Laplace's Equation & Poisson's Equation

Introduction to Electromagnetism with Practice

Theory & Applications

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Poisson & Laplace Equations – Introduction

Remind: Differential Form

$$\nabla \times \mathbf{E} = \mathbf{O}$$

$$\nabla \cdot \mathbf{D} = \rho$$

$$\mathbf{D} = \varepsilon_0 \varepsilon_r \mathbf{E}$$

$$\varepsilon_{\mathrm{r}} = 1$$

Vacuum

$$\nabla \times \mathbf{E} = \mathbf{O}$$

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\varepsilon_0}$$

Remind: Poisson’s Equation in Homogeneous Materials

$$\nabla \times \mathbf{E} = \mathbf{O}$$

$$\nabla \cdot \mathbf{D} = \rho$$

Already related...

$$\mathbf{E} = -\nabla V$$

Need to be related...

$$\mathbf{D} = \varepsilon_0 \varepsilon_r \mathbf{E}$$

Poisson’s Equation

$$\nabla \cdot \left(-\nabla V \right) = \frac{\rho}{\varepsilon_0 \varepsilon_r} \Longrightarrow \boxed{-\nabla^2 V = \frac{\rho}{\varepsilon_0 \varepsilon_r}}$$

Governing Eq. for Electrostatics in Homogeneous Materials (with B.C.)

Remind: Equation for Inhomogeneous Materials

$$\nabla \times \mathbf{E} = \mathbf{O} \qquad \nabla \cdot \mathbf{D} = \rho$$

Already related...

Need to be related...

$$\mathbf{E} = -\nabla V$$

$$\mathbf{D} = \varepsilon_0 \varepsilon_r(\mathbf{x})\mathbf{E}$$

$$\nabla \cdot \left(\varepsilon_r(\mathbf{x})\mathbf{E}\right) = \varepsilon_r(\mathbf{x})\nabla \cdot \mathbf{E} + \left(\nabla \varepsilon_r(\mathbf{x})\right) \cdot \mathbf{E}$$

$$\boxed{-\varepsilon_r(\mathbf{x})\nabla^2 V - \left(\nabla \varepsilon_r(\mathbf{x})\right) \cdot \left(\nabla V\right) = \frac{\rho}{\varepsilon_0}}$$

Governing Eq. for Electrostatics in Inhomogeneous Materials (with B.C.)

Laplace & Poisson Equations

Poisson's Equation

$$\boxed{-\nabla^2 V = \frac{\rho}{\varepsilon_0 \varepsilon_r}}$$

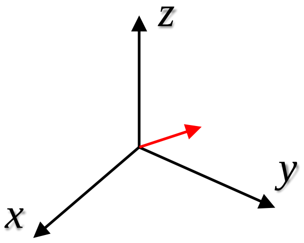
$$\boxed{\rho = 0}$$

Laplace's Equation

$$\boxed{\nabla^2 V = 0}$$

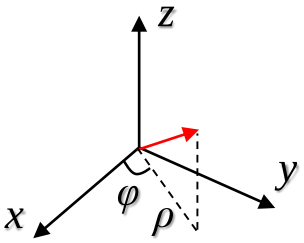
Laplacian Operator in Different Coordinates

$$\boxed{\nabla^2 V = 0}$$



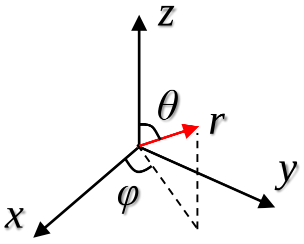
Cartesian Coordinate

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$



Cylindrical Coordinate

$$\nabla^2 = \frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho \frac{\partial}{\partial \rho}\right) + \frac{1}{\rho^2}\frac{\partial^2}{\partial \varphi^2} + \frac{\partial^2}{\partial z^2}$$



Spherical Coordinate

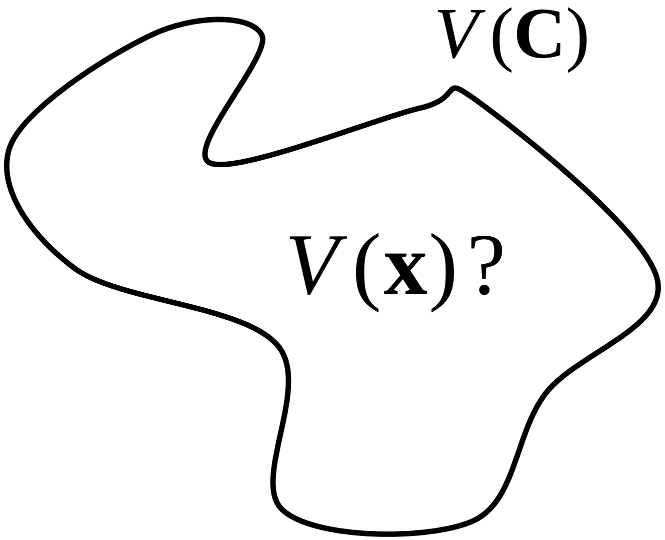
$$\nabla^2 = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial}{\partial r}\right) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial}{\partial \theta}\right) + \frac{1}{r^2 \sin^2\theta}\frac{\partial^2}{\partial \varphi^2}$$

Basic Strategy – General Solution & Boundary Condition

$$\nabla^2 V(x, y, z) = 0$$

$$\nabla^2 V(\rho, \varphi, z) = 0$$

$$\nabla^2 V(r, \theta, \varphi) = 0$$



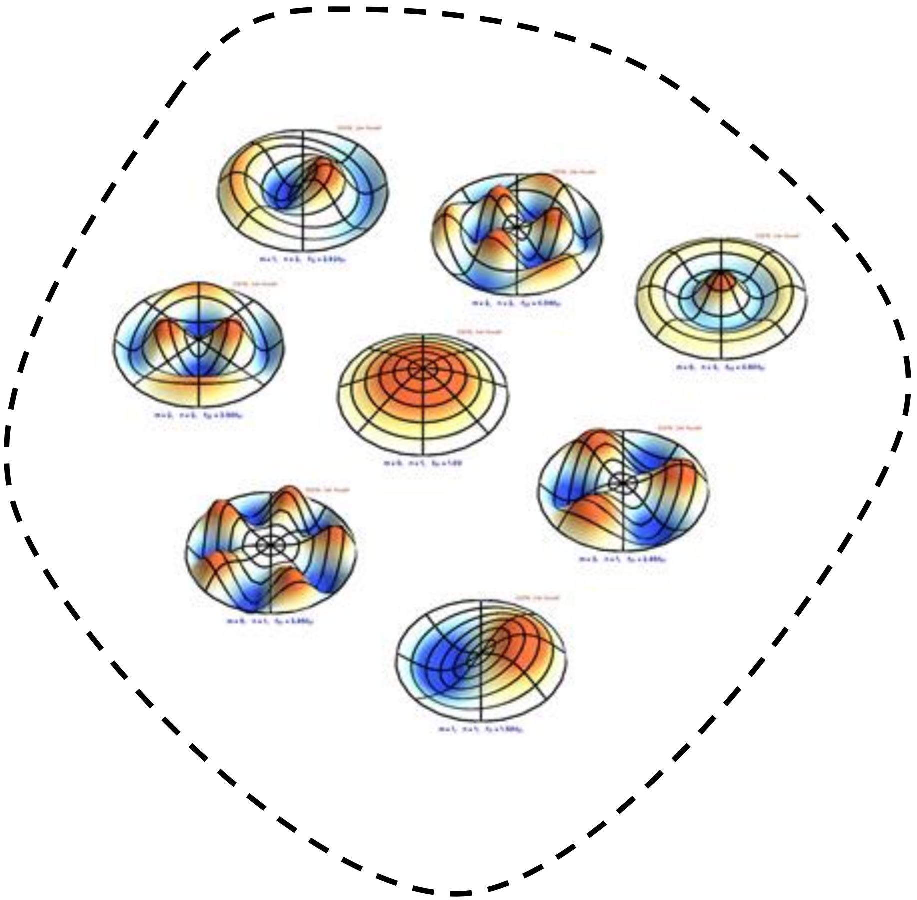
I. Find the mathematical form of ***a general solution***
: it has a “mathematically” different form for each coordinate system
(not physically different)

II. Applying B.C. → Achieving necessary coefficients!
: ***Selecting a proper coordinate system*** ~ ***more easy solving process***

Basic Strategy – Illustration of the Solving Process

I. ***A general solution*** can be expressed with the linear combination of each building block function (which is called an ***eigenmode***)

II. ***Boundary conditions*** then determine the ratio of each component (achieving the ***coefficients of the superposition***)



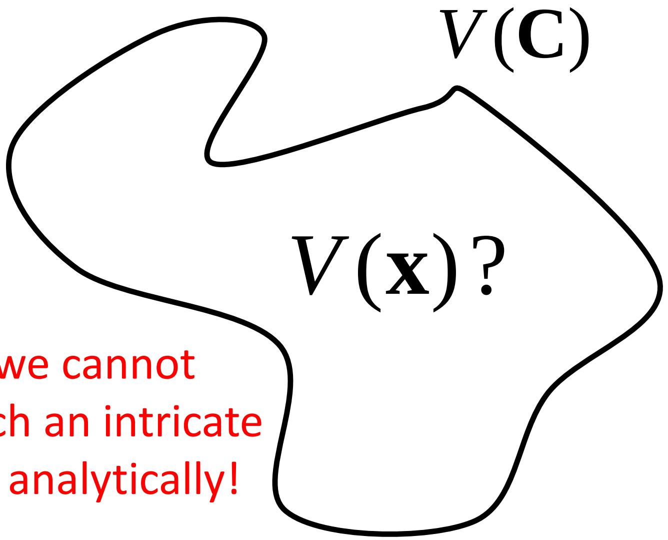
https://www.acs.psu.edu/drussell/Demos/MembraneCircle/Circle.html

Basic Strategy – Focusing on Solvable Problems

$$\nabla^2 V(x, y, z) = 0$$

$$\nabla^2 V(\rho, \varphi, z) = 0$$

$$\nabla^2 V(r, \theta, \varphi) = 0$$



Usually, we cannot solve such an intricate problem analytically!

$$V(\xi_1, \xi_2, \xi_3) = 0$$

$$\Downarrow$$

$$V(\xi_1, \xi_2, \xi_3) = V_1(\xi_1) V_2(\xi_2) V_3(\xi_3)$$

The Separation of Variables (SoV) allows analytical solutions of many important problems!
& also provides efficient numerical assessments!

Notes on SoV

- Not all solutions have the form of separation of variables
  ➔ Some insights are necessary considering the geometry of structures
- How can we have such insights? ➔ Solve many, many problems!
- exp(κx) can have different forms and each problem has a more suitable form that enables easy solving process
  - For real κ ($\kappa^2 > 0$), it has the form of cosh(κx) & sinh(κx)
  - For imaginary κ ($= i\gamma$, $\kappa^2 < 0$), it has the form of cos(γx) & sin(γx)
- How can we select a proper form? ➔ Solve many, many problems!

Laplace Equations – Cartesian Coordinates

Starting from SoV for Cartesian Coordinates

$$\left[\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}\right]V=0$$

Separation of Variables: $V = X(x)Y(y)Z(z)$

$$YZ\frac{\partial^2 X}{\partial x^2}+ZX\frac{\partial^2 Y}{\partial y^2}+XY\frac{\partial^2 Z}{\partial z^2}=0$$

$\times\frac{1}{XYZ}$ ⇓

$$\frac{1}{X}\frac{\partial^2 X}{\partial x^2}+\frac{1}{Y}\frac{\partial^2 Y}{\partial y^2}+\frac{1}{Z}\frac{\partial^2 Z}{\partial z^2}=0$$

$\boxed{=\kappa_x{}^2}$ $\boxed{=\kappa_y{}^2}$ $\boxed{=\kappa_z{}^2}$

$$\boxed{\kappa_x{}^2+\kappa_y{}^2+\kappa_z{}^2=0}$$

Starting from SoV for Cartesian Coordinates

$$\frac{1}{X}\frac{\partial^2 X}{\partial x^2} + \frac{1}{Y}\frac{\partial^2 Y}{\partial y^2} + \frac{1}{Z}\frac{\partial^2 Z}{\partial z^2} = 0$$

$$\boxed{\kappa_x{}^2 + \kappa_y{}^2 + \kappa_z{}^2 = 0}$$

$$\boxed{= \kappa_x{}^2} \quad \boxed{= \kappa_y{}^2} \quad \boxed{= \kappa_z{}^2}$$

(i) $\kappa_x = 0$

$$\frac{1}{X}\frac{\partial^2 X}{\partial x^2} = 0 \Rightarrow \boxed{X = a_x x + b_x}$$

Starting from SoV for Cartesian Coordinates

$$\frac{1}{X}\frac{\partial^2 X}{\partial x^2}+\frac{1}{Y}\frac{\partial^2 Y}{\partial y^2}+\frac{1}{Z}\frac{\partial^2 Z}{\partial z^2}=0$$

$$\boxed{\kappa_x{}^2+\kappa_y{}^2+\kappa_z{}^2=0}$$

$$\boxed{=\kappa_x{}^2} \quad \boxed{=\kappa_y{}^2} \quad \boxed{=\kappa_z{}^2}$$

(ii) $\kappa_x{}^2 > 0$

$$\frac{1}{X}\frac{\partial^2 X}{\partial x^2}=\kappa_x{}^2$$

$$\boxed{X=c_{x+}e^{+\kappa_x x}+c_{x-}e^{-\kappa_x x}=d_{xc}\cosh(\kappa_x x)+d_{xs}\sinh(\kappa_x x)}$$

Starting from SoV for Cartesian Coordinates

$$\frac{1}{X}\frac{\partial^2 X}{\partial x^2}+\frac{1}{Y}\frac{\partial^2 Y}{\partial y^2}+\frac{1}{Z}\frac{\partial^2 Z}{\partial z^2}=0$$

$$\boxed{=\kappa_x{}^2} \quad \boxed{=\kappa_y{}^2} \quad \boxed{=\kappa_z{}^2}$$

$$\boxed{\kappa_x{}^2+\kappa_y{}^2+\kappa_z{}^2=0}$$

(iii) $\kappa_x{}^2 < 0$ ➡ $\kappa_x{}^2 = -\alpha_x{}^2$

Real κ_x & α_x
for “easy” solving process!

$$\frac{1}{X}\frac{\partial^2 X}{\partial x^2}=-\alpha_x{}^2$$

$$\boxed{X=c_{x+}e^{+i\alpha_x x}+c_{x-}e^{-i\alpha_x x}=d_{xc}\cos(\alpha_x x)+d_{xs}\sin(\alpha_x x)}$$

Starting from SoV for Cartesian Coordinates

$$\boxed{\frac{1}{X}\frac{\partial^2 X}{\partial x^2} = \kappa_x{}^2} \Rightarrow \boxed{\begin{array}{ll} X = c_{x+}e^{+i\alpha_x x} + c_{x-}e^{-i\alpha_x x} = d_{xc}\cos(\alpha_x x) + d_{xs}\sin(\alpha_x x) & (\kappa_x{}^2 = -\alpha_x{}^2 < 0) \\ X = c_{x+}e^{+\kappa_x x} + c_{x-}e^{-\kappa_x x} = d_{xc}\cosh(\kappa_x x) + d_{xs}\sinh(\kappa_x x) & (\kappa_x{}^2 > 0) \\ X = a_x x + b_x & (\kappa_x = 0) \end{array}}$$

$$\boxed{\frac{1}{Y}\frac{\partial^2 Y}{\partial y^2} = \kappa_y{}^2} \Rightarrow \boxed{\begin{array}{ll} Y = c_{y+}e^{+i\alpha_y y} + c_{y-}e^{-i\alpha_y y} = d_{yc}\cos(\alpha_y y) + d_{ys}\sin(\alpha_y y) & (\kappa_y{}^2 = -\alpha_y{}^2 < 0) \\ Y = c_{y+}e^{+\kappa_y y} + c_{y-}e^{-\kappa_y y} = d_{yc}\cosh(\kappa_y y) + d_{ys}\sinh(\kappa_y y) & (\kappa_y{}^2 > 0) \\ Y = a_y y + b_y & (\kappa_y = 0) \end{array}}$$

$$\boxed{\frac{1}{Z}\frac{\partial^2 Z}{\partial z^2} = \kappa_z{}^2} \Rightarrow \boxed{\begin{array}{ll} Z = c_{z+}e^{+i\alpha_z z} + c_{z-}e^{-i\alpha_z z} = d_{zc}\cos(\alpha_z z) + d_{zs}\sin(\alpha_z z) & (\kappa_z{}^2 = -\alpha_z{}^2 < 0) \\ Z = c_{z+}e^{+\kappa_z z} + c_{z-}e^{-\kappa_z z} = d_{zc}\cosh(\kappa_z z) + d_{zs}\sinh(\kappa_z z) & (\kappa_z{}^2 > 0) \\ Z = a_z z + b_z & (\kappa_z = 0) \end{array}}$$

$$\boxed{\kappa_x{}^2 + \kappa_y{}^2 + \kappa_z{}^2 = 0}$$

Solution Summary

$$\left[\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}\right]V=0$$

B.C. determines
"***coefficients***"

$$\boxed{V=\sum_{\kappa_x,\kappa_y,\kappa_z} X_{\kappa_x}(x)Y_{\kappa_y}(y)Z_{\kappa_z}(z)}\quad\boxed{\kappa_x{}^2+\kappa_y{}^2+\kappa_z{}^2=0}$$

$$X=c_{x+}e^{+i\alpha_x x}+c_{x-}e^{-i\alpha_x x}=d_{xc}\cos(\alpha_x x)+d_{xs}\sin(\alpha_x x)\quad(\kappa_x{}^2=-\alpha_x{}^2<0)$$

$$X=c_{x+}e^{+\kappa_x x}+c_{x-}e^{-\kappa_x x}=d_{xc}\cosh(\kappa_x x)+d_{xs}\sinh(\kappa_x x)\quad(\kappa_x{}^2>0)$$

$$X=a_x x+b_x\quad(\kappa_x=0)$$

$$Y=c_{y+}e^{+i\alpha_y y}+c_{y-}e^{-i\alpha_y y}=d_{yc}\cos(\alpha_y y)+d_{ys}\sin(\alpha_y y)\quad(\kappa_y{}^2=-\alpha_y{}^2<0)$$

$$Y=c_{y+}e^{+\kappa_y y}+c_{y-}e^{-\kappa_y y}=d_{yc}\cosh(\kappa_y y)+d_{ys}\sinh(\kappa_y y)\quad(\kappa_y{}^2>0)$$

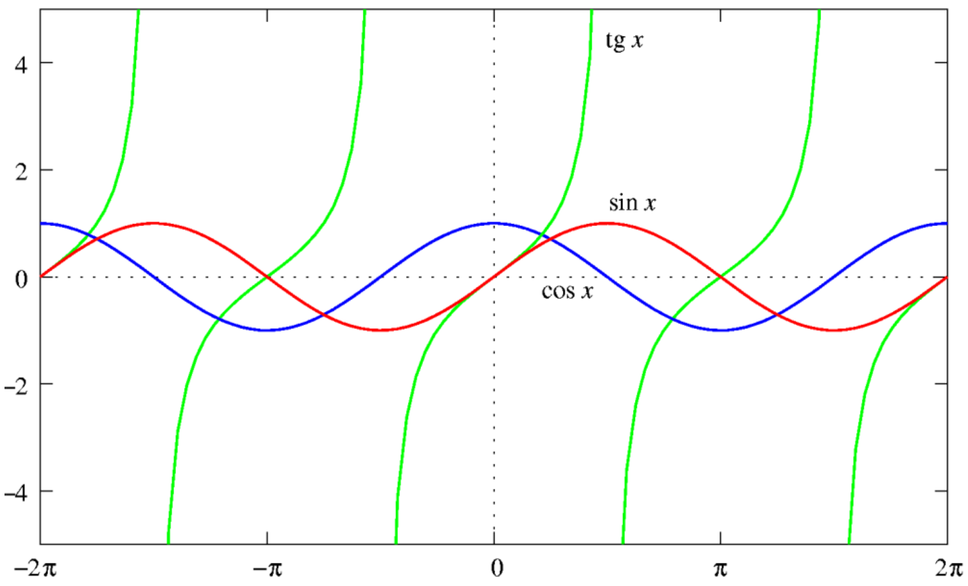
$$Y=a_y y+b_y\quad(\kappa_y=0)$$

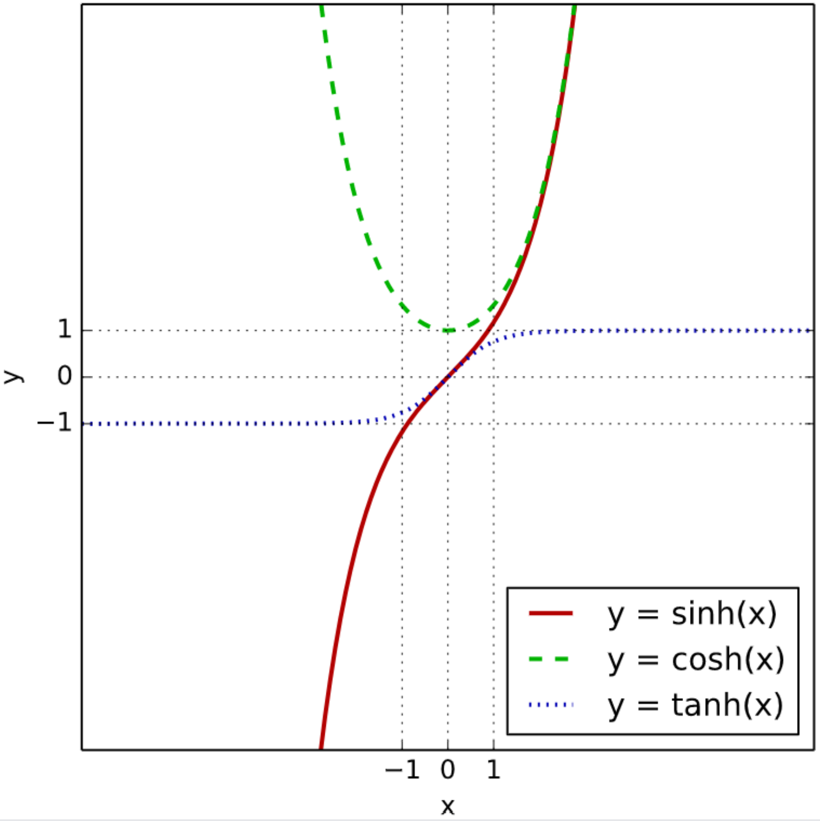
$$Z=c_{z+}e^{+i\alpha_z z}+c_{z-}e^{-i\alpha_z z}=d_{zc}\cos(\alpha_z z)+d_{zs}\sin(\alpha_z z)\quad(\kappa_z{}^2=-\alpha_z{}^2<0)$$

$$Z=c_{z+}e^{+\kappa_z z}+c_{z-}e^{-\kappa_z z}=d_{zc}\cosh(\kappa_z z)+d_{zs}\sinh(\kappa_z z)\quad(\kappa_z{}^2>0)$$

$$Z=a_z z+b_z\quad(\kappa_z=0)$$

Important Tip: Be Familiar with the Function Profiles



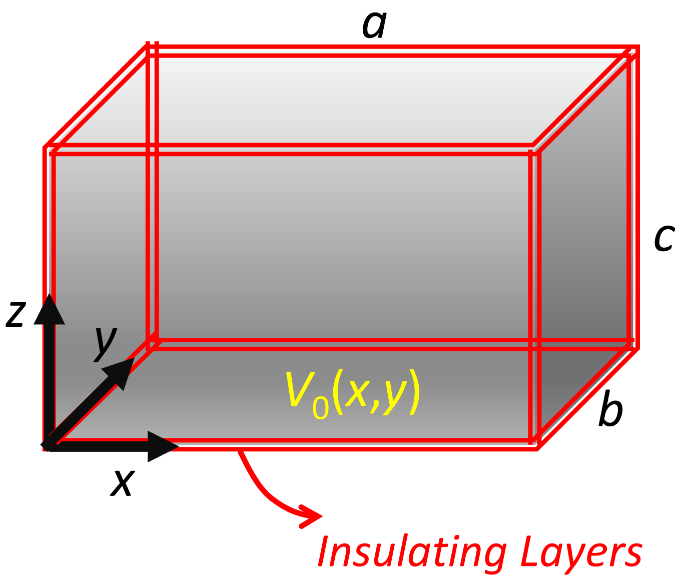


You must be familiar with the function profiles

➔ Selecting proper functions to satisfy B.C.!

Example 016

- Calculate *V*(*x*,*y*,*z*)
  when all the walls are fixed at zero potential except for the *z* = 0 wall, where the potential takes specific values $V_0(x,y)$



For 4 vertical walls, *V* = 0

$$X = d_{xc}\cos(\alpha_x x) + d_{xs}\sin(\alpha_x x) \quad (\kappa_x{}^2 = -\alpha_x{}^2 < 0)$$

$$X = d_{xc}\cosh(\kappa_x x) + d_{xs}\sinh(\kappa_x x) \quad (\kappa_x{}^2 > 0)$$

cannot meet B.C.

$$X = a_x x + b_x \quad (\kappa_x = 0)$$

zero in everywhere

$$Y = d_{yc}\cos(\alpha_y y) + d_{ys}\sin(\alpha_y y) \quad (\kappa_y{}^2 = -\alpha_y{}^2 < 0)$$

$$Y = d_{yc}\cosh(\kappa_y y) + d_{ys}\sinh(\kappa_y y) \quad (\kappa_y{}^2 > 0)$$

$$Y = a_y y + b_y \quad (\kappa_y = 0)$$

$$\kappa_{x,y} \neq 0$$

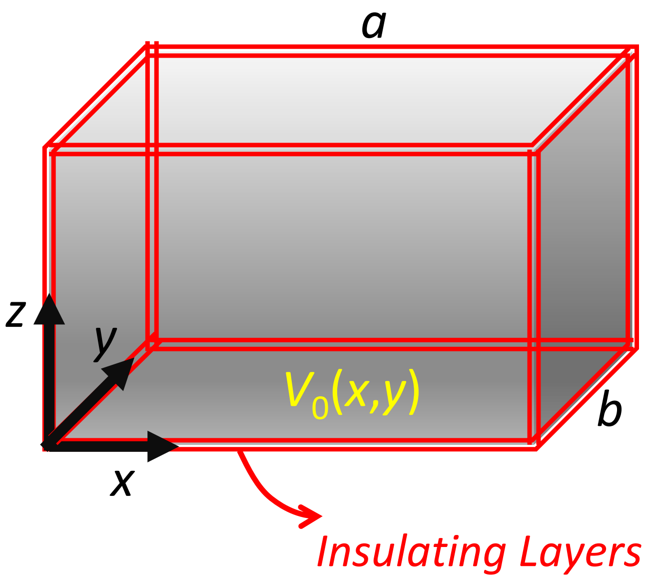
Consider the function profiles, we can set

$$X \sim \sin(\alpha_x x) \quad (\kappa_x{}^2 = -\alpha_x{}^2 < 0)$$

$$Y \sim \sin(\alpha_y y) \quad (\kappa_y{}^2 = -\alpha_y{}^2 < 0)$$

Example 016

- Calculate *V*(*x*,*y*,*z*) when all the walls are fixed at zero potential except for the *z* = 0 wall, where the potential takes specific values $V_0(x,y)$



$$X \sim \sin(\alpha_x x) \quad (\kappa_x{}^2 = -\alpha_x{}^2 < 0)$$

$$Y \sim \sin(\alpha_y y) \quad (\kappa_y{}^2 = -\alpha_y{}^2 < 0)$$

$$\alpha_x a = l\pi, \quad \alpha_y b = m\pi \quad (l, m = 1, 2, 3, \ldots)$$

$$\alpha_x{}^2 + \alpha_y{}^2 = \kappa_z{}^2$$

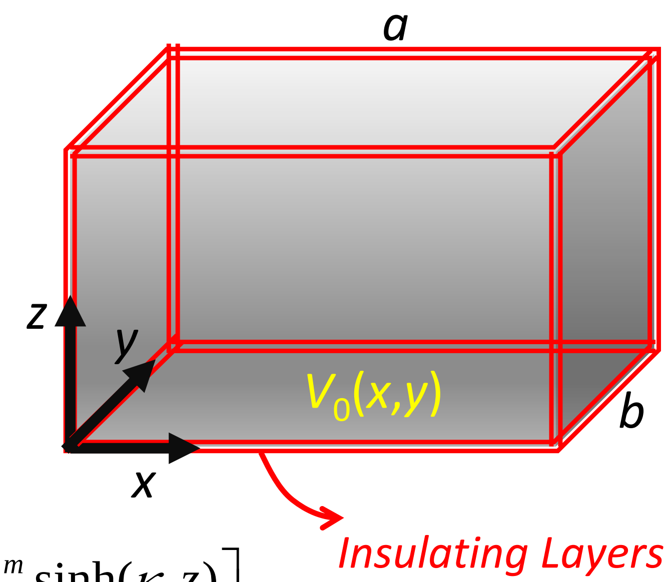
$$\left(\frac{l\pi}{a}\right)^2 + \left(\frac{m\pi}{b}\right)^2 = \kappa_z{}^2 > 0$$

$$\boxed{\begin{aligned} &Z = c_{z+}e^{+i\alpha_z z} + c_{z-}e^{-i\alpha_z z} = d_{zc}\cos(\alpha_z z) + d_{zs}\sin(\alpha_z z) \quad (\kappa_z{}^2 = -\alpha_z{}^2 < 0) \\ &Z = c_{z+}e^{+\kappa_z z} + c_{z-}e^{-\kappa_z z} = d_{zc}\cosh(\kappa_z z) + d_{zs}\sinh(\kappa_z z) \quad (\kappa_z{}^2 > 0) \\ &Z = a_z z + b_z \quad (\kappa_z = 0) \end{aligned}}$$

We set $\kappa_z > 0$

Example 016

- Calculate $V(x,y,z)$ when all the walls are fixed at zero potential except for the $z = 0$ wall, where the potential takes specific values $V_0(x,y)$



$$V = \sum_{l=1}^{\infty}\sum_{m=1}^{\infty}\sin(\frac{l\pi}{a}x)\sin(\frac{m\pi}{b}y)\left[d_{zc}^{l,m}\cosh(\kappa_z z) + d_{zs}^{l,m}\sinh(\kappa_z z)\right]$$

$$\left(\frac{l\pi}{a}\right)^2 + \left(\frac{m\pi}{b}\right)^2 = \kappa_z{}^2 > 0$$

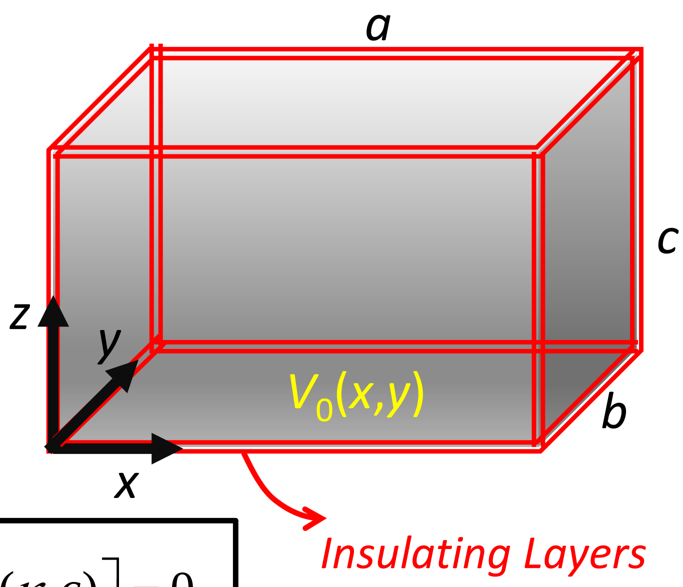
$$\Downarrow$$

$$V(x,y,z=0) = \sum_{l=1}^{\infty}\sum_{m=1}^{\infty}\sin(\frac{l\pi}{a}x)\sin(\frac{m\pi}{b}y)d_{zc}^{l,m} = V_0(x,y)$$

$$V(x,y,z=c) = \sum_{l=1}^{\infty}\sum_{m=1}^{\infty}\sin(\frac{l\pi}{a}x)\sin(\frac{m\pi}{b}y)\left[d_{zc}^{l,m}\cosh(\kappa_z c) + d_{zs}^{l,m}\sinh(\kappa_z c)\right] = 0$$

Example 016

- Calculate $V(x,y,z)$ when all the walls are fixed at zero potential except for the $z = 0$ wall, where the potential takes specific values $V_0(x,y)$



$$\boxed{\sum_{l=1}^{\infty}\sum_{m=1}^{\infty}\sin(\frac{l\pi}{a}x)\sin(\frac{m\pi}{b}y)\left[d_{zc}^{l,m}\cosh(\kappa_z c)+d_{zs}^{l,m}\sinh(\kappa_z c)\right]=0}$$

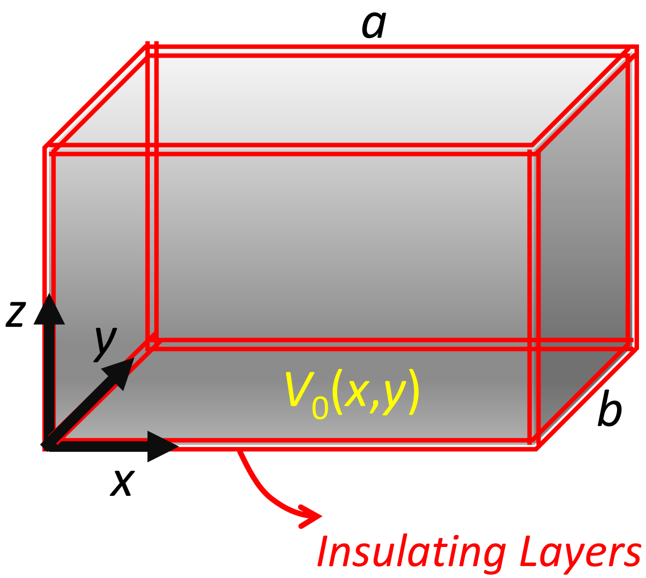
$$d_{zc}^{l,m}\cosh(\kappa_z c)+d_{zs}^{l,m}\sinh(\kappa_z c)=0 \Rightarrow d_{zs}^{l,m}=-\frac{\cosh(\kappa_z c)}{\sinh(\kappa_z c)}d_{zc}^{l,m}$$

$$\boxed{\sum_{l=1}^{\infty}\sum_{m=1}^{\infty}\sin(\frac{l\pi}{a}x)\sin(\frac{m\pi}{b}y)d_{zc}^{l,m}=V_0(x,y)}$$

$$\Rightarrow d_{zc}^{l,m}=\frac{4}{ab}\int_0^a\int_0^b dxdyV_0(x,y)\sin\left(\frac{l\pi}{a}x\right)\sin\left(\frac{m\pi}{b}y\right)=V_{lm}$$

Example 016

- Calculate *V*(*x*,*y*,*z*)
when all the walls are fixed at zero potential except for the *z* = 0 wall, where the potential takes specific values $V_0(x,y)$



$$d_{zc}^{l,m}\cosh(\kappa_z z)+d_{zs}^{l,m}\sinh(\kappa_z z)$$

$$=V_{lm}\cosh(\kappa_z z)-\frac{\cosh(\kappa_z c)}{\sinh(\kappa_z c)}V_{lm}\sinh(\kappa_z z)$$

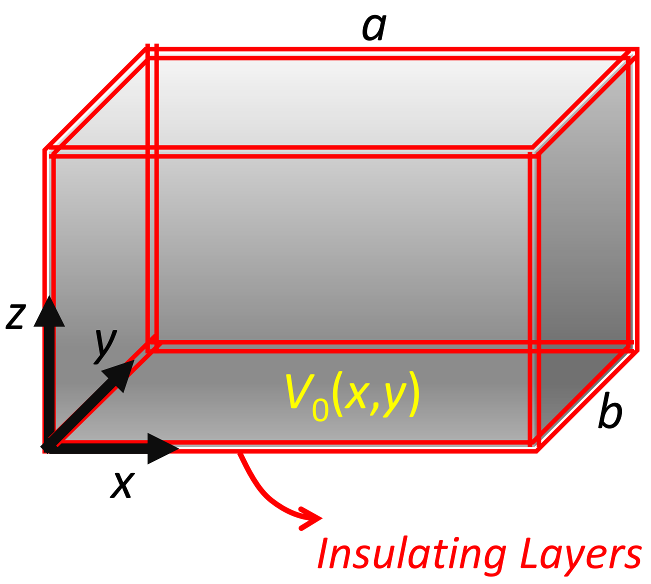
$$=V_{lm}\frac{\sinh(\kappa_z c)\cosh(\kappa_z z)-\cosh(\kappa_z c)\sinh(\kappa_z z)}{\sinh(\kappa_z c)}$$

$$=V_{lm}\frac{\sinh\left[\kappa_z(c-z)\right]}{\sinh(\kappa_z c)}$$

$$\left(\frac{l\pi}{a}\right)^2+\left(\frac{m\pi}{b}\right)^2=\kappa_z{}^2=\kappa_{lm}{}^2>0$$

Example 016

- Calculate *V*(*x*,*y*,*z*)
  when all the walls are fixed at zero potential except for the *z* = 0 wall, where the potential takes specific values $V_0(x,y)$



$$\boxed{V(x,y,z)=\sum_{l=1}^{\infty}\sum_{m=1}^{\infty}V_{lm}\sin(\frac{l\pi}{a}x)\sin(\frac{m\pi}{b}y)\frac{\sinh\left[\kappa_{lm}(c-z)\right]}{\sinh(\kappa_{lm}c)}}$$

$$V_{lm}=\frac{4}{ab}\int_0^a\int_0^b dxdyV_0(x,y)\sin\left(\frac{l\pi}{a}x\right)\sin\left(\frac{m\pi}{b}y\right),\quad \kappa_{lm}=\sqrt{\left(\frac{l\pi}{a}\right)^2+\left(\frac{m\pi}{b}\right)^2}$$

B.C. determines the coefficient of each eigenmode

Superposition for Extension

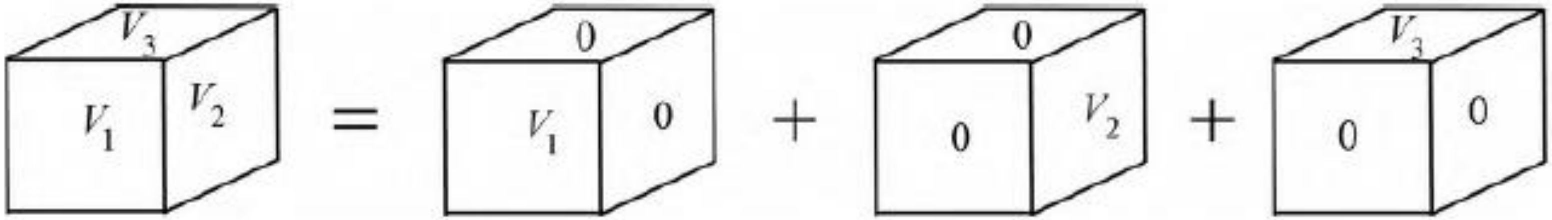


Figure 7.5: The potential in a box with three $\varphi = 0$ walls and three $\varphi \neq 0$ walls represented as the sum of three box potentials, each with five $\varphi = 0$ walls and one $\varphi \neq 0$ wall.

Preview – Cylindrical Coordinates

Starting from SoV for Cylindrical Coordinates

$$\left[\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial}{\partial\rho}\right)+\frac{1}{\rho^2}\frac{\partial^2}{\partial\varphi^2}+\frac{\partial^2}{\partial z^2}\right]V=0$$

Separation of Variables: $V=R(\rho)Q(\varphi)Z(z)$

$$\frac{QZ}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial R}{\partial\rho}\right)+\frac{RZ}{\rho^2}\frac{\partial^2 Q}{\partial\varphi^2}+RQ\frac{\partial^2 Z}{\partial z^2}=0$$

$$\times\frac{\rho^2}{RQZ}$$

$$\frac{\rho}{R}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial R}{\partial\rho}\right)+\boxed{\frac{1}{Q}\frac{\partial^2 Q}{\partial\varphi^2}}+\boxed{\frac{\rho^2}{Z}\frac{\partial^2 Z}{\partial z^2}}=0$$

$$\boxed{=-m^2} \qquad \boxed{=k_z{}^2}$$

$$\rho^2\frac{d^2R}{d\rho^2}+\rho\frac{dR}{d\rho}+\left\{k_z{}^2\rho^2-m^2\right\}R=0$$

Laplace's Equation & Poisson's Equation

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Laplace Equations – Cylindrical Coordinates

Starting from SoV for Cylindrical Coordinates

$$\left[\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial}{\partial\rho}\right)+\frac{1}{\rho^2}\frac{\partial^2}{\partial\varphi^2}+\frac{\partial^2}{\partial z^2}\right]V=0$$

Separation of Variables: $V=R(\rho)Q(\varphi)Z(z)$

$$\frac{QZ}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial R}{\partial\rho}\right)+\frac{RZ}{\rho^2}\frac{\partial^2 Q}{\partial\varphi^2}+RQ\frac{\partial^2 Z}{\partial z^2}=0$$

$$\times\frac{\rho^2}{RQZ}$$

$$\frac{\rho}{R}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial R}{\partial\rho}\right)+\boxed{\frac{1}{Q}\frac{\partial^2 Q}{\partial\varphi^2}}+\frac{\rho^2}{\boxed{Z}}\boxed{\frac{\partial^2 Z}{\partial z^2}}=0$$

$$\boxed{=-m^2}\qquad\boxed{=k_z{}^2}$$

$$\rho^2\frac{d^2R}{d\rho^2}+\rho\frac{dR}{d\rho}+\left\{k_z{}^2\rho^2-m^2\right\}R=0$$

SoV for Cylindrical Coordinates: Height *z*

$$\frac{1}{Z}\frac{\partial^2 Z}{\partial z^2} = k_z{}^2$$

$$\boxed{\begin{aligned}
&Z = c_{z+}e^{+i\alpha_z z} + c_{z-}e^{-i\alpha_z z} = d_{zc}\cos(\alpha_z z) + d_{zs}\sin(\alpha_z z) && (k_z{}^2 = -\alpha_z{}^2 < 0)\\
&Z = c_{z+}e^{+k_z z} + c_{z-}e^{-k_z z} = d_{zc}\cosh(k_z z) + d_{zs}\sinh(k_z z) && (k_z{}^2 > 0)\\
&Z = a_z z + b_z && (k_z = 0)
\end{aligned}}$$

The same condition as Cartesian *z*

SoV for Cylindrical Coordinates: Azimuth φ

$$\frac{1}{Q}\frac{\partial^2 Q}{\partial \varphi^2} = -m^2$$

$$Q(\varphi) = Q(\varphi + 2\pi)$$

(i) $m = 0$

$$\frac{1}{Q}\frac{\partial^2 Q}{\partial \varphi^2} = 0 \Rightarrow Q = a\varphi + b$$

$$a\varphi + b = a\left(\varphi + 2\pi\right) + b$$

$$\boxed{Q = b}$$

SoV for Cylindrical Coordinates: Azimuth φ

$$\frac{1}{Q}\frac{\partial^2 Q}{\partial \varphi^2} = -m^2$$

$$Q(\varphi) = Q(\varphi + 2\pi)$$

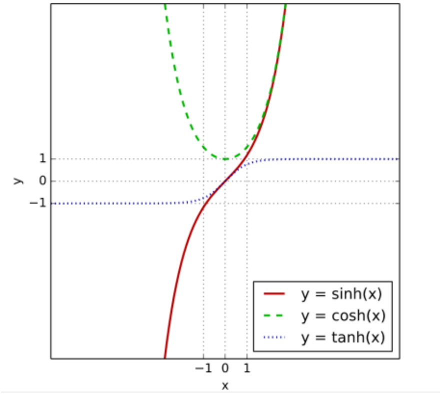
(ii) $m^2 > 0$

$$\frac{1}{Q}\frac{\partial^2 Q}{\partial \varphi^2} = -m^2$$

$$\boxed{Q = c_{\varphi+} e^{+im\varphi} + c_{\varphi-} e^{-im\varphi} = d_{\varphi c}\cos(m\varphi) + d_{\varphi s}\sin(m\varphi)}$$

B.C. Integer *m* ➔ We assume *m* = 1, 2, ...

SoV for Cylindrical Coordinates: Azimuth φ



$$\frac{1}{Q}\frac{\partial^2 Q}{\partial \varphi^2} = -m^2$$

$$Q(\varphi) = Q(\varphi + 2\pi)$$

(iii) $m^2 < 0$ ➡ $m^2 = -\alpha^2$

Real m & α
for "easy" solving process!

$$\frac{1}{Q}\frac{\partial^2 Q}{\partial \varphi^2} = \alpha^2$$

$$\boxed{Q = c_{\varphi+} e^{+\alpha\varphi} + c_{\varphi-} e^{-\alpha\varphi} = d_{\varphi c} \cosh(\alpha\varphi) + d_{\varphi s} \sinh(\alpha\varphi)}$$

$Q(\varphi) = Q(\varphi + 2\pi)$ *is not generally satisfied for the entire* φ

SoV for Cylindrical Coordinates: Azimuth φ

$$\frac{1}{Q}\frac{\partial^2 Q}{\partial \varphi^2} = -m^2$$

$$Q(\varphi) = Q(\varphi + 2\pi)$$

(i) *m* = 0 $Q = b$

(ii) *m*^{2} > 0 $Q = c_{\varphi+}e^{+im\varphi} + c_{\varphi-}e^{-im\varphi} = d_{\varphi c}\cos(m\varphi) + d_{\varphi s}\sin(m\varphi)$

(*m* = 1, 2, ...)

$$\boxed{Q = c_{\varphi+}e^{+im\varphi} + c_{\varphi-}e^{-im\varphi} = d_{\varphi c}\cos(m\varphi) + d_{\varphi s}\sin(m\varphi) \qquad (m = 0,\ 1,\ 2,\ \ldots)}$$

SoV for Cylindrical Coordinates: Radial Distance ρ

$$\rho^2 \frac{d^2 R}{d\rho^2} + \rho \frac{dR}{d\rho} + \left(k_z{}^2 \rho^2 - m^2\right) R = 0$$

$$\Big\Downarrow \quad x = k_z \rho$$

Bessel's Equation

$$\boxed{x^2 \frac{d^2 R}{dx^2} + x \frac{dR}{dx} + \left(x^2 - m^2\right) R = 0}$$

$$Q = c_{\varphi+} e^{+im\varphi} + c_{\varphi-} e^{-im\varphi} = d_{\varphi c} \cos(m\varphi) + d_{\varphi s} \sin(m\varphi) \qquad (m = 0,\ 1,\ 2,\ \ldots)$$

$$Z = c_{z+} e^{+i\alpha_z z} + c_{z-} e^{-i\alpha_z z} = d_{zc} \cos(\alpha_z z) + d_{zs} \sin(\alpha_z z) \quad (k_z{}^2 = -\alpha_z{}^2 < 0)$$

$$Z = c_{z+} e^{+k_z z} + c_{z-} e^{-k_z z} = d_{zc} \cosh(k_z z) + d_{zs} \sinh(k_z z) \quad (k_z{}^2 > 0)$$

$$Z = a_z z + b_z \qquad (k_z = 0)$$

Interim Summary

$$\left[\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial}{\partial\rho}\right)+\frac{1}{\rho^2}\frac{\partial^2}{\partial\varphi^2}+\frac{\partial^2}{\partial z^2}\right]V=0$$

$$\boxed{V=\int_{-\infty}^{\infty}dk_z\sum_{m=0}^{\infty}R_{m,k_z}(\rho)Q_m(\varphi)Z_{k_z}(z)}$$

$$Q_m=d_{\varphi c}\cos(m\varphi)+d_{\varphi s}\sin(m\varphi)$$

$$(m=0,1,2,\ldots)$$

Bessel's Equation

$$\boxed{x^2\frac{d^2R_{m,k_z}}{dx^2}+x\frac{dR_{m,k_z}}{dx}+\left(x^2-m^2\right)R_{m,k_z}=0}$$

$$Z_{k_z}=d_{zc}\cos(\alpha_z z)+d_{zs}\sin(\alpha_z z)$$

$$(k_z{}^2=-\alpha_z{}^2<0)$$

$$Z_{k_z}=d_{zc}\cosh(k_z z)+d_{zs}\sinh(k_z z)$$

$$(k_z{}^2>0)$$

$$Z_{k_z}=a_z z+b_z$$

$$(k_z=0)$$

$$x=k_z\rho$$

The Radial Functions: k_z = 0

$$\rho^2 \frac{d^2R}{d\rho^2} + \rho \frac{dR}{d\rho} + \left(k_z{}^2 \rho^2 - m^2\right) R = 0$$

$$\Big\downarrow \; k_z = 0$$

$$\rho^2 \frac{d^2R}{d\rho^2} + \rho \frac{dR}{d\rho} - m^2 R = 0$$

Diverging at $\rho \to 0$

(i) $m = 0$

$$\rho \frac{d^2R}{d\rho^2} = -\frac{dR}{d\rho} \Rightarrow \boxed{R_{m,k_z}(\rho) = A_{m=0,k_z=0} + B_{m=0,k_z=0} \ln(\rho)}$$

Diverging at $\rho \to \infty$ Diverging at $\rho \to 0$

(ii) $m \neq 0$

$$\boxed{R_{m,k_z}(\rho) = A_{m,k_z=0} \rho^{+m} + B_{m,k_z=0} \rho^{-m}}$$

The Radial Functions: $k_z \neq 0$ – Bessel Functions

Bessel's Equation

$$\boxed{x^2 \frac{d^2R}{dx^2} + x\frac{dR}{dx} + \left(x^2 - m^2\right)R = 0}$$

$$R = \sum_{k=0}^{\infty} a_k x^{k+s}$$

$$(s^2 - m^2)a_0 = 0, \quad \left[(s+1)^2 - m^2\right]a_1 = 0$$

$$a_k = -\frac{1}{(k+s)^2 - m^2}a_{k-2} \qquad (k \geq 2)$$

$$\boxed{a_0 \neq 0, \; a_1 = 0, \; s = \pm m}$$

$k = 2q$

$$a_{2q} = -\frac{a_{2q-2}}{2^2 q(q \pm m)} \quad (q \geq 1, \text{ integer})$$

$$\Gamma(m+1) = m\Gamma(m) \qquad a_0 = \frac{1}{2^m \Gamma(m+1)}$$

The Bessel Function of the first kind of order "*m*"

$s = +m$

$$R = J_m(x) = \sum_{q=0}^{\infty} \frac{(-1)^q}{\Gamma(q+1)\Gamma(q+1+m)}\left(\frac{x}{2}\right)^{2q+m}$$

$s = -m$

$$R = J_{-m}(x) = \sum_{q=0}^{\infty} \frac{(-1)^q}{\Gamma(q+1)\Gamma(q+1-m)}\left(\frac{x}{2}\right)^{2q-m}$$

When *m* is not an integer,
J_m and J_{-m} are linearly independent

But, in our prob. *m* is integer:

Linearly dependent! $J_{-m}(x) = \left(-1\right)^m J_m(x)$

The Bessel Function of the second kind of order "*m*"

$$R = N_m(x) = \frac{\cos(\pi m)J_m(x) - J_{-m}(x)}{\sin(\pi m)}$$

$$\boxed{R_{m,k_z}(x = k_z\rho) = A_{m,k_z} J_m(x) + B_{m,k_z} N_m(x)}$$

Bessel Functions: Modified Bessel Functions

Bessel Functions of the First & Second Kinds

$$\boxed{R_{m,k_z}(x=k_z\rho)=A_{m,k_z}J_m(x)+B_{m,k_z}N_m(x)}$$

~ cos & sin

Two Degrees of Freedom

$x=k_z\rho$ ⇨ **The 1st & 2nd Bessel functions are proper to real k_z (or $k_z^2 > 0$)**

Then, what are the proper forms for imaginary k_z (or $k_z^2 < 0$)

Modified Bessel Functions

$$\boxed{\begin{aligned}&I_m(x)=i^{-m}J_m(ix)\\&K_m(x)=\frac{\pi}{2}i^{m+1}\left[J_m(ix)+iN_m(ix)\right]\end{aligned}}$$

~ cosh & sinh

Bessel Functions: Other Kinds

Bessel Functions of the First & Second Kinds

$$\boxed{R_{m,k_z}(x=k_z\rho)=A_{m,k_z}J_m(x)+B_{m,k_z}N_m(x)}$$

~ cos & sin

Two Degrees of Freedom

Hankel Functions (or Bessel Functions of the Third Kind)

$$\boxed{\begin{aligned}H_m^{(1)}(x)&=J_m(x)+iN_m(x)\\H_m^{(2)}(x)&=J_m(x)-iN_m(x)\end{aligned}}$$

~ exp(±*ix*)

Assume $m=r+\frac{1}{2}\quad(r=0,\ 1,\ 2,\ \ldots)$

Spherical Bessel Functions (Spherical Coordinates!)

$$\boxed{\begin{aligned}j_r(x)&=\sqrt{\frac{\pi}{2x}}J_{r+1/2}(x)=(-1)^r\,x^r\left(\frac{1}{x}\frac{d}{dx}\right)^r\left(\frac{\sin x}{x}\right)\\n_r(x)&=\sqrt{\frac{\pi}{2x}}N_{r+1/2}(x)=-(-1)^r\,x^r\left(\frac{1}{x}\frac{d}{dx}\right)^r\left(\frac{\cos x}{x}\right)\end{aligned}}$$

$$\boxed{\begin{aligned}h_r^{(1)}(x)&=\sqrt{\frac{\pi}{2x}}H_{r+1/2}^{(1)}(x)=-i(-1)^r\,x^r\left(\frac{1}{x}\frac{d}{dx}\right)^r\left(\frac{e^{ix}}{x}\right)\\h_r^{(2)}(x)&=\sqrt{\frac{\pi}{2x}}H_{r+1/2}^{(2)}(x)=i(-1)^r\,x^r\left(\frac{1}{x}\frac{d}{dx}\right)^r\left(\frac{e^{-ix}}{x}\right)\end{aligned}}$$

The Radial Functions: Summary

- $k_z = 0$ (i) $m = 0$ ⇒

Diverging at $\rho \to 0$

$$R_{m,k_z}(\rho) = A_{m=0,k_z=0} + B_{m=0,k_z=0}\ln(\rho)$$

(ii) $m \neq 0$ ⇒

Diverging at $\rho \to \infty$ Diverging at $\rho \to 0$

$$R_{m,k_z}(\rho) = A_{m,k_z=0}\rho^{+m} + B_{m,k_z=0}\rho^{-m}$$

- $k_z^2 > 0$

$$R_{m,k_z}(\rho) = A_{m,k_z} J_m(k_z\rho) + B_{m,k_z} N_m(k_z\rho)$$

- $k_z^2 = -\alpha_z^2 < 0$
  $\alpha_z = ik_z$

$$R_{m,k_z}(\rho) = A_{m,k_z} I_m(\alpha_z\rho) + B_{m,k_z} K_m(\alpha_z\rho)$$

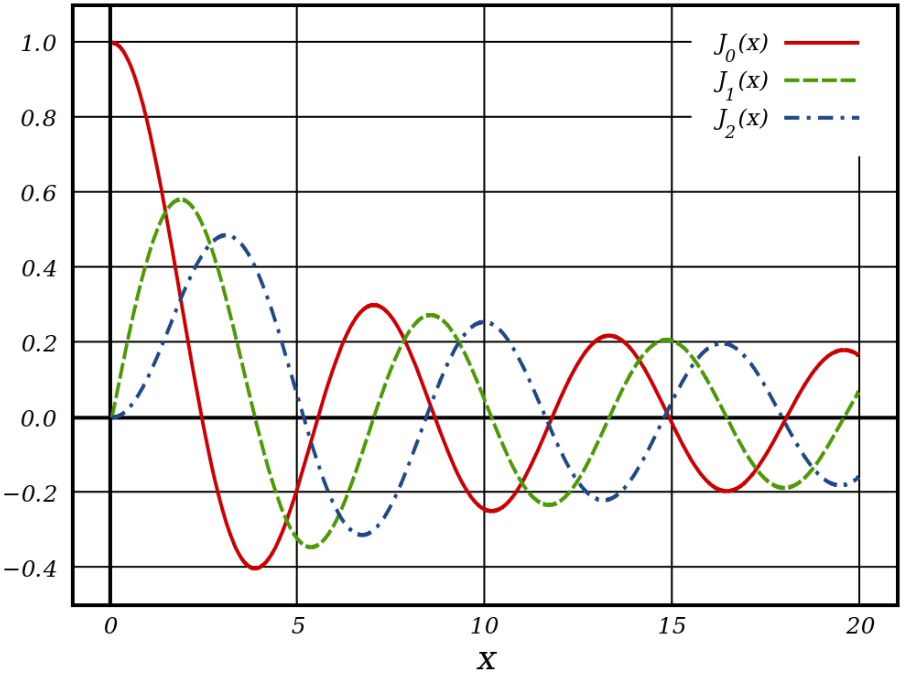
The 1st & 2nd Bessel Functions

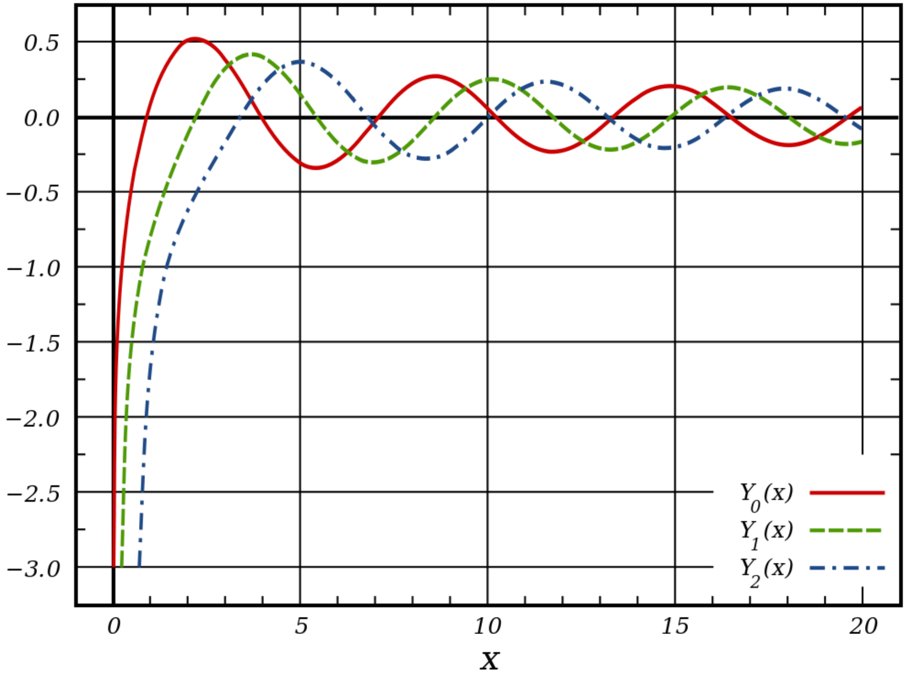
$\boxed{J_m(x)}$

Diverging at $\rho \to 0$

←

$\boxed{N_m(x)}$





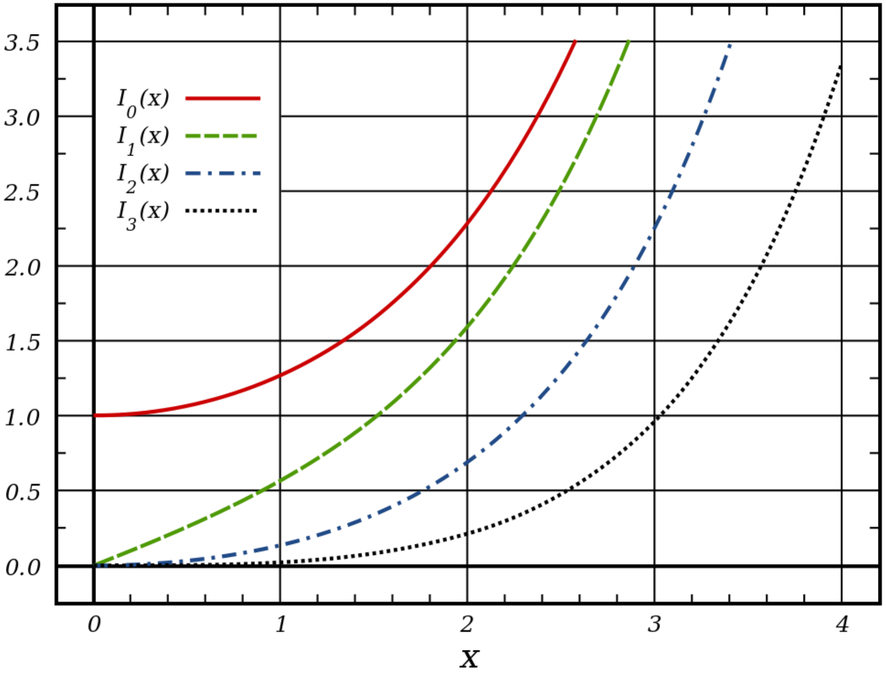
Modified Bessel Functions

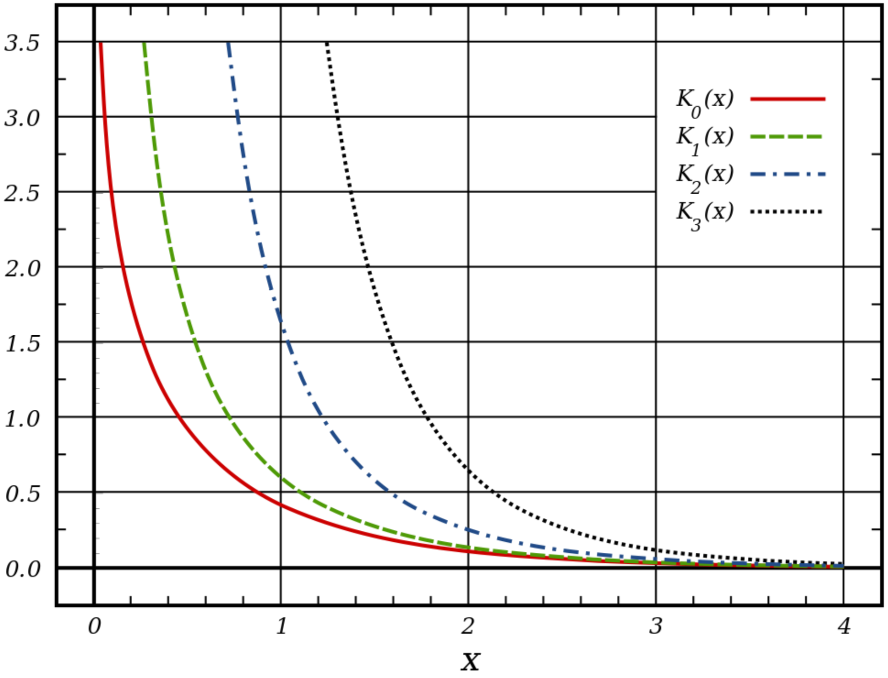
$\boxed{I_m(x)}$

Diverging at $\rho \to \infty$ →

← Diverging at $\rho \to 0$

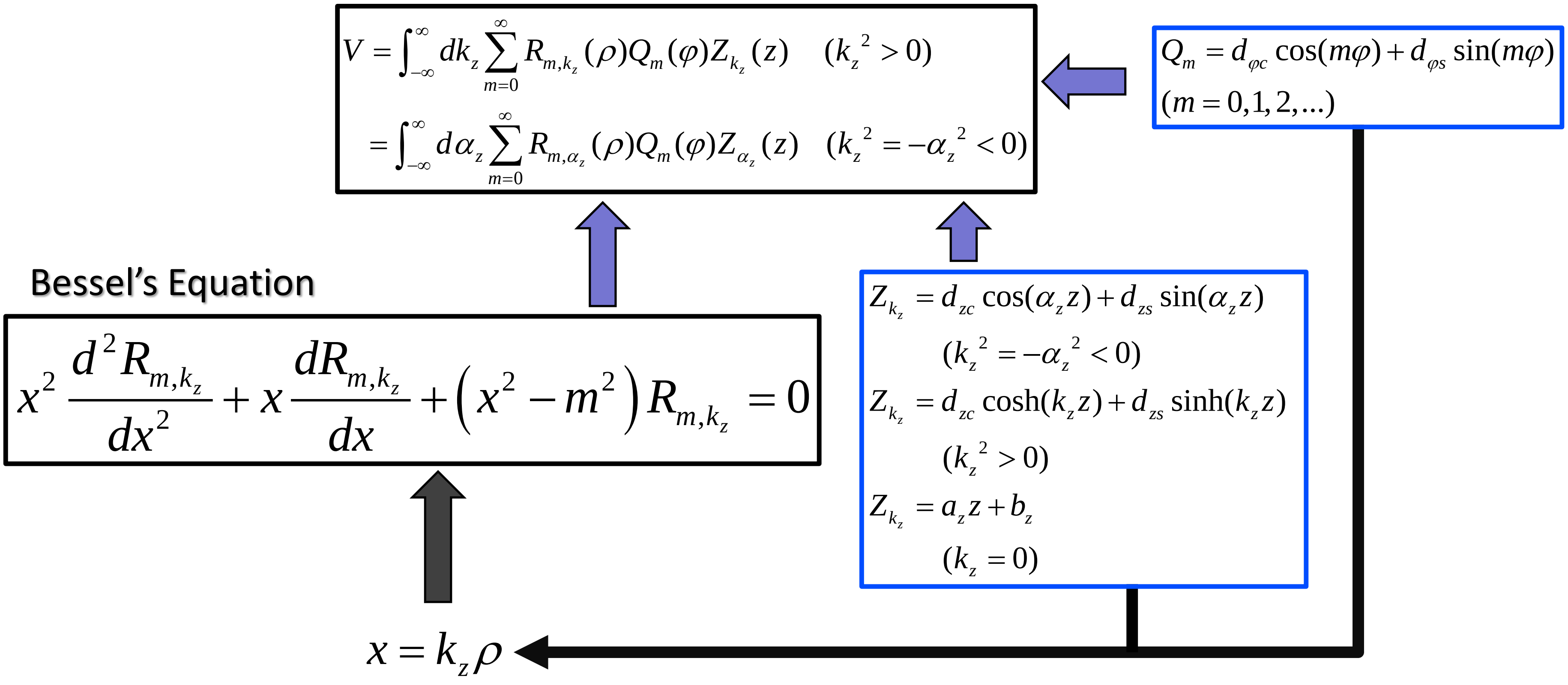
$\boxed{K_m(x)}$





Solution Summary

$$\left[\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial}{\partial\rho}\right)+\frac{1}{\rho^2}\frac{\partial^2}{\partial\varphi^2}+\frac{\partial^2}{\partial z^2}\right]V=0$$



$$V=\int_{-\infty}^{\infty}dk_z\sum_{m=0}^{\infty}R_{m,k_z}(\rho)Q_m(\varphi)Z_{k_z}(z)\quad(k_z{}^2>0)$$

$$=\int_{-\infty}^{\infty}d\alpha_z\sum_{m=0}^{\infty}R_{m,\alpha_z}(\rho)Q_m(\varphi)Z_{\alpha_z}(z)\quad(k_z{}^2=-\alpha_z{}^2<0)$$

$$Q_m=d_{\varphi c}\cos(m\varphi)+d_{\varphi s}\sin(m\varphi)$$

$$(m=0,1,2,\dots)$$

Bessel's Equation

$$x^2\frac{d^2R_{m,k_z}}{dx^2}+x\frac{dR_{m,k_z}}{dx}+\left(x^2-m^2\right)R_{m,k_z}=0$$

$$Z_{k_z}=d_{zc}\cos(\alpha_z z)+d_{zs}\sin(\alpha_z z)$$

$$(k_z{}^2=-\alpha_z{}^2<0)$$

$$Z_{k_z}=d_{zc}\cosh(k_z z)+d_{zs}\sinh(k_z z)$$

$$(k_z{}^2>0)$$

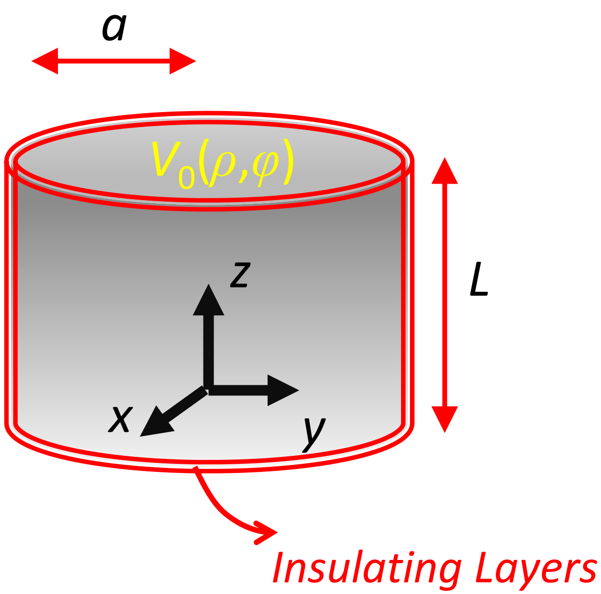
$$Z_{k_z}=a_z z+b_z$$

$$(k_z=0)$$

$$x=k_z\rho$$

Example 017

- Calculate $V(\rho,\varphi,z)$ when all the walls are fixed at zero potential except for the $z = L$ wall, where the potential takes specific values $V_0(\rho,\varphi)$



$$\boxed{\begin{aligned} &Q_m = d_{\varphi c}\cos(m\varphi) + d_{\varphi s}\sin(m\varphi) \\ &(m = 0,1,2,\ldots) \end{aligned}}$$

(i) $k_z = 0$

$$Z_{k_z} = a_z z + b_z$$

$$Z_{k_z}(z=0) = b_z = 0$$

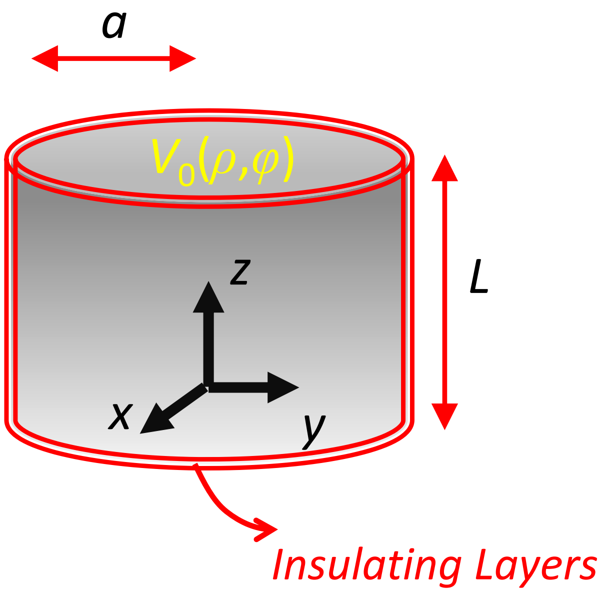
(1) $m = 0$: $Q_m = d_{\varphi c}$ cannot match $z = L$ B.C.

(2) $m \neq 0$: $R_{m,k_z}(\rho) = A_{m,k_z=0}\rho^{+m} + B_{m,k_z=0}\rho^{-m}$

cannot match B.C. at $z = L$

Example 017

- Calculate $V(\rho,\varphi,z)$ when all the walls are fixed at zero potential except for the $z = L$ wall, where the potential takes specific values $V_0(\rho,\varphi)$



$$\boxed{\begin{aligned} &Q_m = d_{\varphi c}\cos(m\varphi) + d_{\varphi s}\sin(m\varphi) \\ &(m = 0, 1, 2, \ldots) \end{aligned}}$$

(ii) $k_z \neq 0$ We set $k_z^2 > 0$; otherwise, the coefficient becomes imaginary

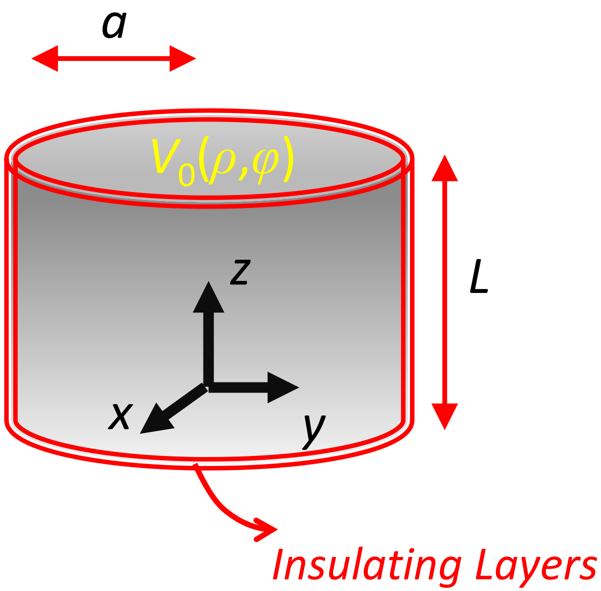
$$Z_{k_z} = d_{zc}\cosh(k_z z) + d_{zs}\sinh(k_z z) \qquad (k_z^{\ 2} > 0)$$

$$Z_{k_z}(z=0) = d_{zc} = 0$$

$$\boxed{Z_{k_z}(z) = d_{zs}\sinh(k_z z) \qquad (k_z^{\ 2} > 0)}$$

Example 017

- Calculate $V(\rho,\varphi,z)$ when all the walls are fixed at zero potential except for the $z = L$ wall, where the potential takes specific values $V_0(\rho,\varphi)$



$$\boxed{\begin{aligned}&Q_m = d_{\varphi c}\cos(m\varphi) + d_{\varphi s}\sin(m\varphi)\\ &(m = 0,1,2,\ldots)\end{aligned}}$$

(ii) $k_z \neq 0$

- $k_z^2 > 0$

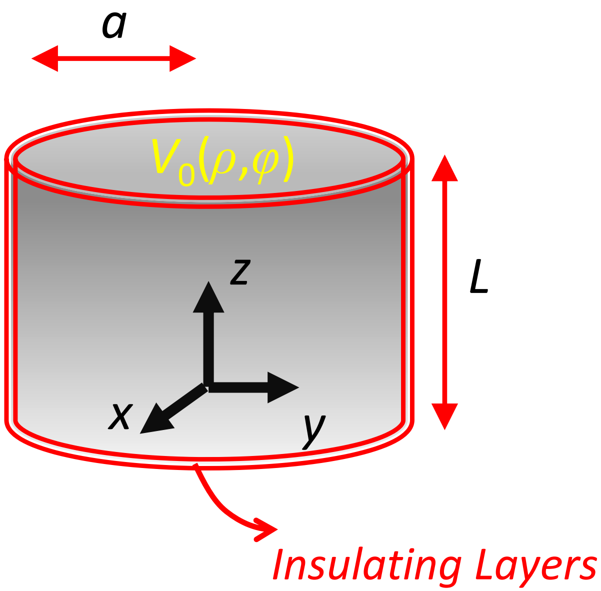
Diverging

$$R_{m,k_z}(\rho) = A_{m,k_z} J_m(k_z\rho) + B_{m,k_z} \cancel{N_m(k_z\rho)}$$

$$J_m(k_z a) = 0: \quad k_z = k_{mn} = \frac{x_{mn}}{a} \qquad (x_{mn}\text{: zeros of } J_m(x),\ n = 1,\ 2,\ 3\ldots)$$

Example 017

- Calculate $V(\rho,\varphi,z)$ when all the walls are fixed at zero potential except for the $z = L$ wall, where the potential takes specific values $V_0(\rho,\varphi)$

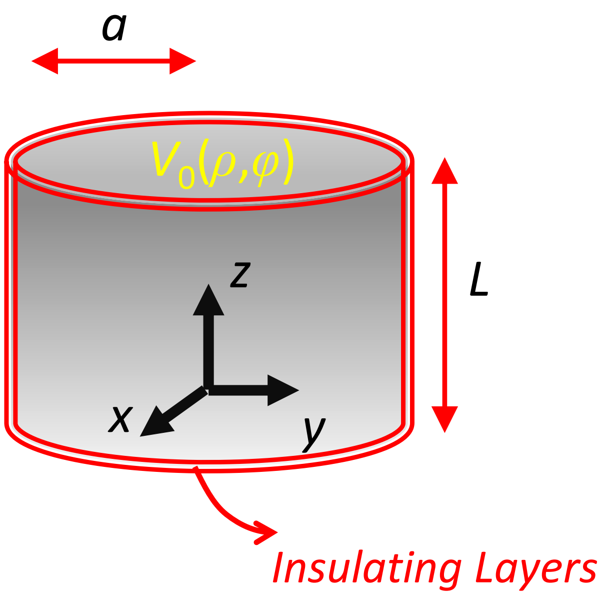


$$\boxed{V=\sum_{n=1}^{\infty}\sum_{m=0}^{\infty}J_m\left(\frac{x_{mn}}{a}\rho\right)\sinh\left(\frac{x_{mn}}{a}z\right)\left[d_{\varphi c}\cos(m\varphi)+d_{\varphi s}\sin(m\varphi)\right]}$$

$$\boxed{V(\rho,\varphi,z=L)=V_0(\rho,\varphi)}$$

Example 017

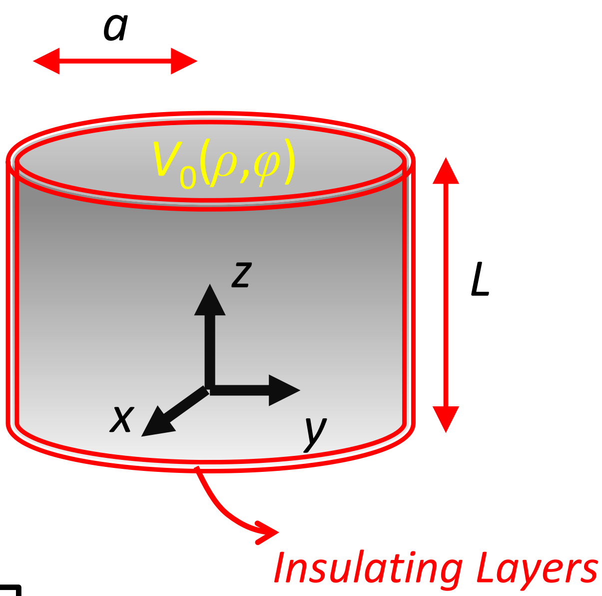
- Calculate $V(\rho,\varphi,z)$ when all the walls are fixed at zero potential except for the $z = L$ wall, where the potential takes specific values $V_0(\rho,\varphi)$



$$V(\rho,\varphi,z=L)=\sum_{n=1}^{\infty}\sum_{m=0}^{\infty}J_m(\frac{x_{mn}}{a}\rho)\sinh(\frac{x_{mn}}{a}L)\left[d_{\varphi c}\cos(m\varphi)+d_{\varphi s}\sin(m\varphi)\right]$$
$$=V_0(\rho,\varphi)$$

Example 017

- Calculate *V*(*ρ*,*φ*,*z*) when all the walls are fixed at zero potential except for the *z* = *L* wall, where the potential takes specific values $V_0(\rho,\varphi)$



$$\boxed{V=\sum_{n=1}^{\infty}\sum_{m=0}^{\infty}J_m(\frac{x_{mn}}{a}\rho)\sinh(\frac{x_{mn}}{a}z)\left[d_{\varphi c}\cos(m\varphi)+d_{\varphi s}\sin(m\varphi)\right]}$$

$$d_{\varphi c}=\frac{2\operatorname{csch}(\frac{x_{mn}}{a}L)}{\pi a^2 J_{m+1}{}^2(\frac{x_{mn}}{a}a)}\int_0^{2\pi}d\varphi\int_0^a d\rho\rho V_0(\rho,\varphi)J_m(\frac{x_{mn}}{a}\rho)\cos(m\varphi)$$

B.C. determines
the coefficient of each eigenmode

$$d_{\varphi s}=\frac{2\operatorname{csch}(\frac{x_{mn}}{a}L)}{\pi a^2 J_{m+1}{}^2(\frac{x_{mn}}{a}a)}\int_0^{2\pi}d\varphi\int_0^a d\rho\rho V_0(\rho,\varphi)J_m(\frac{x_{mn}}{a}\rho)\sin(m\varphi)$$