T_v \tag{3.163}$$

$$L_w^I = -\int\int_A \rho a_z d\eta d\zeta = T_w \tag{3.164}$$

$$\begin{aligned} M_v^I &= -\int\int_A [v'(y_1 - v) + w'(z_1 - w)a_z + (z_1 - w)a_x] d\eta d\zeta \\ &\approx -\int\int_A (z_1 - w)a_x d\eta d\zeta \\ &= -T_w' \end{aligned} \tag{3.165}$$

$$\begin{aligned} M_w^I &= \int\int_A [v'(y_1 - v) + w'(z_1 - w)a_y + (z_1 - w)a_x] d\eta d\zeta \\ &\approx \int\int_A (y_1 - v)a_x d\eta d\zeta \\ &= T_v' \end{aligned} \tag{3.166}$$

$$\begin{aligned} M_u^I &= \int\int_A [(z_1 - w)a_y - (y_1 - v)a_z] d\eta d\zeta \\ &= T_{\hat{\phi}} - v'M_v^I - w'M_w^I \\ &= T_{\hat{\phi}} + v'T_w' - w'T_v' \end{aligned} \tag{3.167}$$

where T_{U_e}, T_v, T_w, T_v', T_w', $T_{\hat{\phi}}$ are identical to those given in equations (3.137) to (3.143). Thus the kinetic energy terms derived before are identical to the inertial terms obtained here. This shows the equivalence of Hamilton's Principle and Newton's Laws.

Let the external loads (air loads and other concentrated loadings if any e.g., a prescribed lag damper force) be denoted by the superscript A. Then the total loads distribution at a section is given by the sum of inertial and external loads

$$\left.\begin{aligned} L_u &= L_u^A + L_u^I \\ L_v &= L_v^A + L_v^I \\ L_w &= L_w^A + L_w^I \\ M_u &= M_{\hat{\phi}}^A + M_u^I \\ M_v &= v' M_{\hat{\phi}}^A + M_v^I \\ M_u &= w' M_{\hat{\phi}}^A + M_w^I \end{aligned}\right\} \tag{3.168}$$

$M_{\hat{\phi}}^A$ is the external pitching moment (e.g. aerodynamic pitching moment) acting in the blade deformed frame. Its components in the x, y, z directions, $M_{\hat{\phi}}^A, v' M_{\hat{\phi}}^A, w' M_{\hat{\phi}}^A$ are obtained using T_{DU} from equation (3.73).

The resultant shear forces and bending moments at any blade section x_0 is given by the following.

$$\left\{\begin{array}{c} f_x \\ f_y \\ f_z \end{array}\right\} = \int_{x_0}^{1} \left\{\begin{array}{c} L_u \\ L_v \\ L_w \end{array}\right\} dx \tag{3.169}$$

$$\left\{\begin{array}{c} m_x \\ m_y \\ m_z \end{array}\right\} = \int_{x_0}^{1} \left\{\begin{array}{c} -L_v(w - w_0) + L_w(v - v_0) + M_u \\ L_u(w - w_0) - L_w(x + u - x_0 - u_0) + M_v \\ -L_u(v - v_0) + L_v(x + u - x_0 - u_0) + M_w \end{array}\right\} dx \tag{3.170}$$

To compute the contribution of the blade loads to the hub loads in the rotating frame, the spanwise integration is carried out from the hub center to the blade tip, and $u_0, v_0, w_0, x_0 = 0$ The hub loads in the fixed frame is calculated using transformation (3.48).

$$\left.\begin{aligned} F_X(\psi) &= \textstyle\sum_{m=1}^{N_b} (f_x^m \cos\psi_m - f_y^m \sin\psi_m - f_z^m \cos\psi_m \beta_p) \\ F_Y(\psi) &= \textstyle\sum_{m=1}^{N_b} (f_x^m \sin\psi_m + f_y^m \cos\psi_m - f_z^m \sin\psi_m \beta_p) \\ F_Z(\psi) &= \textstyle\sum_{m=1}^{N_b} (f_z^m + f_x^m \beta_p) \\ M_X(\psi) &= \textstyle\sum_{m=1}^{N_b} (m_x^m cos\psi_m - m_y^m \sin\psi_m - m_z^m \cos\psi_m \beta_p) \\ M_Y(\psi) &= \textstyle\sum_{m=1}^{N_b} (m_x^m \sin\psi_m + m_y^m \cos\psi_m - m_z^m \sin\psi_m \beta_p) \\ M_Z(\psi) &= \textstyle\sum_{m=1}^{N_b} (m_z^m + m_x^m \beta_p) \end{aligned}\right\} \tag{3.171}$$

where $f_x, f_y, f_z, m_x, m_y, m_z$ are the rotating frame hub loads, i.e., blade loads integrated up to the hub. The steady values of the fixed frame hub loads (3.171) are used for trimming the helicopter. The higher harmonics cause helicopter vibration. For a tracked rotor, with identical structural and aerodynamic behavior, the higher harmonics contain only those frequencies which are integral multiples of rotor frequency. These harmonics are generated by harmonics of rotating frame blade loads which are one higher and one lower than the rotor frequency.

For example, for an N_b bladed rotor, the higher harmonics in the fixed frame hub loads are pN_b/rev, where p is an integer. These harmonics are generated by $pN_b \pm 1$/rev in-plane shear forces (f_x, f_y), pN_b/rev vertical shear force (f_z), $pN_b \pm 1$/rev flap and torsion bending moments (m_x, m_y) and pN_b/rev chord bending moment (m_z). To predict helicopter vibration these rotating frame blade loads must be predicted correctly.

3.8 Hub Reactions

The forces and moments acting on the root of the blade are transmitted to the body. If we sum up all the like forces and like moments from various blades in the fixed frame, these form the