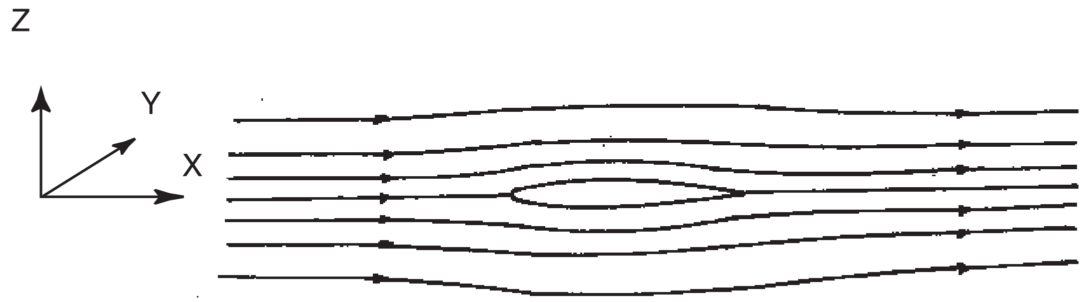
Chapter 4

Unsteady Aerodynamics

The subject of unsteady aerodynamics deals with time dependent aerodynamic forces. The objective of this chapter is to discuss briefly some basic principles of unsteady aerodynamics that concern a rotary wing system. As far as possible, lengthy mathematical derivations are avoided. There are many unique features of the unsteady aerodynamics of the rotary wing as compared to the fixed wing that merit attention. Some of these are: time varying free stream, returning wake, inflow dynamics, radial flow, dynamic stall, reversed flow and complex blade motions. It is important to first understand some of the basic phenomena related to fixed wing unsteady aerodynamics and then expand and reformulate these, wherever possible, to the rotary wing aerodynamics.

4.1 Basic Fluid Mechanics Equations

Assume a wind tunnel case where body is stationary and fluid is flowing over it. The objective is to obtain the state of the fluid at any station x, y, z at a time t. The state consists of six variables and these are pressure (p), fluid density (ρ), temperature (T), and three flow velocity components u, v, w along x, y, and z axes respectively.



There are two forces of interaction between the fluid and the body - (1) pressure force normal to the body surface, and (2) shear force, tangential to the body surface. In addition, a temperature gradient may exist between the fluid and the body. Typically, one would like to determine the pressure distribution and shear force distribution over the body. The shear force depends on the viscosity of the fluid, and velocity gradients near the wall. Without viscosity, the fluid slips past the body without exerting any tangential shear force. The governing partial differential equations are based on conservation of mass, momentum, and energy. Additionally, an algebraic equation of state is used to relate pressure and density to temperature. With the inclusion of viscosity, the equations become involved and it is extremely difficult to solve these equations, even for simple cases.

The governing equations are discussed in the following sections. The derivations of the governing equations can be found in textbooks like Refs. [1, 2].

4.1.1 Navier-Stokes equations

The continuity equation describes the conservation of mass.

1. Conservation of mass

The velocity component in the x direction is u. The velocity components in the y and z direction are u and w.

$$\begin{aligned}&\frac{\partial \rho}{\partial t}+\frac{\partial(\rho u)}{\partial x}+\frac{\partial(\rho v)}{\partial y}+\frac{\partial(\rho w)}{\partial z}=0\\&\frac{\partial \rho}{\partial t}+\nabla\cdot(\rho\mathbf{V})=\mathbf{0}\end{aligned}\tag{4.1}$$

When all spatial terms are expressed in the form of divergence, as above, the equations are said to be in a *conservation form*.

2. Conservation of momentum

Three equations, each for x, y, and z directions. First consider the equation for x direction. In *conservation form* it is given as follows.

$$\frac{\partial(\rho u)}{\partial t}+\frac{\partial(\rho u^2)}{\partial x}+\frac{\partial(\rho uv)}{\partial y}+\frac{\partial(\rho uw)}{\partial z}=\rho f_x+\frac{\partial\sigma_{xx}}{\partial x}+\frac{\partial\sigma_{xy}}{\partial y}+\frac{\partial\sigma_{xz}}{\partial z}\tag{4.2}$$

where the fluid stresses are given as

$$\sigma_{xx}=-p+\tau_{xx}\tag{4.3}$$
$$\sigma_{xy}=\tau_{xy}\tag{4.4}$$
$$\sigma_{xz}=\tau_{xz}\tag{4.5}$$

The viscous stresses can be related to velocity gradients for Newtonian fluids. Air is a Newtonian fluid. Blood is a non-Newtonian fluid.

$$\tau_{xx}=\mu\frac{\partial u}{\partial x}-\frac{2}{3}\mu\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}\right)\tag{4.6}$$
$$\tau_{xy}=\mu\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)\tag{4.7}$$
$$\tau_{xz}=\mu\left(\frac{\partial u}{\partial z}+\frac{\partial w}{\partial x}\right)\tag{4.8}$$

The momentum equations in y and z directions are similarly given as follows.

$$\frac{\partial(\rho v)}{\partial t}+\frac{\partial(\rho uv)}{\partial x}+\frac{\partial(\rho v^2)}{\partial y}+\frac{\partial(\rho vw)}{\partial z}=\rho f_x+\frac{\partial\sigma_{yx}}{\partial x}+\frac{\partial\sigma_{yy}}{\partial y}+\frac{\partial\sigma_{yz}}{\partial z}\tag{4.9}$$

$$\frac{\partial(\rho w)}{\partial t}+\frac{\partial(\rho wu)}{\partial x}+\frac{\partial(\rho wv)}{\partial y}+\frac{\partial(\rho w^2)}{\partial z}=\rho f_x+\frac{\partial\sigma_{xx}}{\partial x}+\frac{\partial\sigma_{xy}}{\partial y}+\frac{\partial\sigma_{xz}}{\partial z}\tag{4.10}$$

The stress tensor σ is given by

$$\sigma=\left(-\mathbf{p}+\frac{\mathbf{2}}{\mathbf{3}}\mu\nabla\cdot\mathbf{V}\right)\mathbf{I}+\mathbf{2}\mu\mathbf{D}\tag{4.11}$$

I is an unit tensor. **D** is the deformation tensor defined as

$$\mathbf{D} = \frac{\mathbf{1}}{\mathbf{2}}\left[\nabla\mathbf{V} + (\nabla\mathbf{V})^{\mathbf{T}}\right] \tag{4.12}$$

where $\nabla\mathbf{V}$ is the gradient tensor.

$$\nabla V = \begin{bmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{bmatrix} \tag{4.13}$$

Alternatively, in Einstein notation we have

$$\sigma_{ij} = -p\delta_{ij} - \frac{2}{3}\mu\nabla\cdot V\delta_{ij} + 2\mu D_{ij} \tag{4.14}$$

Consider equations 4.2, 4.9, and 4.10. The spatial derivatives on the left hand side (LHS) are called the convection fluxes. The spatial derivatives on the right hand side (RHS) are called the diffusive fluxes. ρf, the body forces are called the source terms. The equations can be transformed to *non-conservative* forms as follows. Consider the left hand side of any one equation, say equation 4.2. They can be simplified using the continuity equation as follows.

$$\begin{aligned} LHS =& \frac{\partial\rho u}{\partial t} + \frac{\partial(\rho u^2)}{\partial x} + \frac{\partial(\rho uv)}{\partial y} + \frac{\partial(\rho uw)}{\partial z} \\ =&\rho\frac{\partial u}{\partial t} + \rho u\frac{\partial u}{\partial x} + \rho v\frac{\partial u}{\partial y} + \rho w\frac{\partial u}{\partial z} + u\left[\frac{\partial\rho}{\partial t} + \frac{\partial(\rho u)}{\partial x} + \frac{\partial(\rho v)}{\partial y} + \frac{\partial(\rho w)}{\partial z}\right] \\ =&\rho\left(\frac{\partial}{\partial t} + u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} + w\frac{\partial}{\partial z}\right)u + u(0) \quad \text{using continuity equation} \\ =&\rho\left(\frac{\partial}{\partial t} + V.\nabla\right)u \\ =&\rho\frac{Du}{Dt} \end{aligned} \tag{4.15}$$

where Du/Dt is called the substantial derivative operator. Thus equation 4.2 can be written in non-conservation form as follows.

$$\begin{aligned} \rho\frac{Du}{Dt} =& \frac{\partial\sigma_{xx}}{\partial x} + \frac{\partial\sigma_{xy}}{\partial y} + \frac{\partial\sigma_{xz}}{\partial z} + \rho f_x \\ =& \frac{\partial p}{\partial x} + \frac{\partial\tau_{xx}}{\partial x} + \frac{\partial\tau_{xy}}{\partial y} + \frac{\partial\tau_{xz}}{\partial z} + \rho f_x \end{aligned} \tag{4.16}$$

The three equations along three directions can be written together as

$$\rho\frac{D\mathbf{V}}{Dt} = -\nabla p + \nabla\cdot\tau + \rho\mathbf{f} \tag{4.17}$$

3. Conservation of energy

The conservation of energy equation is based on the first law of thermodynamics. The total energy per unit mass of a fluid, E is given by

$$E = e + \frac{1}{2}\mathbf{v^2}$$

where e is the internal energy, and $1/2v^2$ is the kinetic energy per unit mass. The enthalpy is given by

$$h = e + pv$$

where p is the pressure, and v is specific volume. In general

$$\begin{aligned} e &= e(T, v) \\ h &= h(T, p) \end{aligned}$$

When the intermolecular forces are negligible, and the fluid is not chemically reacting, we obtain a thermally perfect gas

$$\begin{aligned} e &= e(T) \\ h &= h(T) \\ de &= c_v dT \\ dh &= c_p dT \end{aligned}$$

If the specific heats, c_v and c_p are not functions of temperature, we obtain a calorically perfect gas where

$$\begin{aligned} e &= c_v T \\ h &= c_p T \end{aligned}$$

It has been assumed that $h(T = 0) = 0$ and $e(T = 0) = 0$. The energy conservation equation is given as follows

$$\frac{\partial(\rho E)}{\partial t} + \frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z} + \frac{\partial G_x}{\partial x} + \frac{\partial G_y}{\partial y} + \frac{\partial G_z}{\partial z} = B \tag{4.18}$$

where the convective fluxes are

$$\begin{aligned} F_x &= \rho u E + pu \\ F_y &= \rho v E + pv \\ F_z &= \rho w E + pw \end{aligned} \tag{4.19}$$

The diffusive fluxes are

$$\begin{aligned} G_x &= -\tau_{xx} u - \tau_{xy} v - \tau_{xz} w + q_x \\ G_y &= -\tau_{yx} u - \tau_{yy} v - \tau_{yz} w + q_y \\ G_z &= -\tau_{zx} u - \tau_{zy} v - \tau_{zz} w + q_z \end{aligned} \tag{4.20}$$

q are the heat fluxes into the control volume. They can be related to the temperature gradients. For example, from Fourier's law of heat conduction one can write

$$q_x = -k_x A \frac{\partial T}{\partial x}$$

Similary for the other directions. A is the area perpendicular to heat flux.

$$B = \rho \dot{q} + \rho u f_x + \rho v f_y + \rho w f_z$$

where $\rho \dot{q}$ is the heat supply per unit mass. Equation 4.18 is in the conservation form. It can be reduced to the non-conservation form to read

$$\begin{aligned} \rho \frac{DE}{Dt} &= -\nabla \cdot (pV) - \nabla \cdot G + B \\ \rho \frac{De}{Dt} + \rho V \cdot \frac{DV}{Dt} &= -\nabla \cdot (pV) - \nabla \cdot G + B \end{aligned} \tag{4.21}$$

The *conservation* form and the *non-conservation* forms are also called the *strong*, and *weak* conservation forms.

4. Equation of state

The equation of state for a perfect gas (where intermolecular forces are neglected) is given by

$$p = \rho RT \tag{4.22}$$

where R is the Rankine gas constant, 287 Joules/kg.Kelvin.

$$R = \frac{\Re}{M} \tag{4.23}$$

where $\Re$ is the universal gas constant, same for all gases, 8314 Joules/(kg.mole.Kelvin). M is the molecular weight of the gas.

These are six basic equations and there are six unknowns to be evaluated. From outset it appears straight forward. But, in reality, it is not possible to get a closed form solution even for very simple problems. Naturally, we have to depend on an approximate analysis. The equations are non-linear and coupled. There are no analytical solutions.

Numerical solutions can be obtained using either finite difference, or finite element discretization. Both these discretization techniques can be cast into a finite volume method. The finite volume method when applied to the equations in conservation form ensures global conservation of mass, momentum and energy. A finite volume method is implemented in the following manner.

$$\begin{aligned} &\int_\Omega \left(\frac{\partial u}{\partial t} + \frac{\partial F}{\partial x} + \frac{\partial G}{\partial x} - B \right) d\Omega = 0 \\ &\int_\Omega \left(\frac{\partial u}{\partial t} - B \right) d\Omega + \int_\Gamma (F + G)\, n_i d\Gamma = 0 \end{aligned} \tag{4.24}$$

where $\int_\Omega$ is a volume integral, $\int_\Gamma$ is a surface integral. The following two theorems are very useful.

Divergence Theorem

The volume integral of a divergence of a quantity is equal to the surface integral of the component of the quantity normal to the surface.

$$\int_\Omega \nabla \cdot E dV = \int_\Gamma E dA \tag{4.25}$$

Stokes Theorem

The area integral of the curl of a vector is equal to the line integral of the vector around the boundary of the area.

$$\int_\Gamma \nabla \times E dA = \int_L E dl \tag{4.26}$$

The Navier-Stokes equations can be non-dimensionalized. However, for practical applications with complicated geometries, varying fluid properties, and unsteady boundary conditions the number of non-dimensional parameters become very large. Hence for many practical applications are dimensional equations are used. From experiments it has been observed that the Navier-Stokes equations describe the flow of a Newtonian fluid satisfactorily, even though, for a given boundary condition the uniqueness of a solution is difficult to prove.

The Navier-Stokes equations can be simplied for special cases. These simplified cases are also of great value to an analyst. They are incompressible flows (constant density), isothermal flows (constant viscosity), inviscid flows (zero viscosity), potential flow (zero viscosity and irrotational), creeping or Stokes flow (convective fluxes are negligible except for pressure, i.e. flow occurs under viscous, pressure and body forces only), Boussinesq flow (density is constant in the unsteady and convective fluxes, but varying in the gravitational body forces), boundary layer flow (thin shear layer), and steady flows, where the time derivatives are set to zero. The simplfied equations relevant for rotorcraft flows are discussed below.

4.1.2 Euler equations

The fluid is assumed to have no viscosity. The Euler equations are obtained by simply setting the diffusive fluxes to zero in the conservation form of the equations. The continuity equation remains same as equation 4.1.

The momentum equation, in its non-conservation form, equation 4.17, reduces to

$$\rho \frac{D\mathbf{V}}{Dt} = -\nabla p + \rho \mathbf{f} \tag{4.27}$$

The energy equation, equation 4.18, can be re-arranged in several ways. One way is to proceed from equation 4.18 as follows. First set the diffusive fluxes to zero, $G = 0$.

$$\begin{aligned} \rho \frac{De}{Dt} &= -\rho V \cdot \frac{DV}{Dt} - \nabla \cdot (pV) + B \\ &= +V \cdot \nabla p - \nabla \cdot (pV) + B && \text{using the momentum equation} \\ &= -p \nabla \cdot V + B && \text{using } \nabla \cdot (pV) = p\nabla \cdot V + V \cdot \nabla p \end{aligned} \tag{4.28}$$

The continuity equation gives the following

$$\begin{aligned} &\frac{\partial \rho}{\partial t} + (V \cdot \nabla)\rho + \rho \nabla \cdot V = 0 \\ &\nabla \cdot V = -\frac{1}{\rho} \frac{D\rho}{Dt} \end{aligned} \tag{4.29}$$

Thus the energy equation becomes

$$\begin{aligned} &\rho \frac{De}{Dt} = \frac{p}{\rho} \frac{D\rho}{Dt} + B \\ &\rho c_v \frac{De}{Dt} = \frac{p}{\rho} \frac{D\rho}{Dt} \end{aligned} \tag{4.30}$$

Assuming the body forces and heat fluxes to be zero.

The Euler equations cannot be solved analytically. The next level of approximation is to assume irrotational flow. The assumption of irrotational flow leads to the existance of velocity and acceleration potential functions. Thus, these flows are called potential flows.

4.1.3 Velocity Potential Equation for Unsteady Flows

The assumption of irrotationality reduces the momentum equations to the unsteady Bernoulli equation, also called Kelvin's equation. This is the Bernoulli equation in its most non-restrictive form, applicable for unsteady, compressible flows. The governing equation is obtained from the continuity equation. The equations are derived in ref. [3], and given below.

Flow irrotationality means that the curl of the flow velocity is zero. The curl of the flow velocity is vorticity, ξ.

$$\xi = \nabla \times V = 0 \tag{4.31}$$

This is a necessary and sufficient condition for the existance of a velocity potential such that

$$V = \nabla \phi \tag{4.32}$$

For an irrotational flow, the force field must be irrotational. A conservative force field is an irrotational force field. Consider the Euler equation 4.27. The body force per unit mass $\mathbf{f}$ can then be represented as $\nabla \Omega$. The equation is written as

$$\frac{DV}{Dt} = -\frac{\nabla p}{\rho} + \nabla \Omega \tag{4.33}$$

Now

$$\begin{aligned}
\frac{DV}{Dt} &= \frac{\partial V}{\partial t} + (V \cdot \nabla)V \\
&= \frac{\partial V}{\partial t} + \nabla \frac{V^2}{2} - V \times (\nabla \times V) \\
&= \frac{\partial V}{\partial t} + \nabla \frac{V^2}{2} - 0 \\
&= \frac{\partial \nabla \phi}{\partial t} + \nabla \frac{V^2}{2} \\
&= \nabla \left(\phi_t + \frac{V^2}{2} \right)
\end{aligned} \tag{4.34}$$

where $\phi_t = \partial\phi/\partial t$. The momentum conservation equation now becomes

$$\begin{aligned}
\nabla \left(\phi_t + \frac{V^2}{2} \right) &= -\frac{\nabla p}{\rho} + \nabla \Omega \\
&= -\nabla \int \frac{dp}{\rho} + \nabla \Omega
\end{aligned} \tag{4.35}$$

It follows

$$\nabla \left(\phi_t + \frac{V^2}{2} + \int \frac{dp}{\rho} - \Omega \right) = 0 \tag{4.36}$$

The term within the bracket can only be a constant, at most a function of time.

$$\phi_t + \frac{V^2}{2} + \int \frac{dp}{\rho} - \Omega = F(t) \tag{4.37}$$

The above equation is the most non-restrictive form of Bernoulli's equation. The right hand side can be related to conditions at a remote point where ϕ will be a constant. Thus,

$$F(t) = \frac{U_\infty^2}{2} + \int^{p_\infty} \frac{dp}{\rho} - \Omega_\infty \tag{4.38}$$

Thus equation 4.37 can be written as

$$\phi_t + \frac{1}{2}\left(V^2 - U_\infty^2\right) + \int_{p_\infty}^{p} \frac{dp}{\rho} + (\Omega_\infty - \Omega) = 0 \tag{4.39}$$

For isentropic flow

$$\int_{p_\infty}^{p} \frac{dp}{\rho} = \frac{1}{\gamma - 1}\left(a^2 - a_\infty^2\right) \tag{4.40}$$

The local pressure coefficient and the local speed of sound (or absolute temperature T) can be obtained from the above equation.

$$\begin{aligned}
C_p &= \frac{p - p_\infty}{\frac{1}{2}\rho_\infty U_\infty^2} \\
&= \frac{2}{\gamma M^2}\left\{ \left[1 - \frac{\gamma - 1}{a_\infty^2} \left(\phi_t + \frac{V^2 - U_\infty^2}{2} + \Omega_\infty - \Omega \right) \right]^{\frac{\gamma}{\gamma - 1}} - 1 \right\}
\end{aligned} \tag{4.41}$$

From equations 4.40 and 9.18 we have

$$a^2 - a_\infty^2 = -(\gamma - 1)\left[\phi_t + \frac{1}{2}\left(V^2 - U_\infty^2\right) + (\Omega_\infty - \Omega)\right] \tag{4.42}$$

We saw above how the Euler equations for momentum conservation reduce to the unsteady Bernoulli's equation (or Kelvin's equation) under the assumption of flow irrotationality. The continuity equation for mass conservation reduces to the governing governing PDE for the velocity potential ϕ.

$$\begin{aligned}
&\frac{D\rho}{Dt} + \rho\nabla\cdot V = 0\\
&\frac{D\rho}{Dt} + \rho\nabla^2 V = 0\\
&\int_{p_\infty}^{p}\frac{dp}{\rho} = -\phi_t - \frac{1}{2}\left(V^2 - U_\infty^2\right)
\end{aligned} \tag{4.43}$$

Now using Leibnitz theorem we have

$$\frac{d}{dp}\int_{p_\infty}^{p}\frac{dp}{\rho} = \frac{1}{\rho} \tag{4.44}$$

Then

$$\begin{aligned}
\frac{D}{Dt}\int_{p_\infty}^{p}\frac{dp}{\rho} &= \left[\frac{d}{dp}\int_{p_\infty}^{p}\frac{dp}{\rho}\right]\frac{Dp}{Dt}\\
&= \frac{1}{\rho}\frac{dp}{d\rho}\frac{D\rho}{Dt}\\
&= \frac{a^2}{\rho}\frac{D\rho}{Dt}
\end{aligned} \tag{4.45}$$

Using the above result with the third of equation 4.43 we have

$$\begin{aligned}
\frac{1}{\rho}\frac{D\rho}{Dt} &= -\frac{1}{a^2}\frac{D}{Dt}\left(\phi_t + \frac{1}{2}V^2\right)\\
\frac{1}{\rho}\frac{D\rho}{Dt} &= -\frac{1}{a^2}\left(\frac{\partial}{\partial t} + V\cdot\nabla\right)\left(\phi_t + \frac{V^2}{2}\right)
\end{aligned} \tag{4.46}$$

Replace the left hand side of the above expression from the second of equation 4.43. Expand the right hand side to obtain

$$\nabla^2\phi - \frac{1}{a^2}\left[\phi_{tt} + V\cdot\nabla\phi_t + V\cdot\frac{\partial V}{\partial t} + V\cdot\nabla\left(\frac{V^2}{2}\right)\right] \tag{4.47}$$

Thus the governing equation can be written in any of the following forms

$$\begin{aligned}
&\nabla^2\phi - \frac{1}{a^2}\left[\phi_{tt} + \frac{\partial}{\partial t}V^2 + V\cdot\nabla\frac{V^2}{2}\right] = 0\\
&\nabla^2\phi - \frac{1}{a^2}\left(\frac{\partial}{\partial t} + V\cdot\nabla\right)\left(\phi_t + \frac{V^2}{2}\right) = 0\\
&\nabla^2\phi - \frac{1}{a^2}\frac{D}{Dt}\left(\phi_t + \frac{V^2}{2}\right) = 0
\end{aligned} \tag{4.48}$$

These equations describe an inviscid potential flow, but non-stationary and not limited to small disturbances. It has the same form as the equation of wave motion. The disturbance represented by the velocity potential partakes of the local fluid velocity and is propagated as a wave which spreads at a rate equal to the local speed of sound.

At this stage it is important to note the following. Consider the following two assumptions, one by one.

1. The fluid is incompressible, ρ is constant everywhere.
2. The fluid flow is steady, $\partial/\partial t$ terms are zero.

Under each of these two conditions, the governing equation reduces to the same Laplace equation

$$\nabla^2\phi = 0 \tag{4.49}$$

Under the first assumption, the speed of sound goes to infinity. Under the second assumption, the unsteady terms go to zero. Thus for incompressible flow, the governing equation is indistinguishable between steady and unsteady flows. The boundary conditions, however, are to be treated differently. The classical unsteady solutions (obtained during the 1930s by several researchers) are obtained from such differences. This is treated in the section on thin airfoil theory. Thin airfoil theory deals with incompressible potential flow in 2-dimensions.

The above potential flow equations (4.48) is in its invariant form. It can be converted easily o any coordinate system, fixed or moving in space. For example, in cartesian coordinates it reduces to the following form [4]

$$\begin{aligned}\left(1-\frac{u^2}{a^2}\right)\phi_{xx}+\left(1-\frac{v^2}{a^2}\right)\phi_{yy}+\left(1-\frac{w^2}{a^2}\right)\phi_{zz}-2\frac{uv}{a^2}\phi_{xy}-2\frac{vw}{a^2}\phi_{yz}\\ -2\frac{wu}{a^2}\phi_{zx}-2\frac{u}{a^2}\phi_{xt}-2\frac{v}{a^2}\phi_{yt}-2\frac{w}{a^2}\phi_{zt}--\frac{1}{a^2}\phi_{tt}=0\end{aligned} \tag{4.50}$$

where $u=\phi_x$, $v=\phi_y$, $w=\phi_z$, and a is the velocity of propagation of disturbances, i.e. the local speed of sound.

4.1.4 The Acceleration Potential

Like velocity potential ϕ, an acceleration potential Ψ can be defined for potential flows. For an irrotational flow to remain irrotational, the force field must be irrotational. The Euler equation from equation 4.27 is then given by

$$\frac{D\mathbf{V}}{Dt}=-\frac{\nabla p}{\rho}+\frac{\mathbf{f}}{\rho} \tag{4.51}$$

where Ω is the potential of the force field per unit mass. It follows

$$\begin{aligned}\frac{DV}{Dt}&=-\frac{\nabla p}{\rho}+\nabla\Omega\\ &=\nabla\left[\Omega-\int\frac{dp}{\rho}\right]\end{aligned} \tag{4.52}$$

Thus

$$\nabla\times\frac{DV}{Dt}=0 \tag{4.53}$$

which implies DV/Dt can be expressed as a gradient of a potential $\nabla\Psi$. This is called the acceleration potential.

$$\Psi=\Omega-\int\frac{dp}{\rho} \tag{4.54}$$

or at most

$$\Psi=\Omega-\int\frac{dp}{\rho}+G(t) \tag{4.55}$$

where $G(t)$ can be function of time, generally discarded. When disturbances are small, the acceleration potential is useful.

$$\begin{aligned}
&\int_{p_\infty}^{p} \frac{dp}{\rho} \approx \frac{p - p_\infty}{\rho_\infty} \\
&\Psi = -\frac{p - p_\infty}{\rho_\infty}
\end{aligned} \tag{4.56}$$

Thus the acceleration potential Ψ denotes a pressure difference. Doublets of Ψ are useful tools to represent lifting surfaces.

4.1.5 Vorticity Conservation Equation

The Euler equations for momentum conservation can be recast into a vorticity conservation form. This form is often used by researchers for problems where preservation of flow vorticity is of great importance over large flow domains. Consider the substantive derivative of velocity (left hand side of the Euler equation).

$$\begin{aligned}
\frac{DV}{Dt} &= \frac{\partial V}{\partial t} + (V \cdot \nabla)V \\
&= \frac{\partial V}{\partial t} + \nabla \frac{V^2}{2} - V \times (\nabla \times V) \\
&= \frac{\partial V}{\partial t} + \nabla \frac{V^2}{2} - V \times \xi
\end{aligned} \tag{4.57}$$

Now take curl of the equation above

$$\begin{aligned}
\nabla \times \frac{DV}{Dt} &= \nabla \times \frac{\partial V}{\partial t} + \nabla \times \nabla \frac{V^2}{2} - \nabla \times (V \times \xi) \\
&= \frac{\partial \xi}{\partial t} + 0 - \nabla \times (V \times \xi)
\end{aligned} \tag{4.58}$$

Note the following identity

$$\nabla \times (A \times B) = (B \cdot \nabla)A - (A \cdot \nabla)B - B(\nabla \cdot A) + A(\nabla \cdot B) \tag{4.59}$$

It follows

$$\nabla \times (V \times \xi) = (\xi \cdot \nabla)V - (V \cdot \nabla)\xi - \xi(\nabla \cdot V) + V(\nabla \cdot \xi) \tag{4.60}$$

Note that $\nabla \cdot \xi = 0$. This is similar to the Maxwell equation for magnetic induction B, $\nabla \cdot B = 0$. Hence we have

$$\begin{aligned}
\nabla \times \frac{DV}{Dt} &= \frac{\partial \xi}{\partial t} + (V \cdot \nabla)\xi - (\xi \cdot \nabla)V + \xi(\nabla \cdot V) \\
&= \frac{D\xi}{Dt} - (\xi \cdot \nabla)V + \xi(\nabla \cdot V)
\end{aligned} \tag{4.61}$$

The final equation is then

$$\frac{D\xi}{Dt} = (\xi \cdot \nabla)V + \xi(\nabla \cdot V) + \nabla \times \mathbf{f} \tag{4.62}$$

where $\mathbf{f}$ is the body force per unit mass.

4.1.6 Potential Equation for Steady Flow

The potential equation for steady flow can be obtained by setting the time derivatives in equation 4.48 or equation 4.50 to zero. The later gives the following

$$\begin{aligned}\left(1-\frac{\phi_x^2}{a^2}\right)\phi_{xx}+\left(1-\frac{\phi_y^2}{a^2}\right)\phi_{yy}+\left(1-\frac{\phi_z^2}{a^2}\right)\phi_{zz}\\ -2\frac{\phi_x\phi_y}{a^2}\phi_{xy}-2\frac{\phi_y\phi_z}{a^2}\phi_{yz}-2\frac{\phi_z\phi_x}{a^2}\phi_{zx}=0\end{aligned} \tag{4.63}$$

where all velocities have been expressed in terms of the potential. Note that this equation, combines both the continuity and momentum conservation laws. The energy conservation law leads to the following. Conserving enthalpy per unit mass and assuming a calorically perfect fluid we have

$$\begin{aligned}C_pT+\frac{V^2}{2}&=C_pT_0\\ C_p&=\frac{\gamma R}{\gamma-1}\\ a&=\sqrt{\gamma RT}\end{aligned}$$

where T_0 is the stagnation temperature corresponding to zero velocity. Using the above we obtain

$$a^2-a_0^2=-\frac{\gamma-1}{2}\left(\phi_x^2+\phi_y^2+\phi_z^2\right) \tag{4.64}$$

Note that the above equation 4.64 is same as equation 4.42 for a steady case. Only that in the present case the local properties have been related to stagnation properties, whereas in the previous case they were related to un-disturbed conditions at infinity. Equations 4.63 and 4.64, as before, represent one equation for continuity, momentum and energy conservation. The non-linear PDE is applicable to subsonic, transonic, supersonic as well as hypersonic flows. The only assumptions are that of irrotational and isentropic flow.

4.1.7 Potential Equation for Incompressible Flow

In the case of incompressible flow, the continuity equation 4.1 reduces to the following form

$$\nabla\cdot\mathbf{V}=\mathbf{0} \tag{4.65}$$

The condition of flow irrotationality, $\xi=\nabla\times V=0$, gaurantees the existance of a velocity potential $V=\nabla\phi$. The continuity equation then produces the Laplace Equation.

$$\nabla^2\phi=0 \tag{4.66}$$

In cartesian coordinates

$$\phi_{xx}+\phi_{yy}+\phi_{zz}=0 \tag{4.67}$$

Both are readily deducible from equations 4.48 and 4.50. As noted earlier, for an incompressible flow the time dependancies vanish, and hence the governing equation is indistinguishable between steady and unsteady flows. The momentum conservation equation reduces to the Bernoulli's equation for steady flows. From equation 9.18 we have

$$p+\frac{1}{2}\rho V^2=\text{Constant} \tag{4.68}$$

The equations 4.83, 4.67 can be solved with values of the potential or its normal derivatives (or combinations thereof) prescribed at the boundaries. A large volume of knowledge exists on the solution of such classical boundary volume problems, see for example Lamb ([5], 1945), Milne-Thompson ([6], 1960), Thwaites ([7], 1960). The equation cannot be solved directly for arbitrary geometries. Analytical solutions exist for simple flows which can be combined to obtain solutions for practical applications. Such solutions are the uniform flow, source, sink, and a vortex flow. The strength of these solutions are treated as unknowns and determined from boundary conditions. One such application is the basis for the thin airfoil.

4.2 The Rotor Flow Field

The rotor blades operate in a high Reynolds number, typically 1–6 million, highly vortical flowfield. The aerodynamics of a rotor blade differ from that of a fixed wing due to the following phenomenon.

1. Rotor inflow, generated by high RPM of the blades (around 250 for conventional main rotors), necessary for vertical flight.

2. Cyclic variation of blade pitch angle, necessary for control.

3. Time varying, assymetric flow in forward flight with large variations of angle of attack in the advancing and retreating sides.

4. Enormous compressibility effects including shocks on the advancing side and stalled flow on the retreating side.

5. The complex, unsteady wake structure of each blade interacting with following blades.

4.2.1 Wake Structure of a Lifting Wing

Consider an un-twisted wing with symmetric airfoils in steady flight. If the angle of incidence of with respect to the oncoming flow is zero then the flow over the wing is symmetric in the vertical direction. It need not be so in the horizontal direction. If the airfoil sections are thick, and if there is sufficient skin friction the flow will seperate towards the trailing edge of the wing. This assymetry creates skin friction drag and pressure drag forces on the wing. In turn, the disturbance imparted by the wing on the surrounding air can be turbulent and unsteady. The flows on the top and bottom surfaces however, being symmetric, generates no vertical force. This symmetry is broken in the case of a wing at an angle of incidence to the oncoming flow, or in the case of a wing with cambered airfoils. The assymetry between the flow on the top and and flow on the bottom generates a vertical lifting force. The magnitude of this lifting force, or lift, is generally an order of magnitude greater than the drag force. It is the key determinant of wing performance and the performance of the aircraft as a whole. The assymetry between the flow on the top and bottom surfaces also generates an additional horizontal drag force called the induced drag. This horizontal force is produced only in the case of a finite wing. While the net pressure determines lift, the distribution of pressure determines the pitching moment about any point on a wing section. The lift, pitching moment, and drag are called airloads. The pitching moment divided by the lift force gives a distance. This is the distance from the point about which the pitching moment was calculated and the point at which the lift force acts. The later is called the center of pressure. The definition of center of pressure becomes less meaningful when the lift force approaches zero. This happens at small angles of incidences. The center of pressure then approaches infinity, oscillating wildly between positive and negative values depending on the relative signs of the lift and pitching moments. The goal is to calculate the airloads as accurately as possible.

The disturbance imparted by a lifting wing to its surrounding air is highly vortical in nature. Lanchester was the first to show that the effect of the wing on the surrounding air, and the effect of the surrounding air on the wing in terms of the lifting force can be closely simulated by a system of vortices. Lanchester's system of vortices greatly aids in understanding the basic character of the disturbance, or wake, behind finite, lifting wings. The system comprises of what is called a bound vortex system, a trailing vortex system and a starting vortex system. The trailing and starting vortex systems can be related to physical flow characteristics in the wake and are often visible to the naked eye if the conditions are right. The bound vortex system is hypothetical and forms the crux of the model. The bound vortex system replaces the wing. The finite wing theory is concerned with the determination of an equivalent bound vortex system which re-creates the real airloads, and the real disturbances imparted to the fluid, from as close to the wing as possible.

The strength of the bound vortices are related to the strengths of the trailing and starting vortices via the four fundamental theorems of vortex motion, Helmholtz's Theorms. These theorems require that a fluid initially free from vortices remain so permanently - vortices can neither be created nor destroyed. The lift force is related to the strength of the bound vortices via Kutta Joukowski condition. These inter-relations only emphasize the coupled nature of airloads and wake where one determines the other and vice versa.

4.2.2 Coupled Airloads and Wake

For a 2D airfoil, there is no trailing vortex system. If the airfoil is held steady to the oncoming flow, the starting vortex is at infinity. The bound vortex system, which models the airfoil, moves through the flow, and tries to simulate the airloads on the airfoil, and the disburbances imparted by the airfoil on the surrounding air. The strength of the bound vortex system is obtained by zero impenetrability condition, and by ensuring smooth flow over the training edge. The later is also called the Kutta condition. The forms the subject of thin airfoil theory. That is, given an airfoil at an angle of incidence the goal is to determine the bound vortex strength which generates the correct lift.

For a 3D wing, the objective is the same, but there is an added complication. That is the trailing vortex system. Depending on the strength of the trailing vortex system a downwash is induced on the airfoil, which modifies the oncoming flow, and this induced effect must be taken into account while calculating the bound vortex system. However, the strength of the trailing vortex system is itself determined by the bound vortex system. Thus, the wake (trailing vortex system) and the airloads (determined by the bound vortex system) are coupled together and must be treated as a whole.

Prandtl's lifting line model conceptually seperates the two. Its says the following: (1) use thin airfoil theory for each wing section just as if it were 2D, with the only modification, that instead of using the actual angle of incidence use a modified or reduced angle of incidence, and (2) calculate the angle of reduction that is required in step 1. To carry out step 2, the bound vortex distribution used in step 1 is collapsed to a single bound vortex. The wing model then becomes a line of bound vortices extending over the span. The trailing vortex system is then calculated using this line distribution of bound vortices across the span, hence the name lifting-line. The lifting-line forms the wing, which is used to modify the local airfoil incidences. The assumption is that the global induced flow effects of the wake can be simulated disregarding the details of the local airfoil surface flow. It works well for large aspect ratio wings without a sharp spanwise lift variation. As long as the spanwise lift distribution do not show sharp gradients the predictions are satisfactory. This method is called breaking up the wing problem into inner and outer problems. The inner problem deals with the airfoil section. The outer problem deals with the 3D reduction in angle of incidence needed in the inner problem. For wings with low aspect ratio, sweep, delta wings, etc., this approach is no longer effective. Note also, that the spanwise lift change is necessarily sharp at the tip because it drops to zero. Thus a lifting-line model is unsatisfactory near the tip. In general, a lifting model gives good lift variations for aspect ratios above 4, and for straight wings.

In the next level of modeling the wing is not idealized as a bound vortex line, but treated as a distribution of vortices. The vortex system is then treated in an unified manner. For conventional helicopter rotors, blades have a high aspect ratio (8 to 10) and are nominally straight. However the lift variation is not uniform over span due to the rotational motion of the wing. Still, the lifting line model can be successfully used for the calculation of lift. It is often modified to account for measured airfoil properties. The airfoil properties provide 2D lift, drag, and pitching moment data for realistic airfoils. In this case step 1 of the lifting line procedure can be avoided. However, step 2 must be done is a systematic manner such that the bound vortex strengths, trailing vortex strengths and the airfoil properties are all consistent. This means: (1) the predicted lift satisfies Kutta Joukowski theory as well as the airfoil properties, and (2) the reduced angle of incidences are consistent with the trailing vortex system which is consistent with the airfoil lift. A lifting line model is a simplified potential flow solution. The advantage is that the effects of compressibility and viscosity can be easily incorporated externally using prescribed airfoil tables. To understand lifting line models that are classically adopted for rotary wing calculations we begin with the concept of breaking up the problem of calculation of lift into inner and outer problems.

The inner problem is the airfoil response problem. The outer problem is the wake problem. Note that, as mentioned earlier, this break-up is only for the ease of analytical treatment. In reality the 3D wing or rotor problem is a coupled problem of airloads and wake.

4.2.3 Non-steady Excitation in Rotor Blades

In the case of a rotor blade the angle of incidence varies with time. This is due to the time varying control inputs and blade deformations. As a result the trailing vortex system is time varying. In addition, there is a shed vortex system. It can be thought of as a system of starting vortices generated due to a time varying angle of incidence on the blade. In the case of rotor blades

Physically, the problem has two parts - excitation and response. The excitation is due to blade motions and control angles. The response involves the calculation of airloads on the blade and the motion imparted to the surrounding air by the blade, i.e. the wake comprised of the trailed and shed vortex system. The airloads and the wake system, as mentioned earlier, forms a coupled problem. Given the blade excitation, modern CFD methods seek to solve the response problem directly.

For many applications a lifting line approach is adopted to a rotor. In this approach the problem is again be broken into an inner and an outer part. The advantage, as mentioned earlier, in this approach of breaking up the problem into inner and outer parts is the ease of including compressible and viscous effects using corrections to a potential flow solution. The conceptual departure from physics here is the treatment of wake as an agent of excitation, not part of the response. This is discussed in the next subsection.

4.2.4 Trailed and Shed Wake Structure of a Rotor

Consider the wake system one by one. First the trailed wake. As in the case of a fixed wing, a lifting line model can be used to calculate a modified angle of incidence. The key difference here is that the modification must account for the trailed wake from all blades. Next consider the shed wake. The shed wake can be incorporated in two ways. One way is to treat it in the same way as the trailed wake. That is, determine the effect of the shed wake from all blades to modify the angle of incidence. In this case the shed wake is treated as an agent of excitation. A disadvantage here is that, the shed wake being modeled as a vortex system, is necessarily incompressible and inviscid. The second way is to ignore the shed wake while calculating the airfoil excitation. Incorporate the shed wake by modifying the airfoil response. This second approach forms the subject matter of the theory of unsteady aerodynamics. This approach has the advantage that compressibilty, and dynamic stall effects can be incorporated using semi-empirical models. The effect of shed wake is called the circulatory effect. The unsteady airfoil response also contains a non-circulatory or apparant mass effect. This effect must be accounted for in either approach. The effect of trailed and shed wake from all the blades generate the rotor inflow.

If the rotor blades experience a constant angle of incidence, for example in the case of hover with only a collective angle, then there is no shed wake. The trailed wake, all the blades create a steady inflow through the rotor disk, varying over the blade span.

One approximate way to calculate the steady rotor inflow is to use momentum theory. This inflow can then be used to modify the blade angle of incidence. This is the blade element momentum theory. Note that it is not a lifting line model, as the inflow has not been calculated using a blade aerodynamic model, but by simple implementation of energy conservation laws.

4.2.5 Unsteady Aerodynamics

Classical 2D unsteady aerodynamic theory is concerned with calculating the effect of shed wake on airfoil airload response. For example, the effective angle of incidence of the airfoil sections on the rotor blade can be determined based on blade motions, control angles, and trailed wake from all blades. The angle of incidence is also called the effective blade element angle of attack. Based on the angle of attack and its rate of change the airload response can be modified to incorporate the effect of shed wake. The geometry of the shed wake will be different depending on the type of excitation. The effective angle of attack excitation can consist of several parts. For example, the blade pitching motions supply both an angle of attack as well as a rate of change of angle of attack, the flapping motion supplies a plunging motion, the inflow supplies a gust type velocity pattern, the unsteady response to all these stimuli are different and complex. An comprehensive treatment of these effects can be found in Leishman [8]. In general, most rotor simulations combine these effects to define an instantaneous angle of attack for each airfoil section calculated at the 3/4 chord location. The airloads are then calculated using the 'look-up' tables obtained from wind tunnel tests. These are termed quasi-steady airloads. The quasi-steady airloads can then be corrected using unsteady aerodynamic theory. The quasi-steady airloads vary with time. The unsteady correction reduces the peak magnitude and alters the phase. The correction can be made either in the frequency domain or in the time domain. They are equivalent. For example, the classical Theodorsen's theory provides the correction in the frequency domain. On the other hand Wagner's formulation is in the time domain. It is exactly equivalent to Theordorsen's theory in the frequency domain, except that it also includes the non-circulatory forces. These classical 2D theories are based on potential flow solutions to oscillating thin airfoils. Thus they ignore compressibility, viscous effects, affect of airfoil shape and most significantly flow seperation. Over the last three decades, significant improvements have been made in usteady aerodynamic modeling. Oscillating airfoil wind tunnel data have been used to develop semi-empirical models that attempt to capture the real fluid effects, including specific airfoil properties, compressibility, and flow separation. Flow seperation on rotor blades produce the phenomenon of dynamic stall. Whereas the attached flow unsteady effects are produced by continuous vortex shedding from the trailing edge, dynamic stall is characterised by abrupt vortex shedding from the leading edge. For example, semi-empirical indicial models were developed for high sub-sonic (up to Mach number 0.8) 2D unsteady aerodynamics by Leishman and Beddoes [11, 10]. The models were further extended to include nonlinear effects of flow separation [9], dynamic stall [12] and effects of blade sweep on dynamic stall [13].

4.2.6 Dynamic Stall

Dynamic stall is an unsteady flow separation phenomenon that occurs on a heavily loaded rotor or a moderately loaded rotor at high speed. The rotor blades can encounter dynamic stall on the retreating blade because of high angle of attack, and on the advancing blade because of shock